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Positive solutions for nonlinear nonhomogeneous periodic eigenvalue problems

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Dedicated to Professor Jean Mawhin on the occasion of his 70th birthday

Abstract - We consider a nonlinear parametric periodic problem driven by a nonhomogeneous differential operator, and with a Carathéodory reaction which is $(p-1)-$ sublinear in the x- variable, both near $+\infty$ and near 0^{+}. Using variational methods coupled with truncation and comparison techniques, we prove a bifurcation-type theorem describing the existence and multiplicity of positive solutions as the parameter varies.

Key words and phrases : positive solution, bifurcation-type theorem, mountain pass theorem, nonlinear maximum principle, nonhomogeneous differential operator.

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1. Introduction

We consider the following nonlinear periodic problem

$$\begin{cases} -\left(a\left(\left|u'\left(t\right)\right|\right)u'\left(t\right)\right)' + \beta\left(t\right)\left|u'\left(t\right)\right|^{p-2}u\left(t\right) = \lambda f\left(t,u\left(t\right)\right) \text{ on } T=[0,b] \\ u\left(0\right)=u\left(b\right),\ u'\left(0\right)=u'\left(b\right). \end{cases} \tag{P_λ}$$

In this problem, the differential operator $u \to \left(a\left(\left|u'\right|\right)u'\right)'$, where $a\colon(0,\infty)\to(0,\infty)$, needs not be homogeneous and incorporates as special cases the scalar $p-$Laplacian $(1<p<\infty)$, the scalar $(p,q)-$differential operator $(1<q<p<\infty)$, and the generalized $p-$mean curvature operator $(1<p<\infty)$. Also $\beta \in L^{\infty}(T)$, $\beta \geq 0$, $\beta \neq 0$, $\lambda > 0$ is a parameter and $f(t,x)$ is a Carathéodory reaction term (i.e., for all $x \in \mathbb{R}$, $t \to f(t,x)$ is measurable and for a.a. $t \in T$, $x \to f(t,x)$ is continuous) which exhibits $(p-1)-$sublinear growth in x as it approaches $+\infty$ and 0^{+}. Our aim is to establish the existence and multiplicity of positive solutions as the parameter λ varies.

Multiplicity results for positive solutions of equations driven by the $p-$Laplacian were proved for Dirichlet and Sturm-Liouville boundary value problems. We mention the works of Ben Naoum-De Coster [4], De Coster [6], Manasevich-Njoku-Zanolin [8], Njoku-Zanolin [9] and Wang [11]. Positive solutions for periodic problems driven by the scalar $p-$Laplacian were investigated by Aizicovici-Papageorgiou-Staicu in [2] and by Hu-Papageorgiou in [7]. In fact, in [2] the authors considered parametric problems involving competing nonlinearities, i.e., a reaction term of the form

$$\lambda x^{q-1} + f(t,x) \text{ for all } t \in T, \text{ all } x \geq 0,$$

with $\lambda > 0$ being a parameter, $1 < q < p$, and $f(t,x)$ a Carathéodory perturbation which is $(p-1)-$superlinear in x as it approaches $+\infty$. So, the reaction of the equation studied in [2] involves the competing effects of a 'concave' term (λx^{q-1}) and of a 'convex' term $(f(t,x))$. In [2] the authors proved a bifurcation-type theorem for such equations.

In the present paper, we consider a somehow complementary situation, since we deal with a reaction which has an opposite growth pattern near $+\infty$ and near 0^+. Again, we prove a bifurcation-type theorem.

Our approach uses variational methods based on the critical point theory together with suitable truncation and comparison techniques. In the next section, for easy reference, we recall the main mathematical tools which will be used in this work.

2. Mathematical background

Let $(X, \|.\|)$ be a Banach space and X^* its topological dual. By $\langle \cdot, \cdot \rangle$ we denote the duality brackets for the pair (X^*, X). Also $\xrightarrow{w}$ denotes weak convergence in X.

A map $A : X \to X^*$ is said to be *of type* $(S)_+$, if for every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $x_n \xrightarrow{w} x$ in X and

$$\limsup_{n\to\infty} \langle A(x_n), x_n - x \rangle \leq 0,$$

one has

$$x_n \to x \text{ in } X \text{ as } n \to \infty.$$

Let $\varphi \in C^1(X)$. We say that $x^* \in X$ is a critical point of φ if $\varphi'(x^*) = 0$. If $x^* \in X$ is a critical point of φ, then $c := \varphi(x^*)$ is said to be a critical value of φ.

We say that $\varphi \in C^1(X)$ satisfies the *Palais-Smale condition* (*PS-condition*, for short), if the following is true:

> 'every sequence $\{x_n\}_{n\geq 1} \subseteq X$ such that $\{\varphi(x_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and
>
> $$\varphi'(x_n) \to 0 \text{ in } X^* \text{ as } n \to \infty$$
>
> admits a strongly convergent subsequence.'

Using this compactness-type condition, we have the following minimax characterization of certain critical values of φ. The result is known in he literature as the 'mountain pass theorem'.

Theorem 2.1. *If $\varphi \in C^1(X)$ satisfies the PS-condition, $x_0,\ x_1 \in X$ and $r > 0$ are such that*

$$\|x_1 - x_0\| > r, \quad \max\{\varphi(x_0), \varphi(x_1)\} < \inf\{\varphi(x) \mid \|x - x_0\| = r\} =: \eta_r,$$

and $c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \varphi(\gamma(t))$, *where*

$$\Gamma = \{\gamma \in C([0,1], X) \mid \gamma(0) = x_0,\ \gamma(1) = x_1\},$$

then $c \geq \eta_r$ and c is a critical value of φ.

In the study of problem (P_λ), we use the following two Banach spaces:

$$W^{1,p}_{per}(0,b) = \left\{u \in W^{1,p}(0,b) \mid u(0) = u(b)\right\},$$

and

$$\widehat{C^1}(T) = C^1(T) \cap W^{1,p}_{per}(0,b).$$

The space $\widehat{C^1}(T)$ is an ordered Banach space with positive cone given by

$$\widehat{C_+} = \left\{u \in \widehat{C^1}(T) \mid u(t) \geq 0 \text{ for all } t \in T\right\}.$$

This cone has nonempty interior given by

$$int\ \widehat{C_+} = \left\{u \in \widehat{C_+} \mid u(t) > 0 \text{ for all } t \in T\right\}.$$

Throughout this paper, the norm of the Banach space $W^{1,p}_{per}(0,b)$ will be denoted by $\|.\|$, while $\|.\|_p$ will designate the norm of $L^p(0,b)$. Also, for every $x \in \mathbb{R}$, we set $x^{\pm} = \max\{\pm x, 0\}$. Then for every $u \in W^{1,p}_{per}(0,b)$, we define $u^{\pm}(.) = u(.)^{\pm}$.

We know that

$$u^{\pm} \in W^{1,p}_{per}(0,b), \ \ |u| = u^+ + u^-,\ u = u^+ - u^-.$$

If $h : T \times \mathbb{R} \to \mathbb{R}$ is a measurable function, then we set

$$N_h(u)(.) = h(., u(.)) \text{ for all } u \in W^{1,p}_{per}(0,b).$$

Finally, by $|.|_1$ we denote the Lebesgue measure on $\mathbb{R}$.

We next introduce the following conditions on the map $a(.)$:

$\mathbf{H}(a)$: $a : (0,\infty) \to (0,\infty)$ is a C^1-function such that

(i) $x \to a(x)x$ is strictly increasing on $(0,\infty)$ with

$$a(x)x \to 0 \text{ and } \frac{xa'(x)}{a(x)} \to C > -1 \text{ as } x \to 0^+;$$

(ii) there exists $C_0 > 0$ such that

$$C_0 |x|^p \le a(|x|)x^2 \text{ for all } x \in \mathbb{R};$$

(iii) there exist $C_1 > 0$ and $p \in (1,\infty)$ such that

$$|a(|x|)x| \le C_1 \left(1 + |x|^{p-1}\right) \text{ for all } x \in \mathbb{R}.$$

Remark. Let $G_0(x) = \int_0^x a(s)s\,ds$ for all $x > 0$. Evidently, G_0 is strictly convex and strictly increasing. We set

$$G(x) = G_0(|x|) \text{ for all } x \in \mathbb{R}.$$

Then $G(.)$ is convex, $G(0) = 0$ and for all $x \neq 0$ we have

$$G'(x) = G_0'(|x|)\frac{x}{|x|} = a(|x|)x$$

while $G'(0) = 0$ (see $\mathbf{H}(a)(i)$). Therefore $G(.)$ is the primitive of $x \to a(|x|)x$. Evidently we have

$$\frac{C_0}{p}|x|^p \le G(x) \le C_2(1 + |x|^p) \text{ for all } x \in \mathbb{R} \text{ and some } C_2 > 0. \tag{2.1}$$

Examples. The following functions satisfy hypotheses $\mathbf{H}(a)$:

(i) $a(x) = x^{p-2}$ for all $x \in \mathbb{R}$, with $1 < p < \infty$.

In this case the differential operator is the scalar $p-$Laplacian defined by

$$\left(|u'|^{p-2}u'\right)', \text{ for all } u \in W^{1,p}_{per}(0,b).$$

(ii) $a(x) = |x|^{p-2} + \mu|x|^{q-2}$ for all $x > 0$, with $1 < q < p < \infty$, $\mu \ge 0$.

In this case the differential operator is the scalar $(p,q)-$differential operator defined by

$$\left(|u'|^{p-2}u'\right)' + \mu\left(|u'|^{q-2}u'\right)', \text{ for all } u \in W^{1,p}_{per}(0,b).$$

(iii) $a(x) = \left(1 + x^2\right)^{\frac{p-2}{2}}$ for all $x > 0$, with $1 < p < \infty$.

In this case the differential operator is the generalized scalar $p-$mean curvature differential operator defined by

$$\left(\left(1+\left(u'\right)^2\right)^{\frac{p-2}{2}} u'\right)' \text{ for all } u \in W^{1,p}_{per}\left(0,b\right).$$

For problems with scalar mean curvature operator (i.e., $p=1$), see Bereanu-Mawhin [5].

(iv) $a\left(x\right)=x^{p-2}+\frac{x^{p-2}}{1+x^p}$ for all $x>0$, with $1<p<\infty$.

Let $A : W^{1,p}_{per}\left(0,b\right) \to W^{1,p}_{per}\left(0,b\right)^*$ be the nonlinear map defined by

$$\left\langle A\left(u\right),y\right\rangle = \int_0^b a\left(\left|u'\left(t\right)\right|\right)u'\left(t\right)y'\left(t\right)dt \text{ for all } u,\, y \in W^{1,p}_{per}\left(0,b\right). \tag{2.2}$$

From Aizicovici-Papageorgiou-Staicu (see [3]), we have:

Proposition 2.1. *If hypotheses* $\mathbf{H}\left(a\right)$ *hold and* $A : W^{1,p}_{per}\left(0,b\right) \to W^{1,p}_{per}\left(0,b\right)^*$ *is defined by* (2.2), *then* A *is maximal monotone, strictly monotone and of type* $\left(S\right)_+$.

Let $f_0 : T\times\mathbb{R} \to\mathbb{R}$ be a Carathéodory function such that

$$\left|f_0\left(t,x\right)\right| \le \widehat{a}\left(t\right)\left(1+\left|x\right|^{r-1}\right) \text{ for a.a. } t\in T, \text{ all } x\in\mathbb{R},$$

with $1<r<\infty$ and $\widehat{a}\in L^{r'}\left(T\right)_+$ $\left(\frac{1}{r}+\frac{1}{r'}=1\right)$. We set

$$F_0\left(t,x\right)=\int_0^x f_0\left(t,s\right)ds$$

and consider the C^1-functional $\psi_0 : W^{1,p}_{per}\left(0,b\right)\to\mathbb{R}$ defined by

$$\psi_0\left(u\right)=\int_0^b G\left(u'\left(t\right)\right)dt-\int_0^b F_0\left(t,u\left(t\right)\right)dt \text{ for all } u\in W^{1,p}_{per}\left(0,b\right).$$

From Aizicovici-Papageorgiou-Staicu (see [3]), we have the following result relating $\widehat{C^1}\left(T\right)$ and $W^{1,p}_{per}\left(0,b\right)$ local minimizers for the functional ψ_0.

Proposition 2.2. *If hypotheses* $\mathbf{H(a)}$ *hold and* $u_0 \in W^{1,p}_{per}\left(0,b\right)$ *is a local* $\widehat{C^1}\left(T\right)$*-minimizer of* ψ_0 *(i.e., there exists* $\rho_0>0$ *such that* $\psi_0\left(u_0\right)\le\psi_0\left(u_0+h\right)$ *for all* $h\in\widehat{C^1}\left(T\right)$ *with* $\left\|h\right\|_{\widehat{C^1}\left(T\right)}\le\rho_0$*), then* $u_0\in\widehat{C^1}\left(T\right)$ *and it is a local* $W^{1,p}_{per}\left(0,b\right)$*-minimizer of* ψ_0, *(i.e., there exists* $\rho_1>0$ *such that* $\psi_0\left(u_0\right)\le\psi_0\left(u_0+h\right)$ *for all* $h\in W^{1,p}_{per}\left(0,b\right)$ *with* $\left\|h\right\|_{W^{1,p}_{per}\left(0,b\right)}\le\rho_1$*).*

A final auxiliary result is the following simple Lemma (see [3]):

Lemma 2.1. *If $\beta \in L^1(T)$, $\beta(t) \geq 0$ a.e. on T and $\beta \neq 0$, then there exists $\xi_* > 0$ such that*

$$C_0 \left\|u'\right\|_p^p + \int_0^b \beta(t) |u(t)|^p \, dt \geq \xi_* \|u\|^p \ \textit{ for all } u \in W_{per}^{1,p}(0,b).$$

We introduce

$$\widehat{\lambda}_1 := \inf \left\{ \frac{1}{\|u\|_p^p} \left[C_0 \left\|u'\right\|_p^p + \int_0^b \beta(t) |u(t)|^p \, dt \right] \mid u \in W_{per}^{1,p}(0,b), u \neq 0 \right\}. \tag{2.3}$$

It follows from Lemma 2.1 that

$$\widehat{\lambda}_1 \geq \xi_* > 0.$$

3. A bifurcation-type theorem

The hypotheses on $\beta(.)$ and $f(.,.)$ are the following:

$\mathbf{H}(\beta)$: $\beta \in L^\infty(T)$, $\beta(t) \geq 0$ a.e. on T, $\beta \neq 0$.

$\mathbf{H}(f)$: $f : T \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function such that $f(t,0) = 0$ a.e. on T and:

(*i*) for every $\rho > 0$ there exists $a_\rho \in L^\infty(T)$ such that $f(t,x) \leq a_\rho(t)$ for a.a. $t \in T$, all $x \in [0,\rho]$;

(*ii*) $\lim\limits_{x \to \infty} \frac{f(t,x)}{x^{p-1}} = 0$ uniformly for a.a. $t \in T$;

(*iii*) $\lim\limits_{x \to 0^+} \frac{f(t,x)}{x^{p-1}} = 0$ uniformly for a.a. $t \in T$;

(*iv*) for every $\rho > 0$, there exists $\xi_\rho > 0$ such that for a.a. $t \in T$, the map

$$x \to f(t,x) + \xi_\rho x^{p-1}$$

is nondecreasing on $[0,\rho]$;

(v) for every $\tau > 0$, there exists $\widehat{m}_\tau > 0$ such that $f(t,x) \geq \widehat{m}_\tau$ for a.a. $t \in T$, all $x \geq \tau$.

Remark. Since our aim is to produce positive solutions and the above hypotheses concern only the nonnegative half-axis $\mathbb{R}_+ = [0,\infty)$, without any loss of generality we may (and will) assume that $f(t,x) = 0$ for all $t \in T$ and all $x \leq 0$. Note that hypotheses $\mathbf{H}(f)(ii), (iii)$ imply that for a.a. $t \in T$, $x \to f(t,x)$ is $(p-1)$ −sublinear near $+\infty$ and near 0^+.

We introduce the set

$$\mathcal{L} = \{\lambda > 0 \,|\, \text{problem } (P_\lambda) \text{ has a nontrivial positive solution}\}.$$

Also let $\mathcal{S}(\lambda)$ be the set of nontrivial positive solutions of (P_λ). We set

$$\lambda_* = \inf \mathcal{L}.$$

(of course if $\mathcal{L} = \varnothing$, then $\lambda_* = +\infty$).

Proposition 3.1. *If hypotheses $\mathbf{H}(a)$, $\mathbf{H}(\beta)$ and $\mathbf{H}(f)$ hold, then $\mathcal{S}(\lambda) \subset int\ \widehat{C_+}$ and $\lambda_* > 0$.*

Proof. Suppose that $\mathcal{L} \neq \varnothing$ and let $\lambda \in \mathcal{L}$. Then we can find $u \in W^{1,p}_{per}(0,b)$, $u \geq 0$, $u \neq 0$ such that

$$\begin{cases} -\left(a\left(|u'(t)|\right)u'(t)\right)' + \beta(t)\,u(t)^{p-1} = \lambda f(t,u(t)) \text{ a.e. on } T \\ u(0) = u(b),\ u'(0) = u'(b). \end{cases} \tag{3.1}$$

From (3.1) it follows that $u \in \widehat{C_+} \setminus \{0\}$. Then, using hypothesis $\mathbf{H}(f)(v)$ and (3.1), we have

$$-\left(a\left(\left|u'(t)\right|\right)u'(t)\right)' + \beta(t)\,u(t)^{p-1} = \lambda f(t,u(t)) \geq 0 \text{ a.e. on } T.$$

By virtue of the strong maximum principle of Pucci-Serrin ([10, p. 34]), we infer that $u(t) > 0$ for all $t \in (0,b)$. Then, invoking the boundary point theorem of Pucci-Serrin ([10, p. 120]), we conclude that $u \in int\ \widehat{C_+}$. So, we have proved that

$$\mathcal{S}(\lambda) \subset int\ \widehat{C_+}.$$

Hypotheses $\mathbf{H}(f)(i), (ii), (iii)$ imply that we can find $C_3 > 0$ such that

$$f(t,x) \leq C_3 x^{p-1} \text{ for a.a. } t \in T, \text{ all } x \geq 0. \tag{3.2}$$

Recall that $\widehat{\lambda}_1 > 0$ (see Section 2) and let $\lambda_0 \in \left(0, \frac{1}{C_3}\widehat{\lambda}_1\right)$. Suppose that $\lambda_0 \in \mathcal{L}$. Then we can find $u_0 \in \mathcal{S}(\lambda_0) \subset int\ \widehat{C_+}$. We have

$$A(u_0) + \beta u_0^{p-1} = \lambda_0 N_f(u_0). \tag{3.3}$$

On (3.3) we act with $u_0 \in W^{1,p}_{per}(0,b)$. Then

$$\langle A(u_0), u_0\rangle + \int_0^b \beta u_0^p dt = \int_0^b \lambda_0 f(t,u_0) u_0 dt,$$

hence

$$C_0 \left\|u_0'\right\|_p^p + \int_0^b \beta u_0^p dt \leq \lambda_0 C_3 \left\|u_0\right\|_p^p \text{ (see } \mathbf{H}(a)(i) \text{ and } (3.2)),$$

therefore

$$\widehat{\lambda}_1 \left\|u_0\right\|_p^p < \lambda_0 C_3 \left\|u_0\right\|_p^p,$$

a contradiction (see (2.3) and recall the choice of λ_0). Therefore $\lambda_0 \notin \mathcal{L}$, and so $\lambda_* \geq \frac{1}{C_3}\widehat{\lambda}_1 > 0$. $\square$

Let $\varphi_\lambda : W^{1,p}_{per}(0,b) \to \mathbb{R}$ be the energy functional for problem (P_λ) defined by

$$\varphi_\lambda(u)=\int_0^b G\left(u'(t)\right) dt+\frac{1}{p}\int_0^b \beta(t)\left|u(t)\right|^p dt-\lambda\int_0^b F(t,u(t))\, dt, \forall u \in W^{1,p}_{per}(0,b),$$

where

$$F(t,x) = \int_0^x f(t,s)\, ds.$$

We know that $\varphi_\lambda \in C^1\left(W^{1,p}_{per}(0,b)\right)$.

Proposition 3.2. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(\beta)$ *and* $\mathbf{H}(f)$ *hold, then* $\mathcal{L} \neq \varnothing$.

Proof. Hypotheses $\mathbf{H}(f)(ii)$, (iii) imply that given $\varepsilon > 0$, we can find $C_4 = C_4(\varepsilon) > 0$ such that

$$F(t,x) \leq \frac{\varepsilon}{p} x^p + C_4 \text{ for a.a. } t \in T, \text{ all } x \geq 0. \tag{3.4}$$

For $u \in W^{1,p}_{per}(0,b)$, we have

$$\begin{aligned}
\varphi_\lambda(u) &= \int_0^b G(u'(t))\, dt + \tfrac{1}{p}\int_0^b \beta(t)\left|u(t)\right|^p dt - \lambda \int_0^b F(t,u(t))\, dt \\
&\geq \tfrac{C_0}{p}\left\|u'\right\|_p^p + \tfrac{1}{p}\int_0^b \beta(t)\left|u(t)\right|^p dt - \tfrac{\lambda\varepsilon}{p}\left\|u\right\|_p^p - \lambda C_4 b \text{ (see (2.1) and (3.4))} \\
&\geq \tfrac{1}{p}(\xi_* - \lambda\varepsilon)\left\|u\right\|^p - \lambda C_4 b \text{ (see Lemma 2.1).}
\end{aligned} \tag{3.5}$$

Choosing $\varepsilon \in \left(0, \frac{\xi_*}{\lambda}\right)$, from (3.5) it follows that φ_λ is coercive. Also, using the Sobolev embedding theorem, we check easily that φ_λ is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\lambda \in W^{1,p}_{per}(0,b)$ such that

$$\varphi_\lambda(u_\lambda) = \inf\left\{\varphi_\lambda(u) \mid u \in W^{1,p}_{per}(0,b)\right\}. \tag{3.6}$$

If $\xi > 0$, then $\int_0^b F(t,\xi)\,dt > 0$ (see $\mathbf{H}(f)(v)$) and

$$\varphi_\lambda(\xi) = \frac{\xi^p}{p}\|\beta\|_1 - \lambda\int_0^b F(t,\xi)\,dt.$$

Therefore, choosing $\lambda > 0$ big enough, we have $\varphi_\lambda(\xi) < 0$. It follows that

$$\varphi_\lambda(u_\lambda) < 0 = \varphi_\lambda(0) \ \text{(see (3.6))},$$

hence

$$u_\lambda \neq 0.$$

By (3.6) we have $\varphi'_\lambda(u_\lambda) = 0$, hence

$$A(u_\lambda) + \beta|u_\lambda|^{p-2}u_\lambda = \lambda N_f(u_\lambda). \tag{3.7}$$

On (3.7) we act with $-u_\lambda^- \in W^{1,p}_{per}(0,b)$. We obtain

$$C_0\left\|u_\lambda^-\right\|_p^p + \int_0^b \beta(t)\left|u_\lambda^-(t)\right|^p dt \le 0 \ (\text{see } \mathbf{H}(a)(ii)),$$

hence

$$u_\lambda \ge 0,\ u_\lambda \neq 0 \text{ (see Lemma 2.1).}$$

Then, from (3.7) it follows that $u_\lambda \in \mathcal{S}(\lambda) \subset int\,\widehat{C_+}$ and so, $\mathcal{L} \neq \varnothing$. □

Proposition 3.3. *If hypotheses $\mathbf{H}(a)$, $\mathbf{H}(\beta)$ and $\mathbf{H}(f)$ hold and $\lambda \in \mathcal{L}$, then $[\lambda, +\infty) \subset \mathcal{L}$.*

Proof. Let $\eta > \lambda$. Since by hypothesis $\lambda \in \mathcal{L}$, by virtue of Proposition 3.1, we can find $u_\lambda \in \mathcal{S}(\lambda) \subset int\,\widehat{C_+}$. We consider the following Carathéodory function

$$\gamma_\eta(t,x) = \begin{cases} \eta f(t, u_\lambda(t)) & \text{if} \quad x \le u_\lambda(t) \\ \eta f(t,x) & \text{if} \quad u_\lambda(t) < x. \end{cases} \tag{3.8}$$

We set $\Gamma_\eta(t,x) = \int\limits_0^x \gamma_\eta(t,s)\,ds$ and consider the C^1- functional $\sigma_\eta : W^{1,p}_{per}(0,b) \to \mathbb{R}$ defined by

$$\sigma_\eta(u)=\int\limits_0^b G\left(u'(t)\right)dt+\frac{1}{p}\int\limits_0^b \beta(t)\left|u(t)\right|^p dt-\int\limits_0^b \Gamma_\eta(t,u(t))\,dt,\ \forall u\in W^{1,p}_{per}(0,b).$$

As in the proof of Proposition 3.2, we can check that σ_η is coercive. Also, σ_η is sequentially weakly lower semicontinuous. So, by the Weierstrass theorem, we can find $u_\eta \in W^{1,p}_{per}(0,b)$ such that

$$\sigma_\eta(u_\eta) = \inf\left\{\sigma_\eta(u) \mid u \in W^{1,p}_{per}(0,b)\right\}.$$

Then $\sigma_\eta'(u_\eta) = 0$, hence

$$A(u_\eta) + \beta\left|u_\eta\right|^{p-2} u_\eta = N_{\gamma_\eta}(u_\eta). \tag{3.9}$$

On (3.9) we act with $(u_\lambda - u_\eta)^+ \in W^{1,p}_{per}(0,b)$. Then

$$\begin{aligned}
&\left\langle A(u_\eta),(u_\lambda-u_\eta)^+\right\rangle + \int\limits_0^b \beta\left|u_\eta\right|^{p-2}u_\eta(u_\lambda-u_\eta)^+dt\\
&= \int\limits_0^b \gamma_\eta(t,u_\eta)(u_\lambda-u_\eta)^+dt\\
&= \int\limits_0^b \eta f(t,u_\lambda)(u_\lambda-u_\eta)^+dt \text{ (see (3.8))}\\
&\geq \int\limits_0^b \lambda f(t,u_\lambda)(u_\lambda-u_\eta)^+dt \text{ (since } \eta>\lambda \text{ and } f\geq 0))\\
&= \left\langle A(u_\lambda),(u_\lambda-u_\eta)^+\right\rangle + \int\limits_0^b \beta u_\lambda^{p-1}(u_\lambda-u_\eta)^+dt,
\end{aligned}$$

hence

$$\begin{aligned}
&\int\limits_{\{u_\lambda>u_\eta\}} \left(a\left(|u_\lambda'|\right)u_\lambda' - a\left(\left|u_\eta'\right|\right)u_\eta'\right)\left(u_\lambda'-u_\eta'\right)dt\\
&+\int\limits_{\{u_\lambda>u_\eta\}} \beta\left|u_\lambda^{p-1}-u_\eta^{p-2}u_\eta\right|(u_\lambda-u_\eta)\,dt\leq 0.
\end{aligned}$$

We conclude that

$$\left|\{u_\lambda>u_\eta\}\right|_1 = 0$$

(see $\mathbf{H}(a)(i)$), hence $u_\lambda \le u_\eta$. Then, from (3.9) and (3.8) it follows that

$$u_\eta \in \mathcal{S}(\eta) \subset int\ \widehat{C_+}$$

and so $\eta \in \mathcal{L}$. This proves that $[\lambda, +\infty) \subset \mathcal{L}$. □

Note that Proposition 3.3 implies that

$$(\lambda_*, +\infty) \subset \mathcal{L}.$$

Proposition 3.4. *If hypotheses $\mathbf{H}(a)$, $\mathbf{H}(\beta)$ and $\mathbf{H}(f)$ hold and $\lambda > \lambda_*$, then problem (P_λ) has at least two nontrivial positive solutions u_0, $\widehat{u} \in int\ \widehat{C_+}$.*

Proof. As we have already remarked, $(\lambda_*, +\infty) \subset \mathcal{L}$. Let η, λ, $\mu \in (\lambda_*, +\infty)$ with $\eta < \lambda < \mu$. We can find

$$u_\eta \in \mathcal{S}(\eta) \subset int\ \widehat{C_+} \text{ and } u_\mu \in \mathcal{S}(\mu) \subset int\ \widehat{C_+}, \text{ with } u_\eta \le u_\mu$$

(see the proof of Proposition 3.3). Consider the Carathéodory function $\theta_\lambda(t,x)$ defined by

$$\theta_\lambda(t,x) = \begin{cases} \lambda f(t, u_\eta(t)) & \text{if} \quad x < u_\eta(t) \\ \lambda f(t,x) & \text{if} \quad u_\eta(t) \le x \le u_\mu(t) \\ \lambda f(t, u_\mu(t)) & \text{if} \quad u_\mu(t) < x. \end{cases} \tag{3.10}$$

We set $\Theta_\lambda(t,x) = \int_0^x \theta_\lambda(t,s)ds$ and consider the C^1-functional $\xi_\lambda : W^{1,p}_{per}(0,b) \to \mathbb{R}$ defined by

$$\xi_\lambda(u) = \int_0^b G(u'(t))\,dt + \frac{1}{p}\int_0^b \beta(t)\,|u(t)|^p\,dt - \int_0^b \Theta_\lambda(t, u(t))\,dt,\ \ \forall u \in W^{1,p}_{per}(0,b).$$

From (3.10) it is clear that ξ_λ is coercive. Also, it is sequentially weakly lower semicontinuous. So, there exists $u_0 \in W^{1,p}_{per}(0,b)$ such that

$$\xi_\lambda(u_0) = \inf\left\{\xi_\lambda(u) \mid u \in W^{1,p}_{per}(0,b)\right\}.$$

Then

$$\xi'_\lambda(u_0) = 0,$$

hence

$$A(u_0) + \beta\,|u_0|^{p-2}\,u_0 = N_{\theta_\lambda}(u_0). \tag{3.11}$$

On (3.11) we act with $(u_0 - u_\mu)^+ \in W^{1,p}_{per}(0,b)$. Then

$$\begin{aligned}
&\left\langle A(u_0), (u_0-u_\mu)^+\right\rangle + \int_0^b \beta\,|u_0|^{p-2} u_0 (u_0-u_\mu)^+\,dt \\
&= \int_0^b \theta_\lambda(t,u_0)(u_0-u_\lambda)^+\,dt \\
&= \int_0^b \lambda f(t,u_\mu)(u_0-u_\lambda)^+\,dt \text{ (see (3.10))} \\
&\le \int_0^b \mu f(t,u_\mu)(u_0-u_\lambda)^+\,dt \text{ (since } \mu>\lambda \text{ and } f\ge 0)) \\
&= \left\langle A(u_\mu), (u_0-u_\mu)^+\right\rangle + \int_0^b \beta u_\mu^{p-1}(u_0-u_\mu)^+\,dt,
\end{aligned}$$

hence

$$\begin{aligned}
&\int_{\{u_0>u_\mu\}} \left(a(|u_0'|)u_0' - a\left(|u_\mu'|\right)u_\mu'\right)\left(u_0'-u_\mu'\right)dt \\
&+ \int_{\{u_0>u_\mu\}} \beta\left|u_0^{p-1} - u_\mu^{p-1}\right|(u_0-u_\mu)\,dt \le 0.
\end{aligned}$$

We conclude that $|\{u_0>u_\mu\}|_1 = 0$ (see $\mathbf{H}(a)(i)$), hence $u_0 \le u_\mu$. Similarly, acting on (3.11) with $(u_\eta - u_0)^+ \in W^{1,p}_{per}(0,b)$ we obtain $u_\eta \le u_0$. So, it follows that

$$u_0 \in [u_\eta, u_\mu] = \left\{u \in W^{1,p}_{per}(0,b) \mid u_\eta(t) \le u_0(t) \le u_\mu(t) \text{ for all } t\in T\right\}.$$

Let $\rho = \|u_\mu\|_\infty$ and let $\xi_\rho > 0$ be as postulated by hypothesis $\mathbf{H}(f)(iv)$. For $\delta > 0$, we set $u_0^\delta = u_0 + \delta \in int\,\widehat{C_+}$. Then we have

$$\begin{aligned}
&-\left(a\left(\left|(u_0^\delta)'\right|\right)(u_0^\delta)'\right)' + (\beta(t)+\mu\xi_\rho)\left(u_0^\delta\right)^{p-1} \\
&\le -\left(a(|u_0'|)u_0'\right)' + (\beta+\mu\xi_\rho)u_0^{p-1} + \zeta(\delta) \text{ with } \zeta(\delta)\downarrow 0 \text{ as } \delta\downarrow 0 \\
&= \lambda f(t,u_0) + \mu\xi_\rho u_0^{p-1} + \zeta(\delta) \\
&\le \mu f(t,u_0) + \mu\xi_\rho u_0^{p-1} - (\mu-\lambda)\widehat{m}_\tau + \zeta(\delta),
\end{aligned} \tag{3.12}$$

where $\tau = \min_T u_0 > 0$ (recall that $u_0 \in int\,\widehat{C_+}$) and $\widehat{m}_\tau$ is as in hypothesis $\mathbf{H}(f)(v)$. Since $\zeta(\delta)\downarrow 0$ as $\delta \downarrow 0$, we can find $\delta_0>0$ such that

$$\zeta(\delta) \le (\mu-\lambda)\widehat{m}_\tau \text{ for all } \delta\in(0,\delta_0] \text{ (recall that } \mu>\lambda).$$

Therefore, from (3.12) we obtain

$$A\left(u_0^\delta\right)+(\beta+\mu\xi_\rho)\left(u_0^\delta\right)^{p-1}\le A(u_\mu)+(\beta+\mu\xi_\rho)(u_\mu)^{p-1}\ \text{ in } W_{per}^{1,p}(0,b)^*. \tag{3.13}$$

Acting on (3.13) with $\left(u_0^\delta-u_\mu\right)^+\in W_{per}^{1,p}(0,b)$, we show that

$$u_0^\delta\le u_\mu \text{ for all } \delta\in(0,\delta_0],$$

hence $u_\mu-u_0\in int\ \widehat{C_+}$. In a similar fashion, we show that $u_0-u_\eta\in int\ \widehat{C_+}$. Therefore, we have proved that

$$u_0\in int_{\widehat{C^1}(T)}[u_\eta,u_\mu].$$

Note that

$$\varphi_\lambda\mid_{[u_\eta,u_\mu]}=\xi_\lambda\mid_{[u_\eta,u_\mu]}+K_\lambda^* \text{ with } K_\lambda^*\in\mathbb{R} \text{ (see (3.10))},$$

hence u_0 is a $\widehat{C^1}(T)$-local minimizer of φ_λ, therefore u_0 is also a $W_{per}^{1,p}(0,b)$-local minimizer of φ_λ (see Proposition 2.2).

By virtue of hypothesis $\mathbf{H}(f)(iii)$, given $\varepsilon>0$, we can find $\delta=\delta(\varepsilon)>0$ such that

$$F(t,x)\le\frac{\varepsilon}{p}x^p \text{ for a.a. } t\in T,\ \text{all } x\in[0,\delta]. \tag{3.14}$$

Then, for $u\in\widehat{C^1}(T)$ with $\|u\|_{\widehat{C^1}(T)}\le\delta$, we have

$$\begin{aligned}\varphi_\lambda(u)&=\int_0^b G\left(u'(t)\right)dt+\frac{1}{p}\int_0^b\beta(t)|u(t)|^p\,dt-\lambda\int_0^b F(t,u(t))\,dt\\&\ge\frac{C_0}{p}\left\|u'\right\|_p^p+\frac{1}{p}\int_0^b\beta(t)|u(t)|^p\,dt-\frac{\lambda\varepsilon}{p}\|u\|_p^p\ \text{ (see (2.1) and (3.14))}\\&\ge\frac{1}{p}(\xi_*-\lambda\varepsilon)\|u\|^p\ \text{ (see Lemma 2.1).}\end{aligned}$$

Choosing $\varepsilon\in\left(0,\frac{\xi_*}{\lambda}\right)$ we see that $u=0$ is a $\widehat{C^1}(T)$-local minimizer of φ_λ. Hence by virtue of Proposition 2.2, $u=0$ is also a $W_{per}^{1,p}(0,b)$-local minimizer of φ_λ.

Without any loss of generality, we may assume that

$$\varphi_\lambda(0)=0\le\varphi_\lambda(u_0).$$

The analysis is similar if the opposite inequality holds. As in the paper [1] by Aizicovici-Papageorgiou-Staicu (see the proof of Proposition 29), we can find $\rho_0\in(0,1)$ small such that

$$\varphi_\lambda(0)\le\varphi_\lambda(u_0)<\inf\{\varphi_\lambda(u)\mid\|u\|=\rho_0\}=:\eta_0,\ \ \|u_0\|>\rho_0. \tag{3.15}$$

Recall that φ_λ is coercive, hence it satisfies the PS-condition. This fact and (3.15) permit the application of Theorem 2.1 (the mountain pass theorem). So, we can find $\widehat{u} \in W^{1,p}_{per}(0,b)$ such that

$$\varphi'_\lambda(\widehat{u}) = 0 \text{ and } \eta_0 \leq \varphi_\lambda(\widehat{u}). \tag{3.16}$$

From (3.15) and (3.16) it follows that $\widehat{u} \notin \{0, u_0\}$ and

$$A(\widehat{u}) + \beta |\widehat{u}|^{p-2} \widehat{u} = \lambda N_f(\widehat{u}).$$

Acting with $-\widehat{u}^- \in W^{1,p}_{per}(0,b)$, we obtain $\widehat{u} \in \widehat{C_+} \setminus \{0\}$. So, $\widehat{u} \in \mathcal{S}(\lambda) \subset int\ \widehat{C_+}$. □

Proposition 3.5. *If hypotheses* $\mathbf{H}(a)$, $\mathbf{H}(\beta)$ *and* $\mathbf{H}(f)$ *hold, then* $\lambda_* \in \mathcal{L}$.

Proof. Recall that $(\lambda_*, +\infty) \subset \mathcal{L}$. Let $\{\lambda_n\}_{n\geq 1} \subset \mathcal{L}$ be such that

$$\lambda_* < \lambda_{n+1} < \lambda_n \text{ for all } n \geq 1 \text{ and } \lambda_n \downarrow \lambda_* \text{ as } n \to \infty.$$

Then, we can find $u_n \in \mathcal{S}(\lambda_n)$ such that

$$A(u_n) + \beta u_n^{p-1} = \lambda_n N_f(u_n) \text{ for all } n \geq 1. \tag{3.17}$$

In fact, since $\{\lambda_n\}_{n\geq 1}$ is decreasing, from the proof of Proposition 3.3, it follows that

$$u_n \leq u_1 \text{ for all } n \geq 1. \tag{3.18}$$

Then from (3.17), (3.18) and (3.2) we infer that $\{u_n\}_{n\geq 1} \subset W^{1,p}_{per}(0,b)$ is bounded. So, we may assume that

$$u_n \xrightarrow{w} u_* \text{ in } W^{1,p}_{per}(0,b) \text{ and } u_n \to u_* \text{ in } C(T) \text{ as } n \to \infty. \tag{3.19}$$

On (3.17) we act with $u_n - u_* \in W^{1,p}_{per}(0,b)$, pass to the limit as $n \to \infty$ and use (3.19). Then

$$\limsup_{n\to\infty} \langle A(u_n), u_n - u_* \rangle \leq 0,$$

hence

$$u_n \to u_* \text{ in } W^{1,p}_{per}(0,b) \text{ as } n \to \infty \tag{3.20}$$

(see Proposition 2.1). So, if in (3.17) we pass to the limit as $n \to \infty$ and use (3.20), then

$$A(u_*) + \beta u_*^{p-1} = \lambda_* N_f(u_*). \tag{3.21}$$

If we show that $u_* \neq 0$, then $\lambda_* \in \mathcal{L}$. We argue by contradiction.

So, suppose that $u_* = 0$. From (3.20) we have

$$u_n \to 0 \text{ in } C(T) \text{ as } n \to \infty. \tag{3.22}$$

Hypothesis $\mathbf{H}(f)(iii)$ implies that given $\varepsilon > 0$, we can find $\delta = \delta(\varepsilon) > 0$ such that

$$f(t,x) \le \varepsilon x^{p-1} \text{ for a.a. } t \in T, \text{ all } x \in [0,\delta]. \tag{3.23}$$

From (3.22) it follows that we can find $n_0 \ge 1$ such that

$$u_n(t) \in [0,\delta] \text{ for all } t \in T, \text{ all } n \ge n_0,$$

hence

$$-\left(a\left(\left|u_n'(t)\right|\right)u_n'(t)\right)' + \beta(t)\,u_n(t)^{p-1} \le \lambda_n \varepsilon u_n(t)^{p-1} \text{ a.e. on } T, \ \forall n \ge n_0$$

(see (3.23)), therefore

$$C_0 \left\|u_n'\right\|_p^p + \frac{1}{p}\int_0^b \beta(t)\,|u_n(t)|^p\,dt \le \frac{\lambda_n \varepsilon}{p}\|u_n\|_p^p \text{ for all } n \ge n_0.$$

Then, by Lemma 2.1,

$$\frac{\widetilde{\xi}_*}{\lambda_*} \le \varepsilon,$$

for some $\widetilde{\xi}_* > 0$. But $\varepsilon > 0$ is arbitrary. So, letting $\varepsilon \downarrow 0$, we reach a contradiction (recall that $\widetilde{\xi}_*, \lambda_* > 0$). Therefore $u_* \ne 0$, and so $u_* \in \mathcal{S}(\lambda) \subset int\,\widehat{C_+}$ (see (3.21)). Hence $\lambda_* \in \mathcal{L}$. □

Summarizing Propositions 3.1-3.5, we can state the following bifurcation-type theorem:

Theorem 3.1. *If hypotheses $\mathbf{H}(a)$, $\mathbf{H}(\beta)$ and $\mathbf{H}(f)$ hold, then there exists $\lambda_* > 0$ such that*

- (a) *for all $\lambda > \lambda_*$, problem (P_λ) has at least two nontrivial positive solutions u_0, $\widehat{u} \in int\,\widehat{C}_+$;*
- (b) *for $\lambda = \lambda_*$, problem (P_λ) has at least one nontrivial positive solution $u_* \in int\,\widehat{C}_+$;*
- (c) *for all $\lambda \in (0,\lambda_*)$, problem (P_λ) has no nontrivial positive solution.*

Remark. Note that here the bifurcation occurs at large values of $\lambda > 0$, while in [2], where the conditions on the nonlinearity were complementary (both at $+\infty$ and at 0; competing nonlinearities) the bifurcation takes place at small values of $\lambda > 0$.

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