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The fast logarithmic equation with multiplicative Gaussian noise

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Hommage to Professor Jean Mawhin to his seventy years

Abstract - The existence and uniqueness of a strong solution for the stochastic logarithmic diffusion equation with multiplicative noise is proved. This equation is relevant in the description of fast diffusion processes in plasma physics perturbed by a Gaussian noise proportional with mass concentration.

Key words and phrases : Brownian motion, stochastic equation, fast diffusion.

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1. Introduction

Consider the stochastic nonlinear diffusion equation

$$\begin{aligned} &dX_t - \Delta \log X_t\, dt = X_t\, dW_t && \text{in } \ Q_T = (0,T)\times\mathcal{O},\\ &X_0 = x && \text{in } \ \mathcal{O}, \\ &X_t = 1 && \text{on } \ \Sigma_T = (0,T)\times\partial\mathcal{O}. \end{aligned} \tag{1.1}$$

Here $\mathcal{O}$ is a bounded and open subset of $\mathbb{R}^d$ with smooth boundary $\partial\mathcal{O}$ and W_t is a Wiener cylindrical process in a probability space $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$ of the form

$$W_t = \sum_{k=1}^{\infty} \mu_k e_k \beta_k(t), \tag{1.2}$$

where $\mu_k \in \mathbb{R}$, $\{\beta_k\}_{k=1}^{\infty}$ are mutually independent Brownian motion over the stochastic basis $\{\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P}\}$ and $\{e_k\}_{k=1}^{\infty}$ are the eigenfunctions of the Laplace operator Δ with homogeneous boundary value conditions on $\partial\mathcal{O}$. This system is normalized in the space $L^2(\mathcal{O})$. The eigenvalues are denoted λ_k.

We denote by $H_0^1(\mathcal{O})$, $H^1(\mathcal{O})$, $H^{-1}(\mathcal{O})$ the standard Sobolev spaces on $\mathcal{O}$ and by $L^p(\mathcal{O})$, $1 \le p \le \infty$, the space of L^p-summable functions on $\mathcal{O}$ with the norm $|\cdot|_p$. The scalar product in $L^2(\mathcal{O})$ is denoted by $\langle \cdot, \cdot \rangle_2$ and the norm of $H^{-1}(\mathcal{O})$ is denoted by $|\cdot|_{-1}$. We recall that $|u|_{-1}^2 = \big\langle (-\Delta)^{-1}u, u \big\rangle_2$.

Given a Hilbert space H with the norm $\|\cdot\|_H$, we denote by $M^p_{\mathbb{P}}(0,T;H)$ the space of all H-valued progressively measurable processes $X:\Omega\times(0,T)\to H$ such that

$$\mathbb{E}\int_0^T \|X(t)\|^p_H dt<\infty,\ 1\le p<\infty, \tag{1.3}$$

with the usual modification for $p=\infty$. (Here $\mathbb{E}$ is the expectation.) We denote by $C_{\mathbb{P}}([0,T];H)$ the space of all processes $X\in M^2_{\mathbb{P}}(0,T;H)$ which have a modification in $C([0,T];L^2(\Omega,\mathcal{F},\mathbb{P},H))$.

Finally, denote by $L^p_{ad}(\Omega;C([0,T];L^2(\mathcal{O})))$, $1\le p<\infty$, the space of all adapted processes $X:\Omega\times(0,T)\to L^2(\mathcal{O}))$ such that

$$\mathbb{E}\|X\|^p_{C([0,T];L^2(\mathcal{O}))}<\infty.$$

(Here $C([0,T];Y)$, Y a Banach space is the standard space of Y-valued continuous functions on $[0,T]$.)

Definition 1.1. *The process X is called a strong solution to* (1.1) *if the following conditions hold.*

$$X\in C_{\mathbb{P}}([0,T];H^{-1}(\mathcal{O}))\cap M^2_{\mathbb{P}}(0,T;L^2(\mathcal{O})) \tag{1.4}$$

$$X>0,\ \text{a.e. in } Q_T\times\mathcal{O} \tag{1.5}$$

$$\log X\in M^2_{\mathbb{P}}(0,T;H^1_0(\mathcal{O})) \tag{1.6}$$

$$\int_0^t \log X(s)ds\in C_{\mathbb{P}}([0,T];H^1_0(\mathcal{O})) \tag{1.7}$$

$$X(t)=x+\Delta\int_0^t \log X(s)ds+\int_0^t X(s)dW_s,\ \forall t\in[0,T],\ \mathbb{P}\text{-a.s.} \tag{1.8}$$

Here, the last integral is taken in sense of Itô.

There is a growing interest in the theory of nonlinear stochastic equations of the form

$$dX_t-\Delta\psi(X_t)dt=\sigma(X_t)dW_t,\ t\ge 0, \tag{1.9}$$

where $\psi:\mathbb{R}\to\mathbb{R}$ is a monotonically nondecreasing continuous function or, more generally, a maximal monotone, multivalued graph in $\mathbb{R}\times\mathbb{R}$. The standard growth condition on ψ is

$$0\le r\psi(r)\le C|r|^{m+1},\quad \forall r\in\mathbb{R},$$

where $m>0$. The case $m>1$ describes the *low diffusion* processes, while $0<m<1$ models the *fast diffusion*. The case $\psi(r)=\rho\,\mathrm{sign}\,r$ describes processes with *self-organized criticality*. The existence theory in the low diffusion case was developed in [1], [2] and the fast diffusion case in [3], [8].

The self-organized criticality case with stochastic perturbation was treated in [2], [4].

The case $\psi(r) = \log r$ we consider here models superfast diffusion processes in plasma physics as well as in the description of the Carleman model of Boltzman equation ([5], [7], [9]). It can be seen as a limit case $m = 1$ of the porous medium equation

$$dX_t - \operatorname{div}(X_t^{-m}\nabla X_t)dt = X_t dW_t,\ 0 < m < 2. \tag{1.10}$$

The corresponding deterministic equation (1.1) arises also in Riemannian geometry as a model for evolution of a conformally flat metric by its Ricci curvature flow (see [11], [12]).

Our main result here – Theorem 2.1 – is an existence and uniqueness result for the stochastic equation (1.1). Roughly speaking, it amounts to saying that (1.1) has a unique strong solution $X_t = X(t)$ for appropriate initial data.

2. The main result

Everywhere in the following, $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^d$, $d = 1, 2, 3$, and W_t is a cylindrical Wiener process of the form (1.2), where

$$\sum_{k=1}^{\infty} \lambda_k^2 \mu_k^2 < \infty. \tag{2.1}$$

The boundary $\partial\mathcal{O}$ is assumed sufficiently smooth (of class C^2, for instance).

We set $Q_T = (0,T) \times \mathcal{O},\ \Sigma_T = (0,T) \times \partial\mathcal{O}$, where $T > 0$.

Theorem 2.1. *Let* $0<T<\infty$ *be arbitrary but fixed. Then, for each* $x \in L^2(\mathcal{O})$ *such that*

$$x \log x \in L^2(\mathcal{O}), \tag{2.2}$$

equation (1.1) *has a unique strong solution* X*. Moreover, besides* (1.4)–(1.7)*, it also satisfies*

$$X \in L^\infty(0,T; L^2(\Omega; L^2(\mathcal{O}))) \tag{2.3}$$

$$X|\log X| \in L^\infty(0,T; L^2(\Omega \times \mathcal{O})). \tag{2.4}$$

The proof of Theorem 2.1 is given in Section 3.1, via a regularization procedure already used in low and fast diffusion stochastic porous media equations ([1], [2], [3]). It should be said, however, that compared with situations previously encountered in low and fast diffusion cases, the main difficulty here comes from the fact that the logarithmic nonlinearity $\psi(r) = \log r$ is highly singular in origin and its domain is restricted to $(0, \infty)$. This fact requires sharper estimates and techniques on the corresponding approximating equations.

3. Proof of Theorem 2.1

We consider the approximating equation

$$\begin{aligned} &dX_\lambda - \Delta(\psi_\lambda(X_\lambda) + \lambda X_\lambda) = J_\lambda(X_\lambda)dW_t && \text{in } Q_T,\\ &X_\lambda(0) = x && \text{in } \mathcal{O},\\ &\psi_\lambda(X_\lambda) + \lambda X_\lambda = 0 && \text{on } \Sigma_T, \end{aligned} \tag{3.1}$$

where $J_\lambda = (1 + \lambda\psi)^{-1}$ and $\psi_\lambda : \mathbb{R} \to \mathbb{R}$ is the Yosida approximation of the maximal monotone graph $\psi : \mathbb{R} \to \mathbb{R}$

$$\psi(r) = \begin{cases} \log r & \text{if} \quad r > 0 \\ \\ \emptyset & \text{if} \quad r \le 0 \end{cases}, \ r \in \mathbb{R}. \tag{3.2}$$

In other words,

$$\psi_\lambda(r) = \frac{1}{\lambda}\,(r - (1 + \lambda\psi)^{-1}r) = \log((1 + \lambda\psi)^{-1}r),\ \forall \lambda > 0,\ r \in \mathbb{R}. \tag{3.3}$$

We recall that ψ_λ is monotonically increasing, Lipschitzian and $\psi_\lambda(r) \to \psi(r)$ as $\lambda \to 0$.

We set $H = H^{-1}(\mathcal{O})$. By [1], [2], we know that (3.1) has a unique strong solution

$$X_\lambda \in L^2_{ad}(\Omega; C([0,T]; H)) \cap L^2(\Omega \times (0,T) \times \mathcal{O}), \tag{3.4}$$

which is $\mathbb{P}$- a.s. $L^2(\mathcal{O})$-continuous on $[0,T]$.

We set

$$j_\lambda(r) = \int_1^r \psi_\lambda(s)ds,\ j(r) = r\log r - r, \quad \forall r \in \mathbb{R}^+. \tag{3.5}$$

1°. Apriori estimates

By Itô's formula in H, we obtain by (3.1) that

$$\begin{aligned} &\frac{1}{2}\,|X_\lambda(t)|_2^2 + \int_0^t \int_{\mathcal{O}} \nabla(\psi_\lambda(X_\lambda) + \lambda X_\lambda) \cdot \nabla X_\lambda d\xi\, ds \\ &\quad = \frac{1}{2}\,|x|_2^2 + \frac{1}{2}\sum_{k=1}^\infty \mu_k^2 \int_0^t \int_{\mathcal{O}} |J_\lambda(X_\lambda)e_k|^2 d\xi\, ds \\ &\quad + \sum_{k=1}^\infty \mu_k \int_0^t \int_{\mathcal{O}} X_\lambda J_\lambda(X_\lambda)e_k d\xi\, d\beta_k,\ \mathbb{P}\text{-a.s.},\ t \in (0,T). \end{aligned}$$

(As a matter of fact, we apply here the Itô formula to the function $(1 - \nu\Delta)^{-1}X_\lambda = X_\lambda^\nu$, which satisfies the equation

$$\begin{aligned} &X_\lambda^\nu(t) - (1 - \nu\Delta)^{-1}\Delta \int_0^t (\psi_\lambda(X_\lambda(s)) + \lambda X_\lambda(s))ds \\ &\quad = (1 - \nu\Delta)^{-1} \int_0^t J_\lambda(X_\lambda)(s)dW_s \end{aligned}$$

and get

$$\frac{1}{2}\,|X_\lambda^\nu(t)|_2^2+\int_0^t\int_{\mathcal{O}}\nabla(1-\nu\Delta)^{-1}(\psi_\lambda(X_\lambda)+\lambda X_\lambda)\cdot\nabla X_\lambda^\nu d\xi\,ds$$
$$=\frac{1}{2}\,|x|_2^2+\frac{1}{2}\sum_{k=1}^\infty\mu_k^2\int_0^t\int_{\mathcal{O}}|(1-\nu\Delta)^{-1}J_\lambda(X_\lambda)e_k|^2d\xi\,dt$$
$$+\sum_{k=1}^\infty\mu_k\int_0^t\left\langle(1-\nu\Delta)^{-1}J_\lambda(X_\lambda)dW_s,X_\lambda^\nu\right\rangle_2$$

and let $\nu\to 0$ to get the above formula.) This approach will be used several times in the following, when using Itô's formula in $L^2(\mathcal{O})$ to equation (3.1).

Taking into account that $|e_k|_\infty\le C\lambda_k$, $\forall k$, we obtain by (2.1) the estimate

$$|X_\lambda(t)|_2^2+2\lambda\int_0^t|\nabla X_\lambda|^2ds\le|x|_2^2+C\sum_{k=1}^\infty\mu_k^2\lambda_k^2\int_0^t|J_\lambda(X_\lambda(s)|_2^2ds$$
$$+\sum_{k=1}^\infty\mu_k\int_0^t\int_{\mathcal{O}}J_\lambda(X_\lambda)X_\lambda(s)e_k\beta_kd\xi\,ds,\ \mathbb{P}\text{-a.s.},$$

and, finally, by the Burkholder–Davis–Gundy inequality, we obtain (see [2], estimate (3.6))

$$\mathbb{E}\sup_{0\le t\le T}|X_\lambda(t)|_2^2+\lambda E\int_0^T|\nabla X_\lambda(s)|_2^2ds\le C_T|x|_2^2. \tag{3.6}$$

We apply Itô's formula in H to the function $X\to|X|_{-1}^2$. We obtain that

$$\frac{1}{2}\,|X_\lambda(t)|_{-1}^2+\int_0^t\int_{\mathcal{O}}(\psi_\lambda(X_\lambda)+\lambda X_\lambda)X_\lambda d\xi\,ds=\frac{1}{2}\,|x|_{-1}^2$$
$$+\frac{1}{2}\sum_{k=1}^\infty\mu_k^2\int_0^t|J_\lambda(X_\lambda)e_k|_{-1}^2ds+\sum_{k=1}^\infty\mu_k\int_0^t\left\langle X_\lambda,J_\lambda(X_\lambda)e_k\right\rangle_{-1}d\beta_k. \tag{3.7}$$

Next, we apply the Itô formula in (3.1) to the function

$$\phi_\lambda^\varepsilon((1-\varepsilon\Delta)^{-1}x)=\int_{\mathcal{O}}j_\lambda^\varepsilon(x(\xi))d\xi,\ \forall x\in L^2(\mathcal{O}),$$

where j_λ^ε is a smooth regularization of j_λ, for instance,

$$j_\lambda^\varepsilon(r)=(j_\lambda*\rho_\varepsilon)(r),\ \forall r\in\mathbb{R},$$

where $\rho_\varepsilon=\frac{1}{\varepsilon}\,\rho\left(\frac{s}{\rho}\right)$ is a mollifier, that is, $\rho\in C^\infty(\mathbb{R})$, support $\rho\subset[-1,1]$, $\int_{\mathbb{R}}\rho(s)ds=1$. We obtain (by using the same approximating procedure $X_\lambda\to$

$(1-\nu\Delta)^{-1}X_\lambda))$

$$\begin{aligned}\mathbb{E}\phi^\varepsilon_\lambda(X_\lambda(t))+\mathbb{E}\int_0^t\int_{\mathcal{O}}(\nabla\psi_\lambda(X_\lambda)+\lambda\nabla X_\lambda)\cdot\nabla\psi^\varepsilon_\lambda(X_\lambda)d\xi\,ds\\ =\phi^\varepsilon_\lambda(x)+\frac{1}{2}\sum_{k=1}^\infty\mu_k^2\mathbb{E}\int_0^t\int_{\mathcal{O}}|J_\lambda(X_\lambda)e_k|^2(\psi^\varepsilon_\lambda)'(X_\lambda)d\xi\,ds,\end{aligned}\tag{3.8}$$

where

$$\psi^\varepsilon_\lambda(r)=(j^\varepsilon_\lambda)'(r)=(\psi_\lambda*\rho_\varepsilon)(r).$$

Keeping in mind that by (2.2), $\phi^\varepsilon_\lambda(x)\le C$, $\forall\lambda>0$, and $\varepsilon>0$ and also that

$$\psi'_\lambda(r)=\frac{1}{r+(1+\lambda\psi)^{-1}r},\quad\forall r\in\mathbb{R},$$

we obtain by (3.8) that, for $\varepsilon\to0$,

$$\begin{aligned}\mathbb{E}\int_{\mathcal{O}}j_\lambda(X_\lambda(t,\xi))d\xi+\mathbb{E}\int_0^t\int_{\mathcal{O}}|\nabla\psi_\lambda(X_\lambda)|^2d\xi\,ds\\ \le\int_{\mathcal{O}}j_\lambda(x(\xi))d\xi+C\mathbb{E}\int_0^t\int_{\mathcal{O}}|J_\lambda(X_\lambda)|d\xi\,ds\end{aligned}$$

and, taking into account that $j_\lambda(r)=j(1+\lambda\psi)^{-1}r)+(2\lambda)^{-1}(r-(1+\lambda\psi)^{-1}r)^2$, we obtain that

$$\mathbb{E}\textstyle\int_{\mathcal{O}}j_\lambda(X_\lambda(t,\xi)d\xi))+\mathbb{E}\displaystyle\int_0^t\int_{\mathcal{O}}|\nabla\psi_\lambda(X_\lambda)|^2d\xi\,ds\le C_T,\ \forall\lambda>0,\ t\in[0,T].\tag{3.9}$$

Then, again applying the Itô formula in H to $|X_\lambda(t)-X_\mu(t)|^2_{-1}$, we obtain, via the Burkholder-Davis-Gundy inequality (see Lemma 3.1 in [2])

$$\mathbb{E}\sup_{0\le t\le T}|X_\lambda(t)-X_\mu(t)|^2_{-1}e^{-\alpha t}\le C\max(\lambda,\mu).\tag{3.10}$$

2°. Letting $\lambda\to0$

By estimates (3.6), (3.9), (3.10), it follows that there are $X\in L^2_{ad}(\Omega;C([0,T];H)\cap L^2_{ad}(\Omega;L^\infty(0,T;L^2(\mathcal{O}))$ and $\eta\in M^2_{\mathbb{P}}(0,T;H^1_0(\mathcal{O}))$ such that for $\lambda\to0$

$$X_\lambda\to X\qquad\text{weak}^*\text{ in }L^\infty(0,T;L^2(\Omega,L^2(\mathcal{O}))\text{ and strongly in }L^2(\Omega;C([0,T];H)).\tag{3.11}$$

$$\psi_\lambda(X_\lambda)\to\eta\qquad\text{weakly in }L^2(0,T;L^2(\Omega,H^1_0(\mathcal{O})))\tag{3.12}$$

$$(1+\lambda\psi)^{-1}X_\lambda\to X\qquad\text{weakly in }L^2(\Omega\times(0,T)\times\mathcal{O}).\tag{3.13}$$

By (3.9), (3.11), it follows also via Fatou's lemma that

$$\mathbb{E}\sup_{0\le t\le T}\int_{\mathcal{O}} j((t,\xi))d\xi \le C<\infty$$

and, therefore,

$$\mathbb{E}\sup_{0\le t\le T}\int_{\mathcal{O}} |\log X(t,\xi)|X(t,\xi)d\xi \le C_1<\infty. \tag{3.14}$$

Now, if we write (3.1) as

$$X_\lambda(t)-\Delta\int_0^t(\psi_\lambda(X_\lambda(s))+\lambda X_\lambda(s))ds = x+\int_0^t X_\lambda(s)dW_s,\ \forall t\in[0,T], \tag{3.15}$$

and let $\lambda\to 0$, we obtain that

$$X(t)-\Delta\int_0^t \eta(s)ds = x+\int_0^t X(s)dW_s,\ t\in[0,T]. \tag{3.16}$$

To conclude the proof of the existence, it suffices to show that

$$\eta=\log X,\quad \text{a.e. in } \Omega\times(0,T)\times\mathcal{O}. \tag{3.17}$$

Taking into account (3.11), (3.12) and that the realization of operator ψ is maximal monotone in $L^2(\Omega\times Q_T)$ to get (3.17) it suffices to check that

$$\limsup_{\lambda\to 0}\mathbb{E}\int_{Q_T}\psi_\lambda(X_\lambda)X_\lambda d\xi\,dt \le \mathbb{E}\int_{Q_T}\eta X\,d\xi\,dt. \tag{3.18}$$

To this end, we note first that by (3.7) and (3.12) we have

$$\begin{aligned}\limsup_{\lambda\to 0}\mathbb{E}\int_{Q_T}\psi_\lambda(X_\lambda)X_\lambda\,d\xi\,dt &\le -\frac{1}{2}\left(\mathbb{E}|X(T)|_{-1}^2-|x|_{-1}^2\right)\\ &+\frac{1}{2}\sum_{k=1}^\infty \mu_k^2\mathbb{E}\int_0^T |X(t)e_k|_{-1}^2 dt.\end{aligned} \tag{3.19}$$

Now, applying the Itô formula in (3.16) to the function $X\to\frac{1}{2}|X|_{-1}^2$, we obtain that

$$\begin{aligned}&\frac{1}{2}\left(\mathbb{E}|X(T)|_{-1}^2-|x|_{-1}^2\right)+\mathbb{E}\int_0^T {}_{H_0^1(\mathcal{O})}\langle\eta(t),X(t)\rangle_{H^{-1}(\mathcal{O})}\,dt\\ &=\frac{1}{2}\sum_{k=1}^\infty\mu_k^2\mathbb{E}\int_0^T|X(t)e_k|_{-1}^2dt.\end{aligned} \tag{3.20}$$

In order to obtain (3.18) by (3.19) and (3.20), it suffices to note that by (3.11) and (3.12) we have

$$\mathbb{E}\int_0^T {}_{H_0^1(\mathcal{O})}\langle\eta(t),X(t)\rangle_{H^{-1}(\mathcal{O})}\,dt=\mathbb{E}\int_{Q_T}\eta X\,d\xi\,dt. \tag{3.21}$$

Hence, X is a solution to (1.1) in sense of Definition 1.1. We note also that X satisfies (2.2), as claimed.

Uniqueness. If X_1, X_2 are two solutions to (1.1), we obtain by Itô's formula in $H^{-1}(\mathcal{O})$ (we note that it is applicable because $t \to |X(t)|^2_{-1}$ is continuous)

$$\begin{aligned}\frac{1}{2}\,\mathbb{E}|X_1(t)-X_2(t)|^2_{-1} + \mathbb{E}\int_0^t\int_{\mathcal{O}}(\log X_1 - \log X_2)(X_1-X_2)d\xi\,ds \\ = \frac{1}{2}\,\mathbb{E}\sum_{k=1}^{\infty}\mu_k^2\int_0^t\int_{\mathcal{O}}|(X_1-X_2)e_k|^2_{-1}d\xi\,ds,\end{aligned}$$

which yields via Gronwall's formula $X_1 \equiv X_2$, as desired.

Remark 3.1. It is known (see [7]) that the deterministic logarithmic diffusion equation (1.1) with a time-dependent deterministic source term of the form $\alpha(t)X$ has an explicit solution which in $1{-}D$ is of the form

$$X(t,\xi) = (Z(\tau(t))(1+\gamma(\tau(t))\xi^2))^{-1}\exp\left(-\int_0^t \alpha(s)ds\right),$$

where $\gamma(t) = r_0(1+\alpha_0 t)^{-2},\ Z(t) = z_0(1+\gamma_1 z_0 t)$ and this has an extension to $d{-}D$. Roughly speaking, the situation described by equation (1.1) is that where $\alpha(t)$ is the Gaussian noise $W(t)$ and one might expect to obtain on the same lines an explicit solution for (1.1).

Remark 3.2. It is clear that Theorem 2.1 remains true for more general equations of the form

$$\begin{aligned}&dX - \Delta\log X\,dt = (X+a)dW && \text{in } (0,\infty)\times\mathcal{O},\\ &X(0)=x && \text{in } \mathcal{O},\\ &\log X = 0 && \text{on } (0,\infty)\times\partial\mathcal{O},\end{aligned} \tag{3.22}$$

where $a \in \mathbb{R}$. The details of proof are omitted.

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