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Multiple solutions to asymptotically linear hamiltonian systems in $\mathbb{R}^{2N}$: a shooting-type approach

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Dedicated, with admiration and gratefulness, to Professor Jean Mawhin on the occasion of his 70th birthday

Abstract - We prove the existence of multiple solutions to the Dirichlet problem associated to an asymptotically linear hamiltonian system in $\mathbb{R}^{2N}, N \geq 1$. Solutions are distinguished by means of the Maslov index of suitable auxiliary linear systems.

Key words and phrases : multiplicity, Maslov index, phase angles.

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1. Introduction

In this paper we give a multiplicity result (Theorem 2.1) for a boundary value problem of the form

$$\begin{cases} Jz' = S(t,z)z, \ z = (x,y) \in \mathbb{R}^N \times \mathbb{R}^N, \ t \in [0,\pi] \\ \\ x(0) = 0 = x(\pi), \end{cases} \tag{1.1}$$

where S is a continuous function defined in $[0,\pi] \times \mathbb{R}^{2N}$ which takes values in the space of $2N \times 2N$ real symmetric matrices. We are concerned with the case when the nonlinearity S is asymptotically linear (cf. $(H1)$ and (2.2)).

The literature for the above described problem is not very rich. In the framework of variational methods, we refer to the important papers by Benci-Fortunato [2] and Fortunato [6] (for the periodic BVP and for the Dirichlet BVP associated to second order problems, respectively). A continuation method has been used by Manásevich-Mawhin [8] for proving the existence of boundary value problems associated to strongly nonlinear systems of second order equations. In the framework of bifurcation theory, a multiplicity result has been given by the authors in [4]; as for systems of second order problems we refer to [5], which can be considered as the starting point of the present research. All the above quoted results deal with a Morse/Maslov-type index, for whom we refer to [1, 3] and references therein.

For the proof of our main result, we combine the study of the Maslov index associated to some linear systems arising from $(H1)$ and (2.2) with the classical shooting method. More precisely, under the technical assumption $(H2)$, we are led to study an auxiliary linear system in $\mathbb{R}^{2N}$ (cf. (2.9)) which uncouples in N planar systems (cf. (3.8)); to each of these planar systems we can then apply a version of Sturm comparison theorem and complete the proof.

We end this introductory section with a list of notations. For every integer $k \in \{1, \dots, N\}$, we define

$$\Pi_k = \{w \in \mathbb{R}^N \mid w_k = 0\}.$$

For every vector $w \in \mathbb{R}^N$ and for every $j \in \{1, \dots, N\}$, let $ORD_j(w)$ be the j-th component of the vector obtained from w arranging its components from the smallest to the largest.

We also denote by J the usual standard $2N \times 2N$ symplectic matrix and by J_2 the 2×2 symplectic matrix (i.e. J_2 coincides with J when $N = 1$).

Finally, by M_s^{2N} we denote the set of $2N \times 2N$ real symmetric matrices B such that, writing B in the form

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix},$$

where B_{ij} is an $N \times N$ matrix $(i, j = 1, \dots, N)$, the matrix B_{22} is positive definite.

2. Statement of the result

In this Section we consider a boundary value problem of the form

$$\begin{cases} Jz' = S(t,z)z, \;\; z = (x,y) \in \mathbb{R}^N \times \mathbb{R}^N, \;\; t \in [0,\pi] \\ x(0) = 0 = x(\pi), \end{cases} \tag{2.1}$$

where $S : [0,\pi] \times \mathbb{R}^{2N} \longrightarrow M_s^{2N}$ is a continuous function such that uniqueness and global continuability of solutions to initial value problems associated to the equation in (2.1) are guaranteed. We denote by $s_{i,j}(t,z)$, $(t,z) \in [0,\pi] \times \mathbb{R}^{2N}$, $i, j = 1, \dots, 2N$, the coefficients of the matrix $S(t,z)$.

We assume the following hypotheses:

(H1) There exists a continuous map $S_\infty : [0,\pi] \longrightarrow M_s^{2N}$ such that

$$\lim_{|z| \to +\infty} S(t,z) = S_\infty(t), \quad \text{uniformly in } \; t \in [0,\pi].$$

(H2) For every $k \in \{1, \dots, N\}$, the matrix $S_{|_{[0,\pi] \times \Pi_k \times \Pi_k}}$ is diagonal.

Moreover, let

$$S_0(t) = S(t, 0), \quad \forall\, t \in [0, \pi]. \tag{2.2}$$

In order to state our result, we need some preliminary facts about problem (2.1) and some related linear systems.

Indeed, let us first observe that $(H1)$ implies that the function S is bounded; as a consequence, we have the following result (cf. [5, Proposition 4.6]):

Proposition 2.1. *For every $R > 0$ there exists $R' > 0$ such that for every solution z of*

$$Jz' = S(t, z)z$$

we have

$$|z(0)| \leq R \quad \Rightarrow \quad |z(t)| \leq R', \quad \forall\, t \in [0, \pi]. \tag{2.3}$$

Now, let us recall some facts about oscillatory theory for linear hamiltonian systems. Let us consider a continuous map $B : [0, \pi] \to M_s^{2N}$ and the linear system

$$Jz' = B(t)z, \quad z = (x, y) \in \mathbb{R}^N \times \mathbb{R}^N. \tag{2.4}$$

We are interested in solutions of (2.4) such that

$$x(0) = 0 = x(\pi). \tag{2.5}$$

From [3, 5, 7] we know that there exist N continuous functions $\theta_1, \dots, \theta_N : [0, \pi] \to \mathbb{R}$ such that

$$\theta_i(0) = 0, \quad \forall\, i = 1, \dots, N$$

$$\theta_1(t) \leq \dots \leq \theta_N(t), \quad \forall\, t \in [0, \pi],$$

and (2.4)-(2.5) has a nontrivial solution if ond only if there exist an integer $j \in \{1, \dots, N\}$ and $h_j \in \mathbb{N}$ such that

$$\theta_j(\pi) = h_j \pi. \tag{2.6}$$

Moreover, the number of linearly independent solutions of (2.4)-(2.5) is exactly the number of indeces $j \in \{1, \dots, N\}$ such that $\theta_j(\pi)/\pi \in \mathbb{N}$.

Remark 2.1. According to the previous results, every solution of (2.4) such that $x(0) = 0$ satisfies also $x(\pi) = 0$ if and only if

$$\frac{\theta_j(\pi)}{\pi} \in \mathbb{N}, \quad \forall\, j = 1, \dots, N.$$

Remark 2.2. When $N = 1$ the function θ_1 is the usual angular coordinate associated to the planar system (2.4). In this situation θ_1 satisfies a monotonicity property; indeed, let $B_1, B_2 : [0,\pi] \to M_s^{2N}$ be continuous functions and let θ_{1,B_1} and θ_{1,B_2} be the angular coordinates associated to $J_2 z' = B_1(t)z$ and $J_2 z' = B_2(t)z$, respectively.

We then have (cfr. [9, Theorem 16.1])

$$B_1(t) \le B_2(t), \quad \forall\, t \in [0,\pi] \quad \Rightarrow \quad \theta_{1,B_1}(\pi) \le \theta_{1,B_2}(\pi),$$

where the inequality $B_1(t) \le B_2(t)$, for $t \in [0,\pi]$, means that the matrix $B_1(t) - B_2(t)$ is semi-definite negative.

Finally, for every $j \in \{1,\dots,N\}$, let us set

$$\theta_j(\pi) = k_j \pi + \alpha_j,$$

with $k_j \in \mathbb{N}$ and $0 < \alpha_j \le \pi$. Then, the Maslov index associated to (2.4) is the number

$$m(B) = k_1 + \ldots + k_N.$$

Now, for every $\underline{\alpha} \in \mathbb{R}^N$ we denote by $z_{\underline{\alpha}}$ the unique solution of the Cauchy problem

$$\begin{cases} Jz' = S(t,z)z, \ t \in (0,\pi), \ z = (x,y) \in \mathbb{R}^N \times \mathbb{R}^N \\ \\ x(0) = 0, \ y(0) = \underline{\alpha}, \end{cases} \tag{2.7}$$

and let

$$S_{\underline{\alpha}}(t) = S(t, z_{\underline{\alpha}}(t)), \quad \forall\, t \in [0,\pi]. \tag{2.8}$$

Consider the linearized system

$$Jz' = S_{\underline{\alpha}}(t)z \tag{2.9}$$

and denote by $\theta_{1,\underline{\alpha}}, \dots, \theta_{N,\underline{\alpha}}$ its phase-angles. The following result is a straightforward variant of the corresponding Propositions in [5]:

Proposition 2.2. *Under the previous assumptions we have:*

$$\theta_{j,\underline{\alpha}} \to \theta_{j,\infty}, \quad |\underline{\alpha}| \to +\infty, \quad in \quad L^\infty([0,\pi]), \quad j = 1,\dots,N,$$

$$t_{j,\underline{\alpha}} \to \theta_{j,0}, \quad |\underline{\alpha}| \to 0, \quad in \quad L^\infty([0,\pi]), \quad j = 1,\dots,N.$$

Now, let us denote by m_0 and m_∞ the numbers $m(S_0)$ and $m(S_\infty)$, respectively, and assume that the set

$$\mathcal{S} = \big\{(h_1,\dots,h_N) \in \mathbb{N}^N \mid m_0 + N < h_1 + \ldots + h_N < m_\infty,\ h_1 \le \ldots \le h_N\big\}$$

is not empty. A completely analogous result holds true if the set

$$\mathcal{S}' = \left\{(h_1, \ldots, h_N) \in \mathbb{N}^N \mid m_\infty + N < h_1 + \ldots + h_N < m_0,\, h_1 \le \ldots \le h_N\right\}$$

is not empty. Then, there exist $\epsilon > 0$ such that

$$\epsilon < (h_1 + \ldots + h_N)\pi - (m_0 + N)\pi, \quad \epsilon < m_\infty \pi - (h_1 + \ldots + h_N)\pi, \tag{2.10}$$

for every $(h_1, \ldots, h_N) \in \mathcal{S}$. From Proposition 2.2 we deduce that there exists $\alpha_\infty > 0$ such that

$$|\underline{\alpha}| \ge \alpha_\infty \quad \Rightarrow \quad \theta_{1,\underline{\alpha}}(\pi) + \ldots + \theta_{N,\underline{\alpha}}(\pi) > \theta_{1,\infty}(\pi) + \ldots + \theta_{N,\infty}(\pi) - \epsilon. \tag{2.11}$$

With α_∞ as above, let $\alpha'_\infty > 0$ as in Proposition 2.1 and define

$$\widetilde{\Pi}_k = \{w \in \Pi_k \mid \ |w| \le \alpha'_\infty\}, \quad \forall\ k = 1, \ldots, N.$$

Now, for every $k = 1, \ldots, N$, consider the diagonal matrix

$$S_{|_{[0,\pi]\times\widetilde{\Pi}_k\times\widetilde{\Pi}_k}}$$

(recall assumption $(H2)$) and denote by $s^k_{i,i}$ its non zero elements, $i = 1, \ldots, 2N$; define then

$$\lambda^k_i = \max_{(t,z)\in[0,\pi]\times\widetilde{\Pi}_k\times\widetilde{\Pi}_k} s^k_{i,i}(t,z), \quad i = 1, \ldots, 2N,$$

and the matrix

$$\Lambda^k_i = \begin{pmatrix} \lambda^k_i & 0 \\ 0 & \lambda^k_{i+N} \end{pmatrix}, \quad i = 1, \ldots, N.$$

Let θ^k_i be the angular coordinate of the planar system

$$J_2 w' = \Lambda^k_i w, \qquad w \in \mathbb{R}^2,\ i = 1, \ldots, N,$$

and let

$$\mu^k_i = \frac{\theta^k_i(\pi)}{\pi}, \quad i, k = 1, \ldots, N, \tag{2.12}$$

and

$$n_k = ORD_k(\mu^k), \quad k = 1, \ldots, N. \tag{2.13}$$

We are now in position to state our result:

Theorem 2.1. *Assume conditions* $(H1)$ *and* $(H2)$*. Moreover, let* m_0, m_∞, $n_1, \ldots, n_N$ *as above.*

Let us define

$$\mathcal{T} = \left\{ (h_1, \dots, h_N) \in \mathbb{N}^N \mid \begin{array}{l} m_0 + N < h_1 + \ldots + h_N < m_\infty, \\ \\ h_1 > n_1, \dots, h_N > n_N, \\ \\ h_1 \leq \ldots \leq h_N \end{array} \right\}.$$

Then, for every $h = (h_1, \dots, h_N) \in \mathcal{T}$ *there exist* 2^N *nontrivial solutions* z_h *of* (2.1) *such that the linear system*

$$Jz' = S(t, z_h(t))z$$

has Maslov index $h_1 + \ldots + h_N$.

3. Proof and remarks

In order to prove Theorem 2.1 we use a shooting argument; to this aim, let us fix $h = (h_1, \dots, h_N) \in \mathcal{T}$ and let $\underline{\alpha} \in \mathbb{R}^N$. Let $z_{\underline{\alpha}}$ be, as in Section 2, the solution of (2.7) and let $S_{\underline{\alpha}}$ and $\theta_{1,\underline{\alpha}}, \dots, \theta_{N,\underline{\alpha}}$ be as in the previous Section.

According to Remark 2.1, if $\underline{\alpha}$ is such that

$$\theta_{j\,a}(\pi) - h_j\pi = 0, \quad \forall\, j = 1, \dots, N, \tag{3.1}$$

then every solution of (2.9) satisfies also $x(0) = 0 = x(\pi)$; this is also true for $z_{\underline{\alpha}}$ (which, trivially, is a solution of (2.9)). Hence $z_{\underline{\alpha}}$ satisfies the boundary value problem (2.1); moreover, by the definition of the Maslov index,

$$m(S_{\underline{\alpha}}) = h_1 + \ldots + h_N.$$

Therefore, in order to prove the result, it is sufficient to find $\underline{\alpha} \in \mathbb{R}^N$ such that (3.1) holds true. This will be a consequence of the application of the following result on the existence of a zero of a vector field:

Theorem 3.1. (cfr. [5, Theorem 2.1]) *Let* $0 < r < R$ *and consider the conical shell*

$$\mathcal{W}_r^R = \{w \in \mathbb{R}^N \mid r \leq |w| \leq R,\ x_i \geq 0, \quad \forall\, i = 1, \dots, N\}.$$

Assume that $f : \mathcal{W}_r^R \to \mathbb{R}^N$ *is a continuous vector field. Let us suppose that the vector field* f *satisfies the following conditions:*

$$\begin{cases} f_1(\underline{\alpha}) + \ldots + f_N(\underline{\alpha}) < 0, & \textit{if} \ \ \underline{\alpha} \in \mathcal{W}_r^R,\ ||\underline{\alpha}|| = r, \\ \\ f_1(\underline{\alpha}) + \ldots + f_N(\underline{\alpha}) > 0, & \textit{if} \ \ \underline{\alpha} \in \mathcal{W}_r^R,\ ||\underline{\alpha}|| = R. \end{cases} \tag{3.2}$$

Moreover, let us assume that we have

$$f_k(\underline{\alpha}) < 0, \qquad \text{if } \ \underline{\alpha} \in \mathcal{W}_r^R \cap \Pi_k, \tag{3.3}$$

for every $k \in \{1, \ldots, N\}$. Then, there is at least a point $\hat{\underline{\alpha}}$ in the interior of $\mathcal{W}_r^R$ such that

$$f(\hat{\underline{\alpha}}) = 0.$$

Remark 3.1. According to [5], an analogous result holds true if $\mathcal{W}_r^R$ is replaced by the set

$$\tilde{\mathcal{W}}_r^R = \{w \in \mathbb{R}^N \mid \ r \leq |w| \leq R\} \cap O_N, \tag{3.4}$$

where O_N is any octant of $\mathbb{R}^N$.

Now, we will apply Theorem 3.1 to the vector field f whose components are defined by

$$f_i(\underline{\alpha}) = \theta_{i,\underline{\alpha}}(\pi) - h_i \pi, \tag{3.5}$$

for every $\underline{\alpha} \in \mathbb{R}^N$ and $i \in \{1, \ldots, N\}$. For $\epsilon > 0$ as in (2.10), from Proposition 2.2 we deduce that there exists $\alpha_0 > 0$, $\alpha_0 < \alpha_\infty$ such that

$$|\underline{\alpha}| \leq \alpha_0 \quad \Rightarrow \quad \theta_{1,\underline{\alpha}}(\pi) + \ldots + \theta_{N,\underline{\alpha}}(\pi) < \theta_{1,0}(\pi) + \ldots + \theta_{N,0}(\pi) + \epsilon. \tag{3.6}$$

Let us consider the conical shell $\mathcal{W}_{\alpha_0}^{\alpha_\infty}$; if $\underline{\alpha} \in \mathcal{W}_{\alpha_0}^{\alpha_\infty}$ and $|\underline{\alpha}| = \alpha_0$, then the definition of the Maslov index $m_0 = m(S_0)$ and (3.6) imply that

$$\theta_{1,\underline{\alpha}}(\pi) + \ldots + \theta_{N,\underline{\alpha}}(\pi) < (m_0 + N)\pi + \epsilon,$$

i.e.

$$f_1(\underline{\alpha}) + \ldots f_N(\underline{\alpha}) = \theta_{1,\underline{\alpha}}(\pi) + \ldots + \theta_{N,\underline{\alpha}}(\pi) - (h_1 + \ldots + h_N)\pi <$$

$$< (m_0 + N)\pi - (h_1 + \ldots + h_N)\pi + \epsilon < 0,$$

by (2.10). This proves the validity of the first inequality in (3.2).

Similarly, when $\underline{\alpha} \in \mathcal{W}_{\alpha_0}^{\alpha_\infty}$ and $|\underline{\alpha}| = \alpha_\infty$, from (2.10) and (2.11) we infer that

$$f_1(\underline{\alpha}) + \ldots f_N(\underline{\alpha}) = \theta_{1,\underline{\alpha}}(\pi) + \ldots + \theta_{N,\underline{\alpha}}(\pi) - (h_1 + \ldots + h_N)\pi >$$

$$> m_\infty \pi - \epsilon - (h_1 + \ldots + h_N)\pi > 0;$$

as a consequence, also the second inequality in (3.2) is satisfied.

We now check the validity of (3.3); we consider the case $k = 1$ (the other cases can be proved in a very similar way). Hence, let $\underline{\alpha} \in \mathcal{W}_{\alpha_0}^{\alpha_\infty} \cap \Pi_1$; from an easy computation it is possible to show that

$$\underline{\alpha} \in \Pi_1 \quad \Rightarrow \quad z_{\underline{\alpha}} \in \Pi_1 \times \Pi_1.$$

Moreover, from Proposition 2.1 we also know that

$$|\underline{\alpha}| \leq \alpha_\infty \quad \Rightarrow \quad |z_{\underline{\alpha}}(t)| \leq \alpha'_\infty, \quad \forall\, t \in [0,\pi];$$

hence, we have

$$|\underline{\alpha}| \leq \alpha_\infty,\ \underline{\alpha} \in \Pi_1 \quad \Rightarrow \quad z_{\underline{\alpha}}(t) \in \widetilde{\Pi}_1 \times \widetilde{\Pi}_1, \quad \forall\, t \in [0,\pi].$$

This implies that the matrix

$$S_{\underline{\alpha}}(t) = S(t, z_{\underline{\alpha}}(t))$$

is diagonal and its entries $s^1_{\underline{\alpha};i,i}$ satisfy

$$s^1_{\underline{\alpha};i,i}(t) \leq \lambda^1_i, \quad \forall\, t \in [0,\pi],\ \forall\, i = 1,\ldots,N. \tag{3.7}$$

Moreover, it is possible to see that the linearized system (2.9) uncouples into N planar systems

$$J_2 w' = \tilde{S}_{\underline{\alpha},i}(t) w, \quad w \in \mathbb{R}^2,\ i = 1,\ldots,N, \tag{3.8}$$

where $\tilde{S}_{\underline{\alpha},i}$ is the matrix

$$\tilde{S}_{\underline{\alpha},i} = \begin{pmatrix} s^1_{\underline{\alpha};i,i} & 0 \\ 0 & s^1_{\underline{\alpha};i+N,i+N} \end{pmatrix}, \quad i = 1,\ldots,N.$$

As a consequence, up to a reordering, the phase-angles $\theta_{1,\underline{\alpha}},\ldots,\theta_{N,\underline{\alpha}}$ coincide with the phase-angles $\tilde{\theta}_{1,\underline{\alpha}},\ldots,\tilde{\theta}_{N,\underline{\alpha}}$ of the planar systems (3.8). Recalling (3.7) and Remark 2.2, we infer that

$$\tilde{\theta}_{i,\underline{\alpha}}(\pi) \leq \theta^1_i(\pi), \quad \forall\, i = 1,\ldots,N;$$

hence

$$\frac{\tilde{\theta}_{i,\underline{\alpha}}(\pi)}{\pi} \leq \frac{\theta^1_i(\pi)}{\pi} = \mu^1_i, \quad \forall\, i = 1,\ldots,N,$$

by (2.12). In particular, from (2.13) we deduce that there exists $i^* \in \{1,\ldots,N\}$ such that

$$\mu^1_{i^*} = n_1;$$

as a consequence, we obtain

$$\theta_{1,\underline{\alpha}}(\pi) \leq \tilde{\theta}_{i^*,\underline{\alpha}}(\pi) \leq n_1\pi.$$

This implies that

$$f_1(\underline{\alpha}) = \theta_{1,\underline{\alpha}}(\pi) - h_1\pi \leq (n_1 - h_1)\pi < 0,$$

proving (3.3) when $k = 1$.

An application of Theorem 3.1 ensures the existence of $\underline{\alpha} \in \mathcal{W}^{\alpha_\infty}_{\alpha_0}$ such that $f(\underline{\alpha}) = 0$; the corresponding function $z_{\underline{\alpha}}$ is a solution of (2.1).

To obtain the other $2^N - 1$ solutions of (2.1) it is sufficient to apply Theorem 3.1 to any set of the form (3.4).

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