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Nontrivial solutions for a class of one-parameter problems with singular ϕ-Laplacian

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Dedicated to Jean Mawhin for his 70th anniversary

Abstract - We study the mixed boundary value problem with singular ϕ-Laplacian

$$[r^{N-1}\phi(u')]' = r^{N-1}[\alpha(r)u^{q-1} - \lambda p(r,u)] \quad \text{in} \quad [0,R], \quad u'(0) = 0 = u(R),$$

where $\lambda > 0$ is a parameter, $q > 1$, $\alpha : [0,R] \to \mathbb{R}$ is positive on $(0,R)$ and the function $p : [0,R] \times [0,A] \to \mathbb{R}$ is positive on $(0,R) \times (0,A)$, with $p(r,0) = 0 = p(r,A)$ for all $r \in [0,R]$. Using a variational approach, we provide sufficient conditions ensuring the existence of at least one or at least two nontrivial solutions, for large enough values of the parameter.

Key words and phrases : mean extrinsic curvature operator, Dirichlet problem, radial solution, critical point, Palais-Smale condition.

Mathematics Subject Classification (2010) **:** 35J20, 35J60, 35J93, 35J87.

1. Introduction

In this paper we deal with the mixed boundary value problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[\alpha(r)u^{q-1} - \lambda p(r,u)] \text{ in } [0,R],\ u'(0) = 0 = u(R), \quad (1.1)$$

where $\lambda > 0$ is a real parameter and $\phi := \Phi' : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$; the continuous function $\Phi : [-a,a] \to \mathbb{R}$ is of class C^1 on $(-a,a)$ and $\Phi(0) = 0$. The real number $q > 1$ is fixed, $\alpha : [0,R] \to \mathbb{R}$ is continuous, positive on $(0,R)$ and the function $p : [0,R] \times [0,A] \to \mathbb{R}$ is continuous, positive on $(0,R) \times (0,A)$ and satisfies $p(r,0) = 0 = p(r,A)$ for all $r \in [0,R]$.

This study is essentially motivated by the existence of radial solutions to Dirichlet problems involving the *mean extrinsic curvature operator in Minkowski space* (see e.g. [1]), of the type:

$$\operatorname{div}\left(\frac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + g(|x|,v) = 0 \quad \text{in} \quad \mathcal{B}(R), \quad v = 0 \quad \text{on} \quad \partial\mathcal{B}(R), \ (1.2)$$

where $\mathcal{B}(R) = \{x \in \mathbb{R}^N \mid |x| < R\}$ and $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^N$. Setting $r = |x|$ and $v(x) = u(r)$, problem (1.2) becomes

$$\left[r^{N-1}\left(\frac{u'}{\sqrt{1-u'^2}}\right)\right]' + r^{N-1}g(r,u) = 0 \text{ in } [0,R],\ u'(0) = 0 = u(R), \quad (1.3)$$

hence, finding radial solutions of (1.2) reduces to solving the mixed boundary value problem (1.3). So, the interest in studying problem (1.1) comes from the fact that (1.3) is nothing else but a problem of type (1.1) (one takes $\phi(s) = \dfrac{s}{\sqrt{1-s^2}}$; $\Phi(s) = 1-\sqrt{1-s^2}$), with the λ-parameterized nonlinearity $g(r,u) = \lambda p(r,u) - \alpha(r)u^{q-1}$. Problems of type (1.3) were recently studied in the papers [3], [4]. More precisely, when $g(r,u) = \lambda u^{q-1} + h(r,u)$ with $q \in (1,2)$ and $h : [0,R] \times [0,\infty) \to [0,\infty)$ continuous, it is proved in [3] by a Leray-Schauder degree argument that (1.3) has at least one positive solution. Also, if $g(r,u) = \lambda u^{q-1}$, with $q \geq 2$, it is shown in the same paper [3] by a minimization procedure, that (1.3) has at least one positive solution for λ sufficiently large. Finally, in [4] it is proved, using the last mentioned result, in combination with the upper and lower solutions method and degree theory, that there exist $\Lambda > 0$ such that (1.3), with $g(r,u) = \lambda u^{q-1}$ with $q > 2$, has zero, at least one and at least two positive solutions according to $\lambda \in (0,\Lambda), \lambda = \Lambda$ and $\lambda > \Lambda$.

The rest of the paper is organized as follows. In Section 2 we introduce the main hypotheses and the variational setting. The main result is proved in Section 3; some examples of applications are also provided.

2. Hypotheses and a variational approach

Throughout this paper we shall assume that $N \geq 1$ and the following hypotheses on the data in the positive λ–parameter problem (1.1):

(H_ϕ) $\phi : (-a,a) \to \mathbb{R}$ is an increasing homeomorphism with $\phi(0) = 0$, so that there exists $\Phi : [-a,a] \to \mathbb{R}$ continuous, of class C^1 on $(-a,a)$, with $\Phi(0) = 0$ and satisfying $\Phi' = \phi$;

(H_p) the continuous function $p : [0,R] \times [0,A] \to \mathbb{R}$ is such that $p(r,0) = 0 = p(r,A)$ for all $r \in [0,R]$ and $p(r,s) > 0$ for all $(r,s) \in (0,R) \times (0,A)$;

$(H_{\alpha,q})$ $\alpha : [0,R] \to \mathbb{R}$ is continuous, $\alpha > 0$ on $(0,R)$ and $q > 1$.

Next, we introduce the function $f : [0,R] \times \mathbb{R} \to [0,\infty)$ defined by

$$f(r,s) = \begin{cases} 0 & \text{if } s < 0 \text{ or } s > A, \\ p(r,s) & \text{if } s \in [0,A] \end{cases} \quad (2.1)$$

and consider the auxiliary problem

$$[r^{N-1}\phi(u')]' = r^{N-1}[\alpha(r)|u|^{q-2}u - \lambda f(r,u)] \text{ in } [0,R],\ u'(0) = 0 = u(R). \quad (2.2)$$

Proposition 2.1. *If u is a solution of problem* (2.2) *then $0 \le u \le A$, hence u solves problem* (1.1).

Proof. Suppose that there exists some $r_0 \in [0,R)$ such that

$$\min_{[0,R]} u = u(r_0) < 0.$$

If $r_0 \in (0,R)$ then $u'(r_0) = 0$ and there is a sequence $\{r_k\}$ in $(0,r_0)$ converging to r_0 such that $u'(r_k) \le 0$. Then

$$\frac{r_k^{N-1}\phi(u'(r_k)) - r_0^{N-1}\phi(u'(r_0))}{r_k - r_0} = \frac{r_k^{N-1}\phi(u'(r_k))}{r_k - r_0} \ge 0,$$

implying that

$$[r^{N-1}\phi(u'(r))]'_{r=r_0} \ge 0.$$

This yields the contradiction

$$0 \le [r^{N-1}\phi(u'(r))]'_{r=r_0} = r_0^{N-1}\alpha(r_0)|u(r_0)|^{q-2}u(r_0) < 0.$$

If $r_0 = 0$ then there exists $r_1 \in (0,R)$ such that $u(r) < 0$ for all $r \in [0,r_1]$ and $u'(r_1) \ge 0$. Integrating the equation in (2.2) from 0 to r_1, we obtain

$$0 \le r_1^{N-1}\phi(u'(r_1)) = \int_0^{r_1} r^{N-1}\alpha(r)|u(r)|^{q-2}u(r)dr < 0$$

a contradiction, again. Consequently, $u(r) \ge 0$ for all $r \in [0,R]$.

A quite similar reasoning shows that $u(r) \le A$ for all $r \in [0,R]$ and the proof is complete. □

Remark 2.1. (i) According to [4], the functions $\underline{u} = 0$ and $\overline{u} = A$ are lower, respectively upper solutions for problem (2.2).

(ii) The reader will emphasize that if instead of (H_p) it is assumed "$p : [0,R] \times [0,\infty) \to [0,\infty)$ is such that $p(r,0) = 0$ for all $r \in [0,R]$" then, changing f from (2.1) into

$$f(r,s) = \begin{cases} 0 & \text{if } s < 0, \\ p(r,s) & \text{if } s \ge 0 \end{cases} \qquad ((r,s) \in [0,R] \times \mathbb{R}), \quad (2.3)$$

each solution u of problem (2.2) is ≥ 0, hence u solves problem (1.1).

Now, using the general setting from [3, Section 3] we introduce a variational approach for problem (2.2). With this aim, the space $C := C[0, R]$ will be endowed with the usual supremum norm $|| \cdot ||$ and the corresponding open ball of center 0 and radius $\sigma > 0$ will be denoted by B_σ. We set $W^{1,\infty} := W^{1,\infty}(0, R)$.

According to [2] we know that the convex set

$$K := \{v \in W^{1,\infty} \mid \|v'\| \leq a\}$$

is closed in C. This implies that

$$K_0 := \{v \in K \mid v(R) = 0\}$$

is also a convex, closed subset of C. On the other hand, as

$$||v|| \leq aR \quad \text{for all} \quad v \in K, \tag{2.4}$$

K_0 is bounded in $W^{1,\infty}$. Then, by the compactness of the embedding $W^{1,\infty} \subset C$, we infer that K_0 is compact in C.

Let $\Psi : C \to (-\infty, +\infty]$ be given by

$$\Psi(v) = \begin{cases} \displaystyle\int_0^R r^{N-1} \Phi(v')\, dr, & \text{if } v \in K_0, \\ \\ +\infty, & \text{if } v \in C \setminus K_0. \end{cases}$$

We have that is proper, convex and lower semicontinuous (also see [2]).

Setting

$$F(r, s) = \int_0^s f(r, \xi) d\xi, \qquad (r, s) \in [0, R] \times \mathbb{R},$$

we define $\mathcal{F}_\lambda : C \to \mathbb{R}$ by

$$\mathcal{F}_\lambda(v) = \int_0^R r^{N-1} \left[\frac{\alpha(r)}{q}|v|^q - \lambda F(r, v)\right] dr, \qquad v \in C,$$

which is of class C^1 on C. Then, the energy functional $I_\lambda := \Psi + \mathcal{F}_\lambda$ has the structure required by Szulkin's critical point theory (see [6]). Accordingly, a function $u \in C$ is *a critical point* of I_λ if $u \in K_0$ and

$$\Psi(v) - \Psi(u) + \langle \mathcal{F}'_\lambda(u), v - u\rangle \geq 0 \quad \text{for all} \quad v \in C.$$

Also, I_λ is said to satisfy the *Palais-Smale* (in short, (PS)) *condition* if any sequence $\{u_n\} \subset C$ such that $I_\lambda(u_n) \to c \in \mathbb{R}$ and

$$\Psi(v) - \Psi(u_n) + \langle \mathcal{F}'_\lambda(u_n), v - u_n\rangle \geq -\varepsilon_n\|v - u_n\| \quad \text{for all} \quad v \in C,$$

where $\varepsilon_n \to 0$, possesses a convergent subsequence.

Lemma 2.1. *The functional I_λ satisfies the (PS) condition and each critical point of I_λ is a solution of problem* (1.1).

Proof. The compactness of $K_0 \subset C$ implies that I_λ satisfies the (PS) condition, while from [3, Proposition 4] and Proposition 2.1 we have that each critical point of I_λ is a solution of problem (1.1). □

3. Main result

In this section we prove the following main result.

Theorem 3.1. *(i) Under the assumptions (H_ϕ), (H_p) and $(H_{\alpha,q})$, there exists $\Lambda > 0$ such that problem* (1.1) *has at least one nontrivial solution for all $\lambda > \Lambda$.*

(ii) If, in addition, one has $\alpha_m := \min_{[0,R]} \alpha > 0$ and p satisfies

$$\lim_{s\to 0+} \frac{p(r,s)}{s^{q-1}} = 0 \quad \text{uniformly with } r \in [0,R], \tag{3.1}$$

then, problem (1.1) *has at least two nontrivial solutions for all $\lambda > \Lambda$.*

Proof. (i) Let $u_0 \in K_0$ be defined by

$$u_0(r) = \min\left\{A, \frac{2Ra}{\pi}\right\} \cos\frac{\pi r}{2R} \quad (r \in [0,R]).$$

It is clear that

$$F(r, u_0(r)) = \int_0^{u_0(r)} p(r,\xi)d\xi > 0 \quad \text{for all } r \in (0,R).$$

We take

$$\Lambda := \frac{\Psi(u_0) + \frac{1}{q}\int_0^R r^{N-1}\alpha(r)u_0^q}{\int_0^R r^{N-1}F(r,u_0)}.$$

Then, for $\lambda > \Lambda$, from

$$I_\lambda(u_0) = \Psi(u_0) + \frac{1}{q}\int_0^R r^{N-1}\alpha(r)u_0^q dr - \lambda\int_0^R r^{N-1}F(r,u_0)dr < 0$$

it follows that

$$c_\lambda := \inf_C I_\lambda < 0.$$

Since I_λ is bounded from below (this follows from (2.4)) and satisfies the (PS) condition (Lemma 2.1), by virtue of Theorem 7.6.1 in [5], c_λ is a critical value of I_λ. Consequently, there exists a critical point $u_{\lambda,1}$ of I_λ such that

$I_\lambda(u_{\lambda,1}) = c_\lambda < 0$, for all $\lambda > \Lambda$. Clearly, $u_{\lambda,1} \neq 0$ and from Lemma 2.1 we have that $u_{\lambda,1}$ is a solution of problem (1.1).

(ii) Let $\lambda > \Lambda$ be fixed. We shall produce a second nontrivial critical point of I_λ by the Mountain Pass Theorem. To do this, it suffices to prove that

$$\inf_{K_0 \cap \partial B_\rho} I_\lambda > 0 \tag{3.2}$$

for some $\rho \in (0, \|u_{\lambda,1}\|/2)$.

Using (3.1) we can find $\delta > 0$, such that

$$F(r,s) \leq \frac{\alpha_m}{2q\lambda}|s|^q \quad \text{for all } (r,s) \in [0,R] \times [0,\delta].$$

Then, for $u \in C$ with $\|u\| \leq \delta$, we have

$$\begin{aligned} I_\lambda(u) &\geq \int_0^R r^{N-1}\left[\frac{\alpha(r)}{q}|u|^q - \lambda F(r,u)\right] dr \\ &\geq \int_0^R r^{N-1}\left[\frac{\alpha(r)}{q} - \frac{\alpha_m}{2q}\right]|u|^q dr \geq \frac{\alpha_m}{2q}\int_0^R r^{N-1}|u|^q dr. \end{aligned}$$

Let $\rho \in (0, \min\{\delta, \|u_{\lambda,1}\|/2\})$. We *claim* that

$$\inf_{K_0 \cap \partial B_\rho} \int_0^R r^{N-1}|u|^q dr > 0, \tag{3.3}$$

which will imply (3.2) with appropriate ρ. To prove (3.3), suppose by contradiction that there exists a sequence $\{u_n\} \subset K_0 \cap \partial B_\rho$ with

$$\int_0^R r^{N-1}|u_n|^q dr \to 0 \quad \text{as } n \to \infty.$$

As $\{u_n\}$ is bounded in $W^{1,\infty}$, passing to a subsequence if necessary, we may assume that $\{u_n\}$ is convergent in C to some u. This implies that $\|u\| = \rho$ and

$$\int_0^R r^{N-1}|u_n|^q dr \to \int_0^R r^{N-1}|u|^q dr \quad \text{as } n \to \infty.$$

It follows that $u = 0$, a contradiction with $\|u\| = \rho > 0$. So, (3.3) holds true, as claimed.

Now, by the Mountain Pass Theorem [6, Theorem 3.2] there exists a nontrivial critical point $u_{\lambda,2}$ of I_λ with $I_\lambda(u_{\lambda,2}) > 0$. Clearly, $u_{\lambda,1} \neq u_{\lambda,2}$ and the conclusion follows from Lemma 2.1. □

Example 3.1. If $q > 1$ and $m, n > 0$, then there exists $\Lambda > 0$ such that the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \lambda |x|^n \sqrt{\sin v} - |x|^m v^{q-1} = 0 \quad \text{in} \quad \mathcal{B}(R), \\ v = 0 \quad \text{on} \quad \partial\mathcal{B}(R) \end{cases}$$

has at least one nontrivial radial solution for all $\lambda > \Lambda$.

Example 3.2. If $q \in (1, 1.5)$ and $\mu > 0$, then there exists $\Lambda > 0$ such that the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \lambda \sqrt{v(1-v)} - \mu v^{q-1} = 0 \quad \text{in} \quad \mathcal{B}(R), \\ v = 0 \quad \text{on} \quad \partial\mathcal{B}(R) \end{cases}$$

has at least two nontrivial radial solutions for all $\lambda > \Lambda$.

Remark 3.1. On account of Remark 2.1 (ii), it is easy to see that Theorem 3.1 still remains true (with the same proof, but with f defined by (2.3)) if hypothesis (H_p) is replaced by "$(H_p)'$ the continuous function $p : [0, R] \times [0, \infty) \to [0, \infty)$ is such that $p(r, 0) = 0$ for all $r \in [0, R]$ and $p(r, s) > 0$ for all $(r, s) \in (0, R) \times (0, A)$, with some $A > 0$".

Example 3.3. If $q > 1$ and $m, n, l > 0$, then there exists $\Lambda > 0$ such that the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \lambda |x|^n v^l - |x|^m v^{q-1} = 0 \quad \text{in} \quad \mathcal{B}(R), \\ v = 0 \quad \text{on} \quad \partial\mathcal{B}(R) \end{cases}$$

has at least one nontrivial radial solution for all $\lambda > \Lambda$.

Example 3.4. If $\mu, l > 0$ and $q \in (1, l + 1)$, then there exists $\Lambda > 0$ such that the problem

$$\begin{cases} \operatorname{div}\left(\dfrac{\nabla v}{\sqrt{1-|\nabla v|^2}}\right) + \lambda v^l - \mu v^{q-1} = 0 \quad \text{in} \quad \mathcal{B}(R), \\ v = 0 \quad \text{on} \quad \partial\mathcal{B}(R) \end{cases}$$

has at least two nontrivial radial solutions for all $\lambda > \Lambda$.

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