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A variational approach to nonlinear diffusion equations with time periodic coefficients

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Dedicated to Professor Jean Mawhin, on the occasion of his 70^{th} anniversary

Abstract - The paper studies the validity of the equivalent formulation of a nonlinear diffusion equation with periodic data, as the minimization of a certain convex functional, by using the Legendre-Fenchel relations between two conjugated functions, j and j^*. The function j, occurring in the equation, is proper, convex and lower semicontinuous and it represents the potential related to the diffusion coefficient. In this paper we assume that j has a polynomial growth. It is proved that the diffusion equation has a unique solution if and only if this is the solution to the associated minimization problem. In fact it is shown that the dual formulation of our problem verifies the Brezis-Ekeland variational principle.

Key words and phrases : convex optimization problems, variational inequalities, Brezis-Ekeland principle, nonlinear diffusion equations, periodic solutions.

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1. Introduction

We are concerned with the study of the existence of a periodic solution to a diffusion problem with time periodic coefficients, by verifying the application of the Brezis-Ekeland principle. We consider the problem

$$\begin{aligned} \frac{\partial y}{\partial t} - \Delta \partial j(t,x,y) &\ni f && \text{in } \mathbb{R}_+ \times \Omega, \\ y(t,x) &= 0 && \text{on } \mathbb{R}_+ \times \Gamma, \\ y(t,x) &= y(t+T,x) && \text{in } \Omega, \end{aligned} \tag{1.1}$$

where Ω is an open bounded subset of $\mathbb{R}^N$, with the boundary Γ sufficiently smooth, and T is positive and finite. We assume that:

(a) $j(t,x,\cdot) : \mathbb{R} \to (-\infty, \infty]$ is proper, convex and lower semicontinuous (l.s.c. for short), for $(t,x) \in \mathbb{R}_+ \times \Omega$,

(b) the function $(t,x) \to j(t,x,r)$ is measurable for $(t,x) \in \mathbb{R}_+ \times \Omega$, for any $r \in \mathbb{R}$,

(c) f and j are periodic with respect to t, with the period T,

$$f(t,x)=f(t+T,x),\ j(t,x,r)=j(t+T,x,r),\ \text{a.e. } (t,x)\in\mathbb{R}_+\times\Omega,\ r\in\mathbb{R}. \tag{1.2}$$

By (a) it follows that the subdifferential of j,

$$\partial j(t,x,\cdot)=\beta(t,x,\cdot) \text{ a.e. } (t,x)\in\mathbb{R}_+\times\Omega, \tag{1.3}$$

is a maximal monotone graph (possibly multivalued) on $\mathbb{R}$, (see [3, p. 47]). From the physical point of view β is related to the coefficient of diffusion of the process modeled by (1.1).

Moreover we assume that

$$C_1\,|r|^m+C_1^0\le j(t,x,r)\le C_2\,|r|^m+C_2^0,\ \text{for any } r\in\mathbb{R},\ \text{a.e. } (t,x)\in\mathbb{R}_+\times\Omega, \tag{1.4}$$

with $C_1>0$,

$$r\to(\partial j)^{-1}(t,x,r) \text{ is single valued a.e. } (t,x)\in Q, \tag{1.5}$$

and $r\to\partial j(t,x,r)$ is strongly monotone, i.e., there exists $\rho>0$ such that

$$(\eta(t,x,r)-\overline{\eta}(t,x,\overline{r}))(r-\overline{r})\ge\rho\,|r-\overline{r}|^2\,, \tag{1.6}$$

$\forall r,\overline{r}\in\mathbb{R}$, $\eta(t,x,r)\in\partial j(t,x,r)$, $\overline{\eta}(t,x,\overline{r})\in\partial j(t,x,\overline{r})$ a.e. $(t,x)\in\mathbb{R}_+\times\Omega$.

The conjugate of j denoted by j^* is defined as

$$j^*(t,x,\omega)=\sup_{r\in\mathbb{R}}(\omega r-j(t,x,r)),\ \text{a.e. on } \mathbb{R}_+\times\Omega \tag{1.7}$$

and it is proper, convex, l.s.c. (see [3, p. 6]) and periodic. Moreover, the following two Legendre-Fenchel relations take place (see [3, p. 8]),

$$j(t,x,r)+j^*(t,x,\omega)\ge r\omega \text{ for any } r,\omega\in\mathbb{R},\ \text{a.e. on } \mathbb{R}_+\times\Omega, \tag{1.8}$$

$$j(t,x,r)+j^*(t,x,\omega)=r\omega \text{ if and only if } \omega\in\partial j(t,x,r),\ \text{a.e. on } \mathbb{R}_+\times\Omega. \tag{1.9}$$

Our purpose is to prove an equivalence between (1.1) and a certain minimization problem for a functional defined using j, j^* and (1.8)-(1.9).

This idea originates in the papers of Brezis and Ekeland from 1976 (see [4] and [5]) where a minimum principle for certain evolution equations (in particular for the classical heat equation) was formulated. More exactly, they considered the evolution equation with a potential operator

$$\begin{aligned}\frac{du}{dt}(t)+\partial\varphi(u(t)) &\ni f(t) \text{ a.e. } t\in(0,T),\\ u(0)&=u_0,\end{aligned} \tag{1.10}$$

where φ is proper, convex, l.s.c. on H a Hilbert space, $u_0 \in D(\varphi)$, $f \in L^2(0,T;H)$. One defines

$$K = \left\{ \begin{array}{c} v \in C([0,T];H); \frac{dv}{dt} \in L^2(0,T;H),\ \varphi(v) \in L^1(0,T), \\ \varphi^*\left(f - \frac{dv}{dt}\right) \in L^1(0,T),\ v(0) = u_0 \end{array} \right\}$$

and

$$J(v) = \int_0^T \left\{ \varphi(v) + \varphi^*\left(f - \frac{dv}{dt}\right) - (f,v) \right\} dt + \frac{1}{2}\|v(T)\|_H^2 .$$

It was stated that the solution to (1.10) is the solution to the minimization problem

$$\text{Minimize } J(v) \text{ for all } v \in K, \tag{1.11}$$

and viceversa. The necessary implication is rather obvious due to (1.8)-(1.9), written for φ. The proof of the sufficient implication was possible due the fact that it was known (by a semigroup approach) that (1.10) has a solution.

Otherwise, the implication (1.11)$\Longrightarrow$ (1.10) can be hardly proved and this has been a challenge for many researchers in the past decades. We cite a few results on this subject, in which the study of (1.11)$\Longrightarrow$ (1.10) has been approached for particular assumptions on φ : [1], [2], [6] [7], [8], [9], [10].

Our approach investigates the existence of a periodic solution and it is based on the Legendre-Fenchel relations between j and j^* (and not using φ and φ^*). We shall introduce a minimization problem (P) for a functional defined on the basis of j and j^* and we shall show that (1.1) has a solution if and only if (P) has a solution.

1.1. Functional setting

First, we specify the functional setting of the abstract formulation.

Let m and m' such that $\frac{1}{m'} + \frac{1}{m} = 1$. For any $m \geq 2$, we denote by X_m the space

$$X_m := \left\{\psi \in W^{2,m}(\Omega),\ \psi = 0 \text{ on } \Gamma\right\}$$

and let

$$B_{0,m}\psi = -\Delta\psi,\ B_{0,m} : D(B_{0,m}) = X_m \subset L^m(\Omega) \to L^m(\Omega).$$

We extend the operator $B_{0,m}$ from $L^m(\Omega)$ to the dual of X_m by the relation

$$\langle B_m y, \psi\rangle_{X'_m, X_{m'}} = \left(y, B_{0,m'}\psi\right) \text{ for } y \in L^m(\Omega) \text{ and any } \psi \in X_{m'}, \tag{1.12}$$

where

$$B_m : D(B_m) = L^m(\Omega) \subset X'_m \to X'_m,$$

and

$$(y,z) := \int_\Omega y(x)z(x)dx, \text{ for } y \in L^m(\Omega), \ z \in L^{m'}(\Omega).$$

The dual X'_m is the completion of $L^m(\Omega)$ in the norm

$$|||y||| = \left\| B_{0,m}^{-1} y \right\|_{L^m(\Omega)}, \text{ for } y \in L^m(\Omega),$$

and the norm in X'_m is defined by

$$\|y\|_{X'_m} = \|\phi\|_{L^m(\Omega)} \tag{1.13}$$

where ϕ is the unique solution to $B_m\phi = y$.

Also, we consider the Hilbert space $V = H_0^1(\Omega)$ endowed with the standard Hilbertian norm, $\|\phi\|_V = \|\nabla\phi\|_{L^2(\Omega)}$. The scalar product on its dual V' is defined as

$$(y, \overline{y})_{V'} = \left\langle y, B_{0,2}^{-1}\overline{y} \right\rangle_{V',V}, \text{ for } \theta, \overline{\theta} \in V'.$$

The results will be proved in the cases

$$\begin{aligned} N &= 1 \text{ for all } m \geq 2, \\ N &\geq 2 \text{ for } 2 \leq m \leq \frac{2N}{N-1}, \end{aligned} \tag{1.14}$$

which ensure the Sobolev inequality $W^{1,m'}(\Omega) \subset L^m(\Omega)$.

1.2. Statement of the problem

As usually, the problem will be studied first on a time period $(0,T)$, and then extended to all $\mathbb{R}_+$, so that we consider (1.1) on $Q = (0,T) \times \Omega$, i.e.,

$$\begin{aligned} \frac{\partial y}{\partial t} - \Delta \partial j(t,x,y) &\ni f && \text{in } Q = (0,T) \times \Omega, \\ y(t,x) &= 0 && \text{on } \Sigma = (0,T) \times \Gamma, \\ y(0,x) &= y(T,x) && \text{in } \Omega. \end{aligned} \tag{1.15}$$

Definition 1.1. *Let* $T > 0$,

$$f \in L^{m'}(0,T; X'_{m'}). \tag{1.16}$$

We call a **periodic weak solution** *to* (1.15) *a pair* (y, η), *such that*

$$y \in L^m(Q) \cap W^{1,m'}([0,T]; X'_{m'}),$$

$$\eta \in L^{m'}(Q), \ \eta(t,x) \in \partial j(t,x,y(t,x)) \ a.e. \ (t,x) \in Q,$$

which satisfies the equation

$$\int_0^T \left\langle \frac{dy}{dt}(t), \psi(t) \right\rangle_{X'_{m'}, X_m} dt + \int_Q \eta B_{0,m}^{-1} \psi dx dt = \int_0^T \langle f(t), \psi(t) \rangle_{X'_{m'}, X_m} dt \tag{1.17}$$

for any $\psi \in L^m(0,T;X_m)$*, and the condition* $y(0,x) = y(T,x)$*, a.e.* $(t,x) \in Q$.

We consider the abstract Cauchy problem

$$\begin{aligned} \frac{dy}{dt}(t) + B_{m'} \partial j(t,x,y(t)) &\ni f(t), \text{ a.e. } t \in (0,T) \\ y(0) &= y(T), \end{aligned} \tag{1.18}$$

and we introduce the functional

$$J : L^m(Q) \times L^{m'}(Q) \to (-\infty, \infty],$$

$$J(y,w) = \begin{cases} \int_Q \left(j(t,x,y(t,x)) + j^*(t,x,w(t,x)) - w(t,x) y(t,x) \right) dx dt, & \\ & \text{if } (y,w) \in U, \\ +\infty, & \text{otherwise,} \end{cases} \tag{1.19}$$

where

$$U = \left\{ \begin{array}{l} (y,w) \mid \ y \in L^m(Q) \cap W^{1,m'}([0,T]; X'_{m'}), \ y(0) = y(T) \in V', \\ w \in L^{m'}(Q), \ j(\cdot,\cdot,y), \ j^*(\cdot,\cdot,w) \in L^1(Q), \ (y,w) \text{ satisfies (1.20)} \end{array} \right\},$$

$$\begin{aligned} \frac{dy}{dt}(t) + B_{m'} w(t) &= f(t) \text{ a.e. } t \in (0,T), \\ y(0) &= y(T). \end{aligned} \tag{1.20}$$

The set U is not empty, since it contains any constant in time function y and $w(t) = B_{m'}^{-1} f(t)$, a.e. $t \in (0,T)$.

A *solution* to (1.20) is a pair (y,w), $y \in L^m(Q) \cap W^{1,m'}([0,T]; X'_{m'})$, $w \in L^{m'}(Q)$, which satisfies

$$\int_0^T \left\langle \frac{dy}{dt}(t), \psi(t) \right\rangle_{X'_{m'}, X_m} dt + \int_0^T \int_\Omega w B_{0,m} \psi dx dt = \int_0^T \langle f(t), \psi(t) \rangle_{X'_{m'}, X_m} dt \tag{1.21}$$

for all $\psi \in L^m(0,T;X_m)$ and $y(0) = y(T)$.

Finally, the minimization problem to be studied is

$$\text{Minimize } J(y,w) \text{ for all } (y,w) \in U. \tag{P}$$

It can be easily seen that if (y,η) is the unique periodic weak solution to (1.15) (equivalently the strong solution to (1.18)) then the minimum in (P) exists and it is zero, due to (1.8) and (1.9). We shall focus on the converse assertion meaning that (P) has a null minimizer (at which J is zero) and it is the unique weak solution to (1.15).

2. Main results

This section is devoted to the study of the existence of a solution to the minimization problem (P), and to the equivalence between (P) and (1.15), or (1.18).
We assume the hypotheses (1.16), (a)-(c), (1.4)-(1.6) replacing $\mathbb{R}_+ \times \Omega$ by Q.

Theorem 2.1. *Problem (P) has at least a solution (y^*, w^*).*

Proof. First, we are going to prove that J is proper, convex and l.s.c. Let $(y, w) \in U$ and test (1.20) by $B_{0,m}^{-1} y(t)$ and integrate over $(0, t)$. We get

$$\begin{aligned} &\int_0^t \left\langle \frac{dy}{d\tau}(\tau), B_{0,m}^{-1} y(\tau) \right\rangle_{X'_{m'}, X_m} d\tau + \int_0^t \int_\Omega wy dx d\tau \\ = &\int_0^t \left\langle f(\tau), B_{0,m}^{-1} y(\tau) \right\rangle_{X'_{m'}, X_m} d\tau. \end{aligned} \tag{2.1}$$

Denoting $B_{0,m}^{-1} y(\tau) = \psi(t)$ a.e. t and computing the first term we find that

$$\int_0^t \left\langle \frac{dy}{d\tau}(\tau), B_{0,m}^{-1} y(\tau) \right\rangle_{X'_{m'}, X_m} d\tau = \frac{1}{2} \left(\|y(t)\|_{V'}^2 - \|y(0)\|_{V'}^2 \right),$$

and we replace in (2.1)

$$\begin{aligned} &-\int_0^t \int_\Omega wy dx d\tau \\ = &\ \frac{1}{2} \|y(t)\|_{V'}^2 - \frac{1}{2} \|y(0)\|_{V'}^2 - \int_0^t \left\langle f(\tau), B_{0,m}^{-1} y(\tau) \right\rangle_{X'_{m'}, X_m} d\tau, \end{aligned} \tag{2.2}$$

a.e. $t \in (0, T)$.

Since $t \to -\int_0^t \int_\Omega yw dx d\tau + \int_0^t \left\langle f(\tau), A_{0,m}^{-1} y(\tau) \right\rangle_{X'_{m'}, X_m} d\tau$ is continuous, the previous equality is true for $t = T$, too. Then, the first two terms on the right-hand side vanish by periodicity and resuming the expression of J we get

$$\begin{aligned} &J(y, w) \\ = &\int_Q j(t, x, y(t, x)) + j^*(t, x, w(t, x))) dx dt - \int_0^T \left\langle f(\tau), B_{0,m}^{-1} y(\tau) \right\rangle_{X'_{m'}, X_m} d\tau. \end{aligned} \tag{2.3}$$

This shows that J is convex and l.s.c. Also, it is proper because by (1.19) and (1.8) it is nonnegative. Therefore, it has an infimum denoted

$$d := \inf\{J(y, w) \,|\, (y, w) \in U\}.$$

We take $(y_n, w_n)_{n\geq 1}$ a minimizing sequence for (P), i.e.,

$$d \leq J(y_n, w_n) \leq d + \frac{1}{n} \tag{2.4}$$

where (y_n, w_n) is the solution to

$$\begin{aligned} \frac{dy_n}{dt}(t) + B_{m'} w_n(t) &= f(t), \text{ a.e. } t \in (0,T), \\ y_n(0) &= y_n(T). \end{aligned} \tag{2.5}$$

By the Young inequality we compute

$$\begin{aligned} &\int_0^T \left\langle f(t), B_{0,m}^{-1} y(t) \right\rangle_{X'_{m'}, X_m} dt \leq \int_0^T \|f(t)\|_{X'_{m'}} \|y(t)\|_{L^m(\Omega)} dt \\ \leq\ & \frac{C_1}{m} \|y\|_{L^m(Q)}^m + \frac{1}{m' {C_1}^{m'/m}} \|f\|_{L^{m'}(0,T;X'_{m'})}^{m'}, \end{aligned}$$

and using (1.4) it follows that

$$\begin{aligned} &J(y,w) \\ \geq\ & \frac{C_1}{m'} \|y\|_{L^m(Q)}^m + C_1^0 + C_3 \|w\|_{L^{m'}(Q)}^{m'} + C_3^0 - \frac{1}{m' {C_1}^{m'/m}} \int_0^T \|f(t)\|_{X'_{m'}}^{m'} dt. \end{aligned} \tag{2.6}$$

By (2.4) and (2.6) we get that

$$\|y_n\|_{L^m(Q)} \leq C, \ \ \|w_n\|_{L^{m'}(Q)} \leq C,$$

with C denoting various constants independent of n.

It follows that we can extract subsequences denoted by the same subscript n, such that

$$y_n \rightharpoonup y^* \text{ in } L^m(0,T; L^m(\Omega)), \text{ as } n \to \infty, \tag{2.7}$$

$$w_n \rightharpoonup w^* \text{ in } L^{m'}(0,T; L^{m'}(\Omega)), \text{ as } n \to \infty. \tag{2.8}$$

We denote by $\rightharpoonup$ and $\to$ the weak and strong convergence, respectively.

Next, by the definition of $B_{m'}$ we have

$$\int_0^T \langle B_{m'}(w_n - w^*)(t), \psi(t) \rangle_{X'_{m'}, X_m} dt = \int_0^T ((w_n - w^*)(t), B_{0,m}\psi(t))\, dt$$

for any $\psi \in L^m(0,T; X_m)$ and so we can deduce that

$$B_{m'} w_n \rightharpoonup B_{m'} w^* \text{ in } L^{m'}(0,T; X'_{m'}), \text{ as } n \to \infty. \tag{2.9}$$

By (2.5) we get that

$$\frac{dy_n}{dt} \rightharpoonup \frac{dy^*}{dt} \text{ in } L^{m'}(0,T; X'_{m'}), \text{ as } n \to \infty \tag{2.10}$$

and we note that

$$\int_0^T \left\langle f(t), B_{0,m}^{-1} y_n(t) \right\rangle_{X'_{m'},X_m} dt \to \int_0^T \left\langle f(t), B_{0,m}^{-1} y^*(t) \right\rangle_{X'_{m'},X_m} dt.$$

Since $L^m(\Omega)$ is compact in V' it follows by (2.7) and (2.10) that

$$y_n \to y^* \text{ in } L^m(0,T;V'), \text{ as } n \to \infty, \tag{2.11}$$

whence

$$y_n(t) \to y^*(t) \text{ in } V', \text{ as } n \to \infty, \text{ a.e. } t \in (0,T). \tag{2.12}$$

Also, by (2.2) we have

$$\begin{aligned} & \|y_n(t)\|_{V'}^2 \le \|y(0)\|_{V'}^2 + \frac{2}{m'} \int_0^T \|w_n(t)\|_{L^{m'}(\Omega)}^{m'} dt \\ & + \frac{2}{m} \int_0^T \|y_n(t)\|_{L^m(\Omega)}^m dt + 2\int_0^T \|f(t)\|_{X'_{m'}} \|y_n(t)\|_{L^m(Q)} dt \\ \le \; & C, \text{ for any } t \in (0,T). \end{aligned}$$

Since $y_n(0) = y_n(T) \in V'$ it follows that $(y_n(t))_n$ is bounded in V' for any $t \in [0,T]$. Therefore, by the Ascoli-Arzelà theorem we can deduce that

$$y_n(t) \to y^*(t) \text{ in } X'_{m'}, \text{ as } n \to \infty, \text{ uniformly in } t \in [0,T] \tag{2.13}$$

and so, by the periodicity condition, we get $y^*(0) = y^*(T)$.

Now, we can pass to the limit in (2.4). We use the fact that the functions $\varphi(y_n) = \int_Q j(t,x,y_n)dxdt$ and $\varphi^*(w_n) = \int_Q j^*(t,x,w_n)dxdt$ are weakly l.s.c. (see [3, p. 56]) and get

$$d \le J(y^*,w^*) \le \liminf_{n\to\infty} J(y_n,w_n) \le d$$

which shows that (y^*,w^*) realizes the minimum in (P). Passing to the limit in the equivalent form of (2.5)

$$\int_0^T \left\langle \frac{dy_n}{dt}(t) + B_{m'}w_n(t), \phi(t) \right\rangle_{X'_{m'},X_m} dt = \int_0^T \langle f(t), \phi(t) \rangle_{X'_{m'},X_m} dt, \tag{2.14}$$

for any $\phi \in L^m(0,T;X_m)$, we obtain

$$\int_0^T \left\langle \frac{dy^*}{dt}(t) + B_{m'}w^*(t), \phi(t) \right\rangle_{X'_{m'},X_m} dt = \int_0^T \langle f(t), \phi(t) \rangle_{X'_{m'},X_m} dt, \tag{2.15}$$

for any $\phi \in L^m(0,T;X_m)$. Therefore (y^*,w^*) is the solution to (1.20) and so it belongs to U.

In conclusion (P) reaches at (y^*,w^*) its minimum on U. □

We mention that by some algebra, hypothesis (1.4) implies

$$C_3 |\omega|^{m'} + C_3^0 \leq j^*(t,x,\omega) \leq C_4 |\omega|^{m'} + C_4^0, \tag{2.16}$$

for any $\omega \in \mathbb{R}$, a.e. $(t,x) \in \mathbb{R}_+ \times \Omega$, with $C_3 > 0$,

$$|q(t,x)| \leq C_5 |r|^{m-1} + C_5^0,\ q(t,x) \in \partial j(t,x,r) \text{ a.e. on } \mathbb{R}_+ \times \Omega, \tag{2.17}$$

$$|z(t,x)| \leq C_6 |\omega|^{m'-1} + C_6^0,\ z(t,x) \in \partial j^*(t,x,\omega) \text{ a.e. on } \mathbb{R}_+ \times \Omega, \tag{2.18}$$

and we recall that $\partial j^*(t,x,\omega) = (\partial j)^{-1}(t,x,\omega)$ (see [3, p. 8]).

Moreover, by (2.17) and (2.18) we get

$$z \in L^m(Q),\ q \in L^{m'}(Q). \tag{2.19}$$

The next result shows the equivalence between (1.15) and (P), i.e., that the minimum in (P) is zero and that it is reached at the solution to (1.15).

Theorem 2.2. *Let the pair (y^*, w^*) be a solution to (P). Then,*

$$w^*(t,x) \in \partial j(t,x,y^*(t,x)) \ \text{a.e. } (t,x) \in Q, \tag{2.20}$$

$$\min_{(y,w)\in U} J(y,w) = 0 = J(y^*,w^*), \tag{2.21}$$

and (y^*, w^*) is the unique weak solution to (1.15).

Proof. Let (y^*, w^*) be a solution to (P). Then (y^*, w^*) satisfies (1.20),

$$\begin{aligned} \frac{dy^*}{dt}(t) + B_{m'} w^*(t) &= f(t) \text{ a.e. } t \in (0,T), \\ y^*(0) &= y^*(T), \end{aligned} \tag{2.22}$$

and it minimizes J, i.e.,

$$J(y^*,w^*) \leq J(y,w) \text{ for any } (y,w) \in U. \tag{2.23}$$

We introduce the variations

$$y^\lambda = y^* + \lambda Y,\ w^\lambda = w^* + \lambda W,\ \lambda > 0$$

with (Y,W) regular enough, e.g., $C^\infty(\overline{Q})$, $Y(0) = Y(T)$, such that $(y^\lambda, w^\lambda) \in U$. Then, the pair (y^λ, w^λ) satisfies (1.20)

$$\begin{aligned} \frac{dy^\lambda}{dt}(t) + B_{m'} w^\lambda(t) &= f(t) \text{ a.e. } t \in (0,T), \\ y^\lambda(0) &= y^\lambda(T) \end{aligned}$$

and so (Y,W) is the solution to the problem

$$\begin{aligned} \frac{dY}{dt}(t) + B_{0,m} W(t) &= 0 \text{ a.e. } t \in (0,T), \\ Y(0) &= Y(T). \end{aligned} \tag{2.24}$$

This implies that

$$W(t) = -B_{0,m}^{-1}\frac{dY}{dt}(t),\ t \in [0,T]. \tag{2.25}$$

We replace (y,w) by (y^λ, w^λ) in (2.23), and taking into account the expression (2.3) we get

$$\int_Q \Big(j(t,x,y^\lambda) - j(t,x,y^*) + j^*(t,x,w^\lambda) - j^*(t,x,w^*)\Big)\,dxdt$$
$$-\int_0^T \Big\langle f(t), B_{0,m}^{-1}(y^\lambda(t) - y^*(t))\Big\rangle_{X'_{m'},X_m} dt \geq 0.$$

Next we divide by $\lambda > 0$ and pass to the limit as $\lambda \to 0$. We use the definition of the directional derivative

$$\lim_{\lambda\to 0}\int_Q \frac{j(t,x,y^* + \lambda Y) - j(t,x,y^*)}{\lambda}dxdt = \int_Q j'(t,x,y^*;Y)dxdt,$$

and we obtain

$$\int_Q (j'(t,x,y^*;Y) + (j^*)'(t,x,w^*;W))dxdt - \int_0^T \Big\langle f(t), B_{0,m}^{-1}Y(t)\Big\rangle_{X'_{m'},X_m} dt \geq 0.$$

Passing from λ to $-\lambda$, using the property of the directional derivative

$$\int_Q j'(t,x,y^*;Y)dxdt \geq \int_Q \eta^* Y dxdt, \text{ for all } \eta^*(t,x) \in \partial j(t,x,y^*),$$

passing then from (Y,W) to $(-Y,-W)$, and repeating all computations we finally obtain that

$$\int_0^T \int_\Omega (\eta^* Y + \zeta^* W)dxdt - \int_0^T \Big\langle f(t), B_{0,m}^{-1}Y(t)\Big\rangle_{X'_{m'},X_m} dt = 0,$$

where $\zeta^*(t,x) = \partial j^*(t,x,w^*) = (\partial j)^{-1}(t,x,w^*)$.

Now, we replace W by (2.25) and rewrite the first term, getting

$$\int_0^T \Big\langle B_{m'}\eta^*(t), B_{0,m}^{-1}Y(t)\Big\rangle_{X'_{m'},X_m} dt - \int_0^T \Big\langle \zeta^*(t), B_{0,m}^{-1}\frac{dY}{dt}(t)\Big\rangle_{X'_{m'},X_m} dt$$
$$= \int_0^T \Big\langle f(t), B_{0,m}^{-1}Y(t)\Big\rangle_{X'_{m'},X_m} dt$$

and next we compute

$$\int_0^T \Big\langle B_{m'}\eta^*(t), B_{0,m}^{-1}Y(t)\Big\rangle_{X'_{m'},X_m} dt + \int_0^T \Big\langle \frac{d\zeta^*}{dt}(t), B_{0,m}^{-1}Y(t)\Big\rangle_{X'_{m'},X_m} dt$$
$$-\Big\langle \zeta^*(T), B_{0,m}^{-1}Y(T)\Big\rangle_{X'_{m'},X_m} + \Big\langle \zeta^*(0), B_{0,m}^{-1}Y(0)\Big\rangle_{X'_{m'},X_m}$$
$$= \int_0^T \Big\langle f(t), B_{0,m}^{-1}Y(t)\Big\rangle_{X'_{m'},X_m} dt.$$

We recall that $Y(0) = Y(T)$, $(\partial j)^{-1}$ is single valued and periodic with respect to t, by hypotheses, and so we deduce that $\zeta^*(T) = \zeta^*(0)$. Then, we can write

$$\int_0^T \left\langle \frac{d\zeta^*}{dt}(t) + B_{m'}\eta^*(t) - f(t), B_{0,m}^{-1}Y(t) \right\rangle_{X'_{m'},X_m} dt = 0 \tag{2.26}$$

which take place for any $Y \in C^\infty(\overline{Q})$. Therefore, we can write the equation

$$\frac{d\zeta^*}{dt}(t) + B_{m'}\eta^*(t) = f(t), \text{ a.e. } t \in (0,T) \tag{2.27}$$

$$\zeta^*(T) = \zeta^*(0).$$

Now we make the difference between (2.27) and (2.22) and denoting

$$p(t,x) = \zeta^*(t,x) - y^*(t,x), \quad p \in L^m(Q)$$

we have

$$\begin{aligned} -\frac{dp}{dt}(t) + B_{m'}(w^* - \eta^*)(t) &= 0, \text{ a.e. } t \in (0,T), \\ p(0) &= p(T). \end{aligned}$$

We test the equation for $B_{0,m}^{-1}p(t)$ and integrate over $(0,t)$, obtaining

$$\int_0^t \int_\Omega (\eta^* - w^*)(y^* - \zeta^*)dxdt = 0.$$

Since $w^*(t,x) \in \partial j(t,x,\zeta^*(t,x))$ and $\eta^*(t,x) \in \partial j(t,x,y^*(t,x))$ a.e. on Q, it follows by the maximal monotony of ∂j that

$$\rho \left\| y^* - \zeta^* \right\|_{L^2(Q)}^2 \le 0$$

which implies

$$y^*(t,x) = \zeta^*(t,x) \in (\partial j)^{-1}(t,x,w^*(t,x)), \text{ a.e. } (t,x) \in Q. \tag{2.28}$$

This turns out into (2.20), as claimed and proves that (y^*, w^*) is a weak solution to (1.15), satisfying $J(y^*, w^*) = 0$.

Now we prove that the solution is unique. Indeed, if there exists another solution $(\widetilde{y}, \widetilde{w})$ to (1.15) corresponding to the same data, we write the equations satisfied by their difference

$$\begin{aligned} \frac{d(y - \widetilde{y})}{dt}(t) + B_{m'}(\eta - \widetilde{\eta})(t) &= 0 \text{ a.e. } t \in (0,T), \\ (y - \widetilde{y})(0) &= (y - \widetilde{y})(T), \end{aligned}$$

where $\eta(t,x) \in \partial j(t,x,y(t,x))$, $\widetilde{\eta}(t,x) \in \partial j(t,x,\widetilde{y}(t,x))$ a.e. on Q, and (y,η) and $(\widetilde{y},\widetilde{\eta})$ belong to U.

We multiply the equation by $B_{0,m}^{-1}(y-\widetilde{y})(t)$ and integrate over $(0,t)$ obtaining

$$\int_0^t (\eta(t)-\widetilde{\eta}(t), y(t)-\widetilde{y}(t))\,dt = 0.$$

We get again $\rho \|y(t)-\widetilde{y}(t)\|^2_{L^2(Q)} \leq 0$, hence $y(t,x)=\widetilde{y}(t,x)$ a.e. on Q. □

Corollary 2.3. *Under the assumptions* (a)-(c), (1.2)-(1.6) *problem* (1.1) *has a unique periodic solution.*

Proof. Problem (1.1) is reduced to (1.15) due to the periodicity of the functions f and j. We make the transformation $t' = t - T$ and by this variable change we denote $\widetilde{y}(t',x) = y(t+T,x)$ with $t' \in [0,T]$. Using now the periodicity of the functions f and j we find again problem (1.18) which has a unique periodic solution $\widetilde{y}(t')$ belonging to $C([0,T];V')$, such that $\widetilde{y}(0) = \widetilde{y}(T)$. Coming back to the variable t we obtain that (1.1) has a continuous periodic solution on $[T,2T]$ such that $y(T) = y(2T)$ and the procedure is continued on each time period. □

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