Nonlocal Hardy type inequalities with optimal constants and remainder terms

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Dedicated to Professor Jean Mawhin on the occasion of his seventieth birthday
with admiration and gratitude

Abstract - Using a groundstate transformation, we give a new proof of the optimal Stein–Weiss inequality of Herbst

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha}{2}}} I_\alpha(x - y) \frac{\varphi(y)}{|y|^{\frac{\alpha}{2}}} \, \mathrm{d}x \, \mathrm{d}y \leq \mathcal{C}_{N,\alpha,0} \int_{\mathbb{R}^N} |\varphi|^2,$$

and of its combinations with the Hardy inequality by Beckner

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha+s}{2}}} I_\alpha(x - y) \frac{\varphi(y)}{|y|^{\frac{\alpha+s}{2}}} \, \mathrm{d}x \, \mathrm{d}y \leq \mathcal{C}_{N,\alpha,1} \int_{\mathbb{R}^N} |\nabla\varphi|^2,$$

and with the fractional Hardy inequality

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha+s}{2}}} \quad I_\alpha(x - y) \frac{\varphi(y)}{|y|^{\frac{\alpha+s}{2}}} \, \mathrm{d}x \, \mathrm{d}y$$

$$\leq \mathcal{C}_{N,\alpha,s} \mathcal{D}_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left|\varphi(x) - \varphi(y)\right|^2}{|x - y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y$$

where I_α is the Riesz potential, $0 < \alpha < N$ and $0 < s < \min(N, 2)$. We also prove the optimality of the constants. The method is flexible and yields a sharp expression for the remainder terms in these inequalities.

Key words and phrases : Hardy-Littlewood-Sobolev inequality, fractional Hardy inequality, ground-state transformation, Stein-Weiss fractional integral inequality, Pitt's inequality, Riesz potential, fractional Laplacian.

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1. Introduction

E. Stein and G. Weiss [21] have proved that for every $N \geq 1$ and $\alpha \in (0, N)$ there exists a constant $C > 0$ such that for every $\varphi \in L^2(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha}{2}}} I_\alpha(x - y) \frac{\varphi(y)}{|y|^{\frac{\alpha}{2}}} \, \mathrm{d}y \, \mathrm{d}x \leq C \int_{\mathbb{R}^N} |\varphi|^2, \tag{1.1}$$

where I_α is the Riesz potential defined for $\alpha \in (0, N)$ and $x \in \mathbb{R}^N \setminus \{0\}$ by

$$I_\alpha(x) = \frac{\mathcal{A}_\alpha}{|x|^{N-\alpha}},$$

and the constant $\mathcal{A}_\alpha$ is

$$\mathcal{A}_\alpha := \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{2^\alpha \pi^{N/2} \Gamma\left(\frac{\alpha}{2}\right)},$$

so that the Riesz potentials satisfy the semigroup property $I_\alpha * I_\beta = I_{\alpha+\beta}$ for $0 < \alpha < N$ and $0 < \beta < N - \alpha$, see [19, p. 19] or [14, chapter 1.1].

The optimal constant in Stein–Weiss inequality (1.1) was computed by I. Herbst [13], who proved the following

Theorem A. (I. Herbst, 1977 [13, Theorem 2.5]) *For every $N \geq 1$ and $\alpha \in (0, N)$ it holds*

$$\mathcal{C}_{N,\alpha,0} := \sup_{\substack{\varphi \in L^2(\mathbb{R}^N) \\ \|\varphi\|_{L^2} \leq 1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha}{2}}} I_\alpha(x - y) \frac{\varphi(y)}{|y|^{\frac{\alpha}{2}}} \, \mathrm{d}x \, \mathrm{d}y = \frac{1}{2^\alpha} \left(\frac{\Gamma(\frac{N-\alpha}{4})}{\Gamma(\frac{N+\alpha}{4})}\right)^2.$$

Herbst's proof consists in writing the associated linear operator on the space $L^2(\mathbb{R}^N)$ as a convolution for the dilation group of simpler operators. A close inspection of his proof suggests that the equality is not achieved and that almost extremizers of the inequality should be similar to the function $x \in \mathbb{R}^N \mapsto |x|^{-N/2}$. The proof was rediscovered independently under the name of Pitt's inequality by W. Beckner [4], who also obtained in [5] for $N \geq 3$ the optimal constant for the combination of the Stein–Weiss inequality with the classical Hardy inequality,

$$\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha+2}{2}}} I_\alpha(x - y) \frac{\varphi(y)}{|y|^{\frac{\alpha+2}{2}}} \, \mathrm{d}x \, \mathrm{d}y \leq \mathcal{C}_{N,\alpha,1} \int_{\mathbb{R}^N} |\nabla \varphi|^2,$$

which holds for every $\varphi \in \dot{H}^1(\mathbb{R}^N)$. Here $\dot{H}^1(\mathbb{R}^N)$ is the homogeneous Sobolev space, obtained by completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{\dot{H}^1}$ defined by $\|\varphi\|_{\dot{H}^1}^2 = \int_{\mathbb{R}^N} |\nabla \varphi|^2 \, \mathrm{d}x$.

Theorem B. (W. Beckner, 2008 [5, Theorem 4]) *For every $N \geq 3$ and $\alpha \in (0, N)$, it holds*

$$\begin{aligned}
\mathcal{C}_{N,\alpha,1} &:= \sup_{\substack{\varphi \in \dot{H}^1(\mathbb{R}^N) \\ \|\nabla \varphi\|_{L^2} \leq 1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha+2}{2}}} I_\alpha(x - y) \frac{\varphi(y)}{|y|^{\frac{\alpha+2}{2}}} \, \mathrm{d}x \, \mathrm{d}y \\
&= \frac{1}{2^{\alpha-2}} \left(\frac{\Gamma(\frac{N-\alpha}{4})}{(N-2)\Gamma(\frac{N+\alpha}{4})}\right)^2.
\end{aligned}$$

In the present note, we give a simple new proof of Theorems A and B. Our proof is based on groundstate transformations in the spirit of Agmon–Allegretto–Piepenbrink [1, 3, 18], which allow to derive sharp remainder

representations in both inequalities. Herbst's inequality (Theorem A) can be deduced from the following identity

Theorem A′. *If $N \geq 1$ and $\alpha \in (0, N)$, then for every $\varphi \in L^2(\mathbb{R}^N)$ it holds*

$$
\mathcal{C}_{N,\alpha,0} \int_{\mathbb{R}^N} |\varphi|^2 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha}{2}}} I_\alpha(x-y) \frac{\varphi(y)}{|y|^{\frac{\alpha}{2}}} \, \mathrm{d}x \, \mathrm{d}y
$$
$$
+ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{I_\alpha(x-y)}{|x|^{\frac{N+\alpha}{2}} |y|^{\frac{N+\alpha}{2}}} \left| \varphi(x)|x|^{\frac{N}{2}} - \varphi(y)|y|^{\frac{N}{2}} \right|^2 \, \mathrm{d}x \, \mathrm{d}y.
$$

Beckner's inequality (Theorem B) is a consequence of its quantitative version:

Theorem B′. *If $N \geq 3$ and $\alpha \in (0, N)$, then for every $\varphi \in \dot{H}^1(\mathbb{R}^N)$ it holds*

$$
\mathcal{C}_{N,\alpha,1} \left(\int_{\mathbb{R}^N} |\nabla \varphi|^2 - \int_{\mathbb{R}^N} \frac{\left| \nabla \left(|x|^{\frac{N-2}{2}} \varphi(x) \right) \right|^2}{|x|^{N-2}} \, \mathrm{d}x \right)
$$
$$
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha+2}{2}}} I_\alpha(x-y) \frac{\varphi(y)}{|y|^{\frac{\alpha+2}{2}}} \, \mathrm{d}x \, \mathrm{d}y
$$
$$
+ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{I_\alpha(x-y)}{|x|^{\frac{N+\alpha}{2}} |y|^{\frac{N+\alpha}{2}}} \left| \varphi(x)|x|^{\frac{N-2}{2}} - \varphi(y)|y|^{\frac{N-2}{2}} \right|^2 \, \mathrm{d}x \, \mathrm{d}y.
$$

In the limiting case $\alpha = 0$, I_α is Dirac's delta and we recover the Agmon–Allegretto–Piepenbrink groundstate representation [1, 3, 18] for the classical local Hardy's inequality,

$$
\int_{\mathbb{R}^N} |\nabla \varphi|^2 \, \mathrm{d}x = \left(\frac{N-2}{2} \right)^2 \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^2} \, \mathrm{d}x + \int_{\mathbb{R}^N} \frac{\left| \nabla \left(|x|^{\frac{N-2}{2}} \varphi(x) \right) \right|^2}{|x|^{N-2}} \, \mathrm{d}x.
$$

Our proof of Theorems A′ and B′ combines previously known groundstate representations with a novel version of a groundstate representations which is designed to handle the nonlocal term on the right-hand side. Our method is flexible enough to establish the optimal constants and sharp remainder representations in a family of nonlocal Hardy type inequalities, which includes Theorems A′ and B′ as the limit cases. We prove

Theorem C. *Let $N \geq 1$, $\alpha \in (0, N)$, $s \in (0, 2)$ and $s < N$. Then*

$$
\mathcal{C}_{N,\alpha,s} := \sup_{\substack{\varphi \in \dot{H}^{s/2}(\mathbb{R}^N) \\ \|\varphi\|_{\dot{H}^{s/2}} \leq 1}} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha+s}{2}}} I_\alpha(x-y) \frac{\varphi(y)}{|y|^{\frac{\alpha+s}{2}}} \, \mathrm{d}x \, \mathrm{d}y
$$
$$
= \frac{1}{2^{\alpha+s}} \left(\frac{\Gamma\left(\frac{N-s}{4} \right) \Gamma\left(\frac{N-\alpha}{4} \right)}{\Gamma\left(\frac{N+s}{4} \right) \Gamma\left(\frac{N+\alpha}{4} \right)} \right)^2.
$$

Here $\dot{H}^s(\mathbb{R}^N)$ denotes the homogeneous Sobolev space obtained by completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\|\cdot\|_{\dot{H}^{\frac{s}{2}}}$ defined by

$$\|\varphi\|_{\dot{H}^{\frac{s}{2}}}^2 = \mathcal{D}_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left|\varphi(x) - \varphi(y)\right|^2}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y,$$

where

$$\mathcal{D}_{N,s} = \frac{\Gamma\left(\frac{N+s}{2}\right)s}{2^{2-s}\pi^{N/2}\Gamma\left(1 - \frac{s}{2}\right)}.$$

The constant $\mathcal{D}_{N,s}$ ensures that $\lim_{s\to 0}\|\varphi\|_{\dot{H}^{s/2}} = \|\varphi\|_{L^2}$ and $\lim_{s\to 2}\|\varphi\|_{\dot{H}^{s/2}} = \|\nabla\varphi\|_{L^2}$ [15]. In the limit $\alpha = 0$, I_α is Dirac's delta and Theorem C yields the fractional Hardy inequality,

$$\int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^s} \, \mathrm{d}x \leq \mathcal{C}_{N,0,s}\|\varphi\|_{\dot{H}^{\frac{s}{2}}}, \tag{1.2}$$

obtained by I. Herbst [13] and independently by D. Yafaev [22].

The quantitative version of Theorem C is

Theorem C′. *Let $N \geq 1$, $\alpha \in (0, N)$, $s \in (0, 2)$ and $s < N$. Then for all $\varphi \in \dot{H}^{s/2}(\mathbb{R}^N)$, it holds*

$$
\begin{aligned}
\mathcal{D}_{N,s} &\left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left|\varphi(x) - \varphi(y)\right|^2}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y \right. \\
&\left. - \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left||x|^{\frac{N-s}{2}}\varphi(x) - |y|^{\frac{N-s}{2}}\varphi(y)\right|^2}{|x|^{\frac{N-s}{2}}|x-y|^{N+s}|y|^{\frac{N-s}{2}}} \, \mathrm{d}x \, \mathrm{d}y \right) \\
&= \frac{1}{\mathcal{C}_{N,\alpha,s}} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{\alpha+s}{2}}} I_\alpha(x-y) \frac{\varphi(y)}{|y|^{\frac{\alpha+s}{2}}} \, \mathrm{d}x \, \mathrm{d}y \right. \\
&\left. + \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{I_\alpha(x-y)}{|x|^{\frac{N+\alpha}{2}}|y|^{\frac{N+\alpha}{2}}} \left|\varphi(x)|x|^{\frac{N-s}{2}} - \varphi(y)|y|^{\frac{N-s}{2}}\right|^2 \, \mathrm{d}x \, \mathrm{d}y \right).
\end{aligned}
$$

The groundstate representation of Theorem C′ implies that the infimum in Theorem C is never achieved in $\dot{H}^{s/2}(\mathbb{R}^N)$. In fact, the form of the remainder terms suggests that optimality in Theorem C is related to functions φ that satisfy $\varphi(x) \approx |x|^{-(N-s)/2}$ for $x \in \mathbb{R}^N \setminus \{0\}$.

In the limiting case $\alpha = 0$ the nonlocal remainder term was obtained by R. Frank, E. Lieb and R. Seiringer [11, section 4] (see also [12] and [6]) in a

nonlocal groundstate representation for the fractional Hardy inequality,

$$\mathcal{D}_{N,s} \left(\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left| \varphi(x) - \varphi(y) \right|^2}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y \right.$$
$$- \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left| |x|^{\frac{N-s}{2}} \varphi(x) - |y|^{\frac{N-s}{2}} \varphi(y) \right|^2}{|x|^{\frac{N-s}{2}} |x-y|^{N+s} |y|^{\frac{N-s}{2}}} \, \mathrm{d}x \, \mathrm{d}y \right)$$
$$= \frac{1}{\mathcal{C}_{N,0,s}} \int_{\mathbb{R}^N} \frac{|\varphi(x)|^2}{|x|^2} \, \mathrm{d}x.$$

A nonlocal groundstate transformation for a local Schrödinger operator is derived in Section 2, from which Theorems A, A$'$, B and B$'$ are deduced in Section 3. A general version of a groundstate representations for fractional operators is then obtained in Section 4 below. The proof of Theorem C and C$'$ is given in Section 5.

2. A nonlocal groundstate representation for a Schrödinger operator

Recall that if $u > 0$ is a solution of the local Schrödinger equation

$$-\Delta u + V u = 0 \qquad \text{in } \mathbb{R}^N, \tag{2.1}$$

then for all $\varphi \in C_c^\infty(\mathbb{R}^N)$ it holds

$$\int_{\mathbb{R}^N} |\nabla \varphi^2| + \int_{\mathbb{R}^N} V \varphi^2 = \int_{\mathbb{R}^N} \left| \nabla \left(\frac{\varphi}{u} \right) \right|^2 u^2. \tag{2.2}$$

This identity can be derived by simply testing the equation (2.1) against $\frac{\varphi^2}{u}$ and by integrating by parts.

Identity (2.2) is known as the *groundstate representation* of the Schrödinger operator $-\Delta + V$ with respect to a positive solution u. It was discovered independently by W. Allegretto [3] and J. Piepenbrink [18]. We refer the readers to the lecture notes [1, 2] by S. Agmon for a review of powerful applications of the groundstate representation in the context of general second order elliptic operators on Riemannian manifolds.

In this section we are going to derive a version of the groundstate representation for the nonlocal equation with a Schrödinger operator and an additional integral operator in the right-hand side,

$$-\Delta u + V u = \int_{\mathbb{R}^N} K(\cdot, y) u(y) \, \mathrm{d}y \qquad \text{in } \mathbb{R}^N.$$

Nonlocal linear equation with such structure occur, for instance, in the analysis of nonlinear Choquard (Schrödinger–Newton) equations, where groundstate representations become an important tool for proving decay bounds on the solutions and nonlinear Liouville theorems [16].

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^N$, $u \in H^1_{\mathrm{loc}}(\Omega)$, $A \in L^\infty_{\mathrm{loc}}(\Omega; \mathrm{Lin}(\mathbb{R}^N; \mathbb{R}^N))$ be self-adjoint almost everywhere in Ω, $V \in L^1_{\mathrm{loc}}(\Omega)$ and $K : \Omega \times \Omega \to [0, \infty)$ measurable such that for every $x, y \in \Omega$, $K(x,y) = K(y,x)$. If $Vu \in L^1_{\mathrm{loc}}(\Omega)$, $u^{-1} \in L^\infty_{\mathrm{loc}}(\Omega)$ and for every nonnegative $\psi \in C^1_c(\Omega)$,*

$$\int_\Omega A[\nabla u] \cdot \nabla \psi + V u \psi = \int_\Omega \int_\Omega K(x,y) u(x) \psi(y) \, \mathrm{d}y \, \mathrm{d}x,$$

then for every $\varphi \in C^1_c(\Omega)$,

$$\int_\Omega A[\nabla \varphi] \cdot \nabla \varphi + V |\varphi|^2 = \int_\Omega \int_\Omega K(x,y) \, \varphi(x) \, \varphi(y) \, \mathrm{d}y \, \mathrm{d}x$$
$$+ \int_\Omega A\Big[\nabla\Big(\frac{\varphi}{u}\Big)\Big] \cdot \nabla\Big(\frac{\varphi}{u}\Big)$$
$$+ \frac{1}{2} \int_\Omega \int_\Omega K(x,y) \, u(x) \, u(y) \Big|\frac{\varphi(x)}{u(x)} - \frac{\varphi(y)}{u(y)}\Big|^2 \, \mathrm{d}y \, \mathrm{d}x.$$

In the local case $s = 2$ an adaptation of groundstate representation (2.2) to distributional solutions $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and singular potentials $V \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ was developed in [8, Lemma 1.4] (see also [7, Theorem 2.12], [10, Lemma B.1], [16, Proposition 3.1]).

Proof of Proposition 2.1. First note that since A is locally bounded and $Vu \in L^1(\Omega)$, by classical regularization arguments, we can take nonnegative and compactly supported $\psi \in H^1(\Omega) \cap L^\infty(\Omega)$ as test functions. In particular, we can thus take $\psi = \varphi^2/u$ as a test function. We compute, since $A(x)$ is self-adjoint for almost every $x \in \Omega$, that

$$A[\nabla \varphi] \cdot \nabla \varphi = A[\nabla u] \cdot \nabla\Big(\frac{\varphi^2}{u}\Big) + A\Big[\nabla\Big(\frac{\varphi}{u}\Big)\Big] \cdot \nabla\Big(\frac{\varphi}{u}\Big)$$

and since K is symmetric,

$$\int_\Omega \int_\Omega K(x,y) \, u(x) \, \frac{\varphi(y)^2}{u(y)} \, \mathrm{d}y \, \mathrm{d}x$$
$$= \frac{1}{2} \int_\Omega \int_\Omega K(x,y) \Big(u(x)\frac{\varphi(y)^2}{u(y)} + u(y)\frac{\varphi(x)^2}{u(x)}\Big) \, \mathrm{d}y \, \mathrm{d}x$$
$$= \int_\Omega \int_\Omega K(x,y) \, \varphi(x) \, \varphi(y) \, \mathrm{d}y \, \mathrm{d}x$$
$$+ \frac{1}{2} \int_\Omega \int_\Omega K(x,y) \, u(x) \, u(y) \Big(\frac{\varphi(x)}{u(x)} - \frac{\varphi(y)}{u(y)}\Big)^2 \, \mathrm{d}y \, \mathrm{d}x;$$

this yields the conclusion.   □

3. Proofs of the inequalities of Herbst and Beckner

We first prove the quantitative version of Herbst's inequality:

Proof of Theorem A$'$. Take $u(x) = |x|^{-\frac{N}{2}}$ and

$$K(x,y) = \frac{I_\alpha(x-y)}{|x|^{\frac{\alpha}{2}}|y|^{\frac{\alpha}{2}}}.$$

By the semigroup property of the Riesz potentials (see [19]), for every $y \in \mathbb{R}^N \setminus \{0\}$,

$$\int_{\mathbb{R}^N \setminus \{0\}} K(x,y)u(x)\,\mathrm{d}x$$

$$= \int_{\mathbb{R}^N \setminus \{0\}} \frac{1}{|x|^{\frac{N+\alpha}{2}}} I_\alpha(x-y)\frac{1}{|y|^{\frac{\alpha}{2}}}\,\mathrm{d}y = \frac{1}{2^\alpha}\Big(\frac{\Gamma(\frac{N-\alpha}{4})}{\Gamma(\frac{N+\alpha}{4})}\Big)^2 \frac{1}{|y|^{\frac{N}{2}}}.$$

By Proposition 2.1 with $\Omega = \mathbb{R}^N \setminus \{0\}$, $A = 0$ and $V = \frac{1}{2^\alpha}\frac{\Gamma(\frac{N-\alpha}{4})^2}{\Gamma(\frac{N+\alpha}{4})^2}$, we have the conclusion for $\varphi \in C_c^1(\mathbb{R}^N \setminus \{0\})$. If $\varphi \in L^2(\mathbb{R}^N)$, one uses a classical density argument, passing to the limit with the help of the inequality. $\square$

We now show how Theorem A can be deduced from Theorem A$'$.

Proof of Theorem A. Take $\eta \in C\big((0,\infty);[0,1]\big)$ such that $\eta = 1$ on $(0,1)$, $\eta = 0$ on $(2,\infty)$. Define for $\lambda \geq 1$,

$$u_\lambda(x) = \eta\Big(\frac{|x|}{\lambda}\Big)\eta\Big(\frac{1}{\lambda|x|}\Big)\frac{1}{|x|^{\frac{N}{2}}}.$$

and estimate

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{I_\alpha(x-y)}{|x|^{\frac{N}{2}}|y|^{\frac{N}{2}}}\big|u_\lambda(x)|x|^{\frac{N}{2}} - u_\lambda(y)|y|^{\frac{N}{2}}\big|^2\,\mathrm{d}x\,\mathrm{d}y$$

$$\leq \int_{\mathbb{R}^{2N}\setminus(B_\lambda\setminus B_{1/\lambda})^2\setminus(B_{1/2\lambda}\cup\mathbb{R}^N\setminus B_{2\lambda})^2} \frac{I_\alpha(x-y)}{|x|^{\frac{N}{2}}|y|^{\frac{N}{2}}}\,\mathrm{d}x\,\mathrm{d}y$$

$$\leq 2\int_{B_{2\lambda}}\int_{\mathbb{R}^N\setminus B_\lambda} \frac{I_\alpha(x-y)}{|x|^{\frac{N}{2}}|y|^{\frac{N}{2}}}\,\mathrm{d}x\,\mathrm{d}y + 2\int_{B_{1/\lambda}}\int_{\mathbb{R}^N\setminus B_{1/2\lambda}} \frac{I_\alpha(x-y)}{|x|^{\frac{N}{2}}|y|^{\frac{N}{2}}}\,\mathrm{d}x\,\mathrm{d}y.$$

By scale invariance, it suffices to note that

$$\int_{B_2}\int_{\mathbb{R}^N\setminus B_1} \frac{1}{|x|^{\frac{N}{2}}|x-y|^{N-\alpha}|y|^{\frac{N}{2}}}\,\mathrm{d}x\,\mathrm{d}y < \infty,$$

to show that

$$\sup_{\lambda\geq 1}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{I_\alpha(x-y)}{|x|^{\frac{N}{2}}|y|^{\frac{N}{2}}}\big|u_\lambda(x)|x|^{\frac{N}{2}} - u_\lambda(y)|y|^{\frac{N}{2}}\big|^2\,\mathrm{d}x\,\mathrm{d}y < \infty.$$

Since

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^N} |u_\lambda|^2 = \infty,$$

the conclusion follows.   □

Now we consider Beckner's inequality:

Proof of Theorem B′. We begin as in the proof of Theorem A′, taking for every $x, y \in \mathbb{R}^N \setminus \{0\}$, $u(x) = |x|^{-\frac{N-2}{2}}$ and

$$K(x, y) = \frac{I_\alpha(x - y)}{|x|^{\frac{\alpha+2}{2}} |y|^{\frac{\alpha+2}{2}}}.$$

We compute now, by the semigroup property of the Riesz potentials, for every $y \in \mathbb{R}^N \setminus \{0\}$,

$$\int_{\mathbb{R}^N \setminus \{0\}} K(x, y) \, u(x) \, \mathrm{d}x = \frac{1}{2^\alpha} \left(\frac{\Gamma(\frac{N-\alpha}{4})}{\Gamma(\frac{N+\alpha}{4})} \right)^2 \frac{1}{|y|^{\frac{N+2}{2}}}.$$

On the other hand, we have for every $x \in \mathbb{R}^N \setminus \{0\}$,

$$-\Delta u(x) = \left(\frac{N-2}{2} \right)^2 \frac{1}{|x|^{\frac{N+2}{2}}},$$

the conclusion follows from Proposition 2.1, taking

$$A(x) = \frac{1}{2^{\alpha-2}} \left(\frac{\Gamma(\frac{N-\alpha}{4})}{(N-2)\Gamma(\frac{N+\alpha}{4})} \right)^2 \text{ and } V = 0$$

and a density argument.   □

We finally show how Theorem B follows:

Proof of Theorem B. Choose η as in the proof of Theorem A, and define now

$$u_\lambda(x) = \eta\left(\frac{|x|}{\lambda} \right) \eta\left(\frac{1}{\lambda|x|} \right) \frac{1}{|x|^{\frac{N-2}{2}}}.$$

One has, as in the proof of Theorem A,

$$\sup_{\lambda \geq 1} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{I_\alpha(x-y)}{|x|^{\frac{N+\alpha}{2}} |y|^{\frac{N+\alpha}{2}}} \left(u_\lambda(x)|x|^{\frac{N-2}{2}} - u_\lambda(y)|y|^{\frac{N-2}{2}} \right)^2 \mathrm{d}x \, \mathrm{d}y < \infty$$

and

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^N} |\nabla u_\lambda|^2 = \infty.$$

In order to conclude, note that if $\lambda \geq 1$,

$$
\int_{\mathbb{R}^N} \frac{\left| \nabla(|x|^{\frac{N-2}{2}} u_\lambda(x)) \right|^2}{|x|^{N-2}} \, \mathrm{d}x
$$

$$
= \int_{B_{2\lambda} \setminus B_\lambda} \frac{\eta'(|x|/\lambda)^2}{\lambda^2 |x|^{N-2}} \, \mathrm{d}x + \int_{B_{1/\lambda} \setminus B_{1/2\lambda}} \frac{\lambda^2 \eta'(\lambda/|x|)^2}{|x|^{N+2}} \, \mathrm{d}x
$$

$$
= \int_{B_2 \setminus B_1} \frac{\eta'(|z|)^2}{|z|^{N-2}} \, \mathrm{d}z + \int_{B_1 \setminus B_{1/2}} \frac{\eta'(1/|z|)^2}{|z|^{N+2}} \, \mathrm{d}z,
$$

whose right-hand side does not depend on λ. □

4. A nonlocal groundstate representation for the fractional Laplacian

A version of a nonlocal groundstate representation for the fractional Laplacian $(-\Delta)^{s/2}$ with $0 < s < 2$ was introduced by R. Frank, E. Lieb and R. Seiringer in [11, section 3] and [12], where (amongst other things) it was used to obtain an alternative proof of the fractional Hardy's inequality (1.2).

In this section we are going to derive a version of the groundstate representation for the nonlocal equation with a fractional Laplacian in the left and an integral operator in the right-hand side.

Proposition 4.1. *Let* $N \geq 1$, $s \in (0,2)$ *and* $s < N$. *Let* $K : \mathbb{R}^N \times \mathbb{R}^N \to [0, \infty)$ *be measurable symmetric, that is* $K(x,y) = K(y,x)$ *for almost every* $x, y \in \mathbb{R}^N$. *Let* $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$ *and assume that*

$$
\int_{\mathbb{R}^N} \frac{u(x)}{1 + |x|^{N+s}} \, dx < \infty \quad \text{and} \quad \int_{\mathbb{R}^N} K(\cdot, y) \, u(y) \, \mathrm{d}y \in L^1_{\mathrm{loc}}(\mathbb{R}^N).
$$

If for every nonnegative $\psi \in C_c^\infty(\mathbb{R}^N)$ *it holds*

$$
\int_{\mathbb{R}^N} \!\! \int_{\mathbb{R}^N} \frac{(u(x) - u(y))(\psi(x) - \psi(y))}{|x - y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^N} \!\! \int_{\mathbb{R}^N} K(x,y) \, u(y) \, \psi(x) \, \mathrm{d}y \, \mathrm{d}x,
$$

and $u^{-1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, *then for every* $\varphi \in \dot{H}^{s/2}(\mathbb{R}^N)$ *it holds*

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{|\varphi(x) - \varphi(y)|^2}{|x - y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x,y) \, \varphi(x) \, \varphi(y) \, \mathrm{d}y \, \mathrm{d}x
$$

$$
+ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x,y) \, u(x) \, u(y) \left(\frac{\varphi(x)}{u(x)} - \frac{\varphi(y)}{u(y)} \right)^2 \mathrm{d}y \, \mathrm{d}x
$$

$$
+ \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \left| \frac{\varphi(x)}{u(x)} - \frac{\varphi(y)}{u(y)} \right|^2 \frac{u(x) \, u(y)}{|x - y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y.
$$

In the case $K(x, y) = 0$ the above result improves upon [11, Proposition 4.1], where instead of $\int_{\mathbb{R}^N} \frac{u(x)}{1+|x|^{N+s}} \, dx < \infty$ a stronger assumption $u \in H^{s/2}(\mathbb{R}^N)$ was required. A similar improvement was obtained recently in [9, Lemma 2.10]. In Section 5 we will use the groundstate representation with respect to a function $u(x) = |x|^{-(N-s)/2} \notin H^{s/2}_{\mathrm{loc}}(\mathbb{R}^N)$, so such improvement is indeed important.

Proof of Proposition 4.1. First note that since $u^{-1} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$, for arbitrary $\varphi \in C^\infty_c(\mathbb{R}^N)$ we have $\psi = \varphi^2/u \in L^\infty_c(\mathbb{R}^N)$.

Let $\eta \in C^\infty_c(\mathbb{R}^N)$ be such that $\operatorname{supp} \eta \subset B_1$, $\int_{\mathbb{R}^N} \eta = 1$ and $\eta \geq 0$. For $\delta > 0$ and $x \in \mathbb{R}^N$, let $\eta_\delta(x) = \delta^{-N} \eta(x/\delta)$ and let $\check{\eta}_\delta(x) = \eta_\delta(-x)$. Given $\varphi \in C^\infty_c(\Omega)$ and $\delta > 0$, we can thus take $\psi_\delta = \check{\eta}_\delta * \frac{\varphi^2}{\eta_\delta * u} \in C^\infty_c(\Omega)$ as a test function in the equation. We will handle each of the terms separately.

Since $u \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$, we have $\eta_\delta * u \to u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$ and almost everywhere in $\mathbb{R}^N$ as $\delta \to 0$. By our assumption and Lebesgue's dominated convergence, we obtain

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) \, u(x) \left(\check{\eta}_\delta * \frac{\varphi^2}{\eta_\delta * u} \right)(y) \, dy \, dx
$$
$$
= \int_{\mathbb{R}^N} \left(\eta_\delta * \int_{\mathbb{R}^N} K(x, y) \, u(x) \, dx \right) \frac{\varphi^2}{\eta_\delta * u}(y) \, dy
$$
$$
\to \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) \, u(x) \frac{\varphi^2}{u}(y) \, dy \, dx,
$$

as $\delta \to 0$. Since K is symmetric, as in the proof of Proposition 2.1, the latter could be transformed as

$$
\int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) \, u(x) \frac{\varphi(y)^2}{u(y)} \, dy \, dx
$$
$$
= \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) \left(u(x) \frac{\varphi(y)^2}{u(y)} + u(y) \frac{\varphi(x)^2}{u(x)} \right) dy \, dx
$$
$$
= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) \, \varphi(x) \, \varphi(y) \, dy \, dx
$$
$$
+ \frac{1}{2} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} K(x, y) \, u(x) \, u(y) \left(\frac{\varphi(x)}{u(x)} - \frac{\varphi(y)}{u(y)} \right)^2 dy \, dx.
$$

In the left-hand side, by a change of variable

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\Big(\big(\check{\eta}_\delta * \tfrac{\varphi^2}{\eta_\delta * u}(x)\big) - \big(\check{\eta}_\delta * \tfrac{\varphi^2}{\eta_\delta * u}\big)(y)\Big)}{|x-y|^{N+s}} \,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(u(x)-u(y)\big)\Big(\eta_\delta(-z)\tfrac{\varphi^2}{\eta_\delta * u}(x-z) - \eta_\delta(-z)\tfrac{\varphi^2}{\eta_\delta * u}(y-z)\Big)}{|x-y|^{N+s}}\,\mathrm{d}z\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(\eta_\delta(w)u(x-w) - \eta_\delta(w)u(y-w)\big)\Big(\tfrac{\varphi^2}{\eta_\delta * u}(x) - \tfrac{\varphi^2}{\eta_\delta * u}(y)\Big)}{|x-y|^{N+s}}\,\mathrm{d}w\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big((\eta_\delta * u)(x) - (\eta_\delta * u)(y)\big)\Big(\tfrac{\varphi^2}{\eta_\delta * u}(x) - \tfrac{\varphi^2}{\eta_\delta * u}(y)\Big)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y.$$

Similarly to [11, Proposition 4.1], we obtain

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big((\eta_\delta * u)(x) - (\eta_\delta * u)(y)\big)\Big(\tfrac{\varphi^2}{\eta_\delta * u}(x) - \tfrac{\varphi^2}{\eta_\delta * u}(y)\Big)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y$$

$$=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^2 - \big|\tfrac{\varphi}{\eta_\delta * u}(x) - \tfrac{\varphi}{\eta_\delta * u}(y)\big|^2 (\eta_\delta * u)(x)\,(\eta_\delta * u)(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y$$

$$\to \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\varphi(x)-\varphi(y)|^2 - \big|\tfrac{\varphi}{u}(x) - \tfrac{\varphi}{u}(y)\big|^2 u(x)u(y)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y,$$

as $\delta \to 0$, again by Lebesgue's dominated convergence theorem. $\qquad\square$

5. Proofs of the inequalities in fractional spaces

In order to deduce the quantitative groundstate representation of Theorem C′ from its more general version of Proposition 4.1.

Proof of Theorem C′. In Proposition 4.1, take

$$u(x) = \frac{1}{|x|^{\frac{N-s}{2}}} \qquad \text{and} \qquad K(x,y) = \mathcal{C}_{N,\alpha,s}\frac{I_\alpha(x-y)}{|x|^{\frac{\alpha+s}{2}}|y|^{\frac{\alpha+s}{2}}}.$$

Since by [15],

$$\mathcal{D}_{N,s}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\big(\varphi(x)-\varphi(y)\big)\big(u(x)-u(y)\big)}{|x-y|^{N+s}}\,\mathrm{d}x\,\mathrm{d}y = \int_{\mathbb{R}^N} \overline{\widehat{\varphi}(\xi)}\,|\xi|^s\,\widehat{u}(\xi)\,d\xi$$

and

$$\hat{I}_\gamma(\xi) = \frac{1}{|\xi|^\gamma},$$

where the Fourier transform $\hat{\varphi}$ is defined for every $\xi \in \mathbb{R}^N$ by

$$\hat{\varphi}(\xi) = \frac{1}{(2\pi)^{\frac{N}{2}}}\int_{\mathbb{R}^N} \varphi(x)e^{-i\xi\cdot x}\,\mathrm{d}x,$$

we compute

$$
\mathcal{D}_{N,s} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\big(\varphi(x) - \varphi(y)\big)\big(u(x) - u(y)\big)}{|x - y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y
$$
$$
= \int_{\mathbb{R}^N} \overline{\widehat{\varphi}(\xi)} \, |\xi|^s \, I_{\frac{N-s}{2}}(\xi) \, d\xi
$$
$$
= 2^s \Big(\frac{\Gamma\big(\frac{N+s}{4}\big)}{\Gamma\big(\frac{N-s}{4}\big)}\Big)^2 \int_{\mathbb{R}^N} \overline{\widehat{\varphi}(\xi)} \, |\xi|^s \, I_{\frac{N+s}{2}}(\xi) \, d\xi
$$
$$
= 2^s \Big(\frac{\Gamma\big(\frac{N+s}{4}\big)}{\Gamma\big(\frac{N-s}{4}\big)}\Big)^2 \int_{\mathbb{R}^N} \frac{\varphi(x)}{|x|^{\frac{N+s}{2}}} \, \mathrm{d}x.
$$

On the other hand, by the semigroup property of the Riesz potentials [19], for $0 < \alpha < \beta < N$

$$
\frac{1}{|x|^{\frac{\alpha+s}{2}}} \Big(I_\alpha * \frac{1}{|x|^{-\frac{N+\alpha}{2}}}\Big) = 2^{-\alpha} \Big(\frac{\Gamma\big(\frac{N-\alpha}{4}\big)}{\Gamma\big(\frac{N+\alpha}{4}\big)}\Big)^2 \frac{1}{|x|^{\frac{N+s}{2}}}.
$$

By Proposition 4.1, we reach the required conclusion for $\varphi \in C_c^\infty(\mathbb{R}^N)$. If $\varphi \in H^{s/2}(\mathbb{R}^N)$, one uses a classical density argument, passing to the limit with the help of inequality (1.1). $\qquad \square$

We now show how optimality of the constant $\mathcal{C}_{N,\alpha,s}$ can be deduced using the remainder terms of the groundstate representation of Theorem C$'$.

Proof of Theorem C from Theorem C$'$. Take $\eta \in C((0, \infty); [0, 1])$ such that $\eta = 1$ on $(0, 1)$, $\eta = 0$ on $(2, \infty)$. Define for $s \in (0, 2)$ and $\lambda \geq 1$,

$$
u_\lambda(x) := \eta\Big(\frac{|x|}{\lambda}\Big) \eta\Big(\frac{1}{\lambda|x|}\Big) \frac{1}{|x|^{\frac{N-s}{2}}}.
$$

We shall estimate the remainders in Theorem C$'$.

For $\alpha \in (0, N)$ we obtain

$$
\mathcal{J}_\alpha(u_\lambda) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{I_\alpha(x - y)}{|x|^{\frac{N+\alpha}{2}} |y|^{\frac{N+\alpha}{2}}} \Big| u_\lambda(x)|x|^{\frac{N-s}{2}} - u_\lambda(y)|y|^{\frac{N-s}{2}} \Big|^2 \, \mathrm{d}x \, \mathrm{d}y
$$
$$
\leq 2 \int_{B_{2\lambda}} \int_{\mathbb{R}^N \setminus B_\lambda} \frac{I_\alpha(x - y)}{|x|^{\frac{N+\alpha}{2}} |y|^{\frac{N+\alpha}{2}}} \, \mathrm{d}x \, \mathrm{d}y
$$
$$
+ 2 \int_{B_{1/\lambda}} \int_{\mathbb{R}^N \setminus B_{1/2\lambda}} \frac{I_\alpha(x - y)}{|x|^{\frac{N+\alpha}{2}} |y|^{\frac{N+\alpha}{2}}} \, \mathrm{d}x \, \mathrm{d}y.
$$

By scale invariance, it suffices to note that

$$
\int_{B_2} \int_{\mathbb{R}^N \setminus B_1} \frac{I_\alpha(x - y)}{|x|^{\frac{N+\alpha}{2}} |y|^{\frac{N+\alpha}{2}}} \, \mathrm{d}x \, \mathrm{d}y < \infty,
$$

in order to conclude that

$$\limsup_{\lambda \to \infty} \mathcal{J}_\alpha(u_\lambda) < \infty.$$

For $0 < s < \min\{2, N\}$ we obtain

$$\mathcal{R}_s(u_\lambda) := \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left| u_\lambda(x)|x|^{\frac{N-s}{2}} - u_\lambda(y)|y|^{\frac{N-s}{2}} \right|^2}{|x|^{\frac{N-s}{2}} |x-y|^{N+s} |y|^{\frac{N-s}{2}}} \, \mathrm{d}x \, \mathrm{d}y,$$

$$\leq \int_{B_{2\lambda}} \int_{\mathbb{R}^N \setminus B_\lambda} \frac{1}{|x|^{\frac{N-s}{2}} |x-y|^{N+s} |y|^{\frac{N-s}{2}}} \, \mathrm{d}x \, \mathrm{d}y$$

$$+ \int_{B_{1/\lambda}} \int_{\mathbb{R}^N \setminus B_{1/2\lambda}} \frac{1}{|x|^{\frac{N-s}{2}} |x-y|^{N+s} |y|^{\frac{N-s}{2}}} \, \mathrm{d}x \, \mathrm{d}y.$$

As before, note that

$$\int_{B_2} \int_{\mathbb{R}^N \setminus B_1} \frac{1}{|x|^{\frac{N-s}{2}} |x-y|^{N+s} |y|^{\frac{N-s}{2}}} \, \mathrm{d}x \, \mathrm{d}y < \infty,$$

in order to conclude by scale invariance that for $s \in (0,2)$,

$$\limsup_{\lambda \to \infty} \mathcal{R}_s(u_\lambda) < \infty.$$

Finally, note that

$$\lim_{\lambda \to \infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{\left| u_\lambda(x) - u_\lambda(y) \right|^2}{|x-y|^{N+s}} \, \mathrm{d}x \, \mathrm{d}y$$

$$= \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{1}{|x-y|^{N+s}} \left| \frac{1}{|x|^{\frac{N-s}{2}}} - \frac{1}{|y|^{\frac{N-s}{2}}} \right|^2 \, \mathrm{d}x \, \mathrm{d}y = \infty,$$

so the conclusion follows.   □

References

[1] S. AGMON, *On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds*, in: D. Greco (ed.), Methods of functional analysis and theory of elliptic equations, Liguori, Naples, 1983, pp. 19-52.

[2] S. AGMON, Bounds on exponential decay of eigenfunctions of Schrödinger operators, Schrödinger operators (Como, 1984), Lecture Notes in Math., vol. 1159, Springer, Berlin, 1985, pp. 1-38.

[3] W. ALLEGRETTO, On the equivalence of two types of oscillation for elliptic operators, *Pacific J. Math.*, **55** (1974), 319-328.

[4] W. BECKNER, Pitt's inequality and the uncertainty principle, *Proc. Amer. Math. Soc.*, **123** (1995), 1897-1905.

[5] W. BECKNER, Pitt's inequality with sharp convolution estimates, *Proc. Amer. Math. Soc.*, **136** (2008), 1871-1885.

[6] W. BECKNER, Pitt's inequality and the fractional Laplacian: sharp error estimates, *Forum Math.*, **24** (2012), 177-209.

[7] H.L. CYCON, R.G. FROESE, W. KIRSCH and B. SIMON, *Schrödinger operators with application to quantum mechanics and global geometry*, Texts and Monographs in Physics, Springer, Berlin, 1987.

[8] M.M. FALL, Nonexistence of distributional supersolutions of a semilinear elliptic equation with Hardy potential, to appear in *J. Funct. Anal.*, arXiv:1105.5886.

[9] M.M. FALL, *Semilinear elliptic equations for the fractional Laplacian with Hardy potential*, arXiv:1109.5530.

[10] M.M. FALL and R. MUSINA, Sharp nonexistence results for a linear elliptic inequality involving Hardy and Leray potentials, *J. Inequal. Appl.*, **2011** (2011), 917201.

[11] R.L. FRANK, E.H. LIEB and R. SEIRINGER, Hardy-Lieb-Thirring inequalities for fractional Schrödinger operators, *J. Amer. Math. Soc.*, **21** (2008), 925-950.

[12] R.L. FRANK and R. SEIRINGER, Non-linear groundstate representations and sharp Hardy inequalities, *J. Funct. Anal.*, **255** (2008), 3407-3430.

[13] I.W. HERBST, Spectral theory of the operator $(p^2 + m^2)^{1/2} - Ze^2/r$, *Comm. Math. Phys.*, **53** (1977), 285-294.

[14] N.S. LANDKOF, *Foundations of modern potential theory*, Grundlehren der Mathematischen Wissenschaften, 180, Springer, New York, 1972.

[15] W. MASJA and J. NAGEL, Über äquivalente Normierung der anisotropen Funktionalräume $H^\mu(\mathbf{R}^n)$, *Beiträge Anal.*, **12** (1978), 7-17.

[16] V. MOROZ and J. VAN SCHAFTINGEN, *Nonexistence and optimal decay of supersolutions to Choquard equations in exterior domains*, arXiv:1203.3154.

[17] V. MOROZ and J. VAN SCHAFTINGEN, *Groundstates of nonlinear Choquard equations: existence, qualitative properties and decay asymptotics*, arXiv:1205.6286.

[18] J. PIEPENBRINK, Nonoscillatory elliptic equations, *J. Differential Equations*, **15** (1974), 541-550.

[19] M. RIESZ, L'intégrale de Riemann-Liouville et le problème de Cauchy, *Acta Math.*, **81** (1949), 1-223.

[20] L. SILVESTRE, Regularity of the obstacle problem for a fractional power of the Laplace operator, *Comm. Pure Appl. Math.*, **60** (2007), 67-112.

[21] E.M. STEIN and G. WEISS, Fractional integrals on n-dimensional Euclidean space, *J. Math. Mech.*, **7** (1958), 503-514.

[22] D. YAFAEV, Sharp constants in the Hardy–Rellich inequalities, *J. Funct. Anal.*, **168** (1999), 121-144.

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