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On Q-homeomorphisms with respect to p-modulus

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Abstract - It is established that a Q-homeomorphism with respect to p-modulus in $\mathbb{R}^n$, $n \geqslant 2$, is finitely Lipschitz if $n-1<p<n$ and if the mean integral value of $Q(x)$ over infinitesimal balls $B(x_0, \varepsilon)$ is finite everywhere.

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1. Introduction

Recall that, given a family of paths Γ in $\mathbb{R}^n$, a Borel function $\varrho : \mathbb{R}^n \to [0, \infty]$ is called *admissible* for Γ, abbr. $\varrho \in adm\,\Gamma$, if

$$\int\limits_{\gamma} \varrho\, ds \ \geq \ 1 \tag{1.1}$$

for all $\gamma \in \Gamma$. The *p-modulus* of Γ is the quantity

$$M_p(\Gamma) \ = \inf_{\varrho \in adm\,\Gamma} \int\limits_{\mathbb{R}^n} \varrho^p(x)\ dm(x)\ . \tag{1.2}$$

Here the notation m refers to the Lebesgue measure in $\mathbb{R}^n$.

Let G and G' be domains in $\mathbb{R}^n$, $n \geq 2$, and let $Q : G \to [0, \infty]$ be a measurable function. A homeomorphism $f : G \to G'$ is called a *$Q-$homeomorphism with respect to the p-modulus* if

$$M_p(f\Gamma) \leq \int\limits_{G} Q(x) \cdot \varrho^p(x)\ dm(x) \tag{1.3}$$

for every family Γ of paths in G and every admissible function ϱ for Γ.

This conception is a natural generalization of the geometric definition of a quasiconformal mapping: if $Q(x) \leq K < \infty$ a.e., then f is quasiconformal under $p=n$, see 13.1 and 34.6 in [20], and local quasiisometric under $n-1<p<n$ (see [5]).

The notion of Q–homeomorphism is closely related to the concept of moduli with weights essentially due to Andreian Cazacu (see, e.g., [1] and references therein).

This class of Q-homeomorphisms with respect to the n-modulus was first considered in the papers [14]-[16], see also the monograph [17]. The main goal of the theory of Q-homeomorphisms is to clear up various interconnections between properties of the majorant $Q(x)$ and the corresponding properties of the mappings themselves. In particular, the problem of the local and boundary behavior of Q-homeomorphisms has been studied in $\mathbb{R}^n$ first in the case $Q\in BMO$ (bounded mean oscillation) in the papers [14]-[16] and then in the case of $Q\in FMO$ (finite mean oscillation) and other cases in the papers [10], [11].

Note that the estimate of the type (1.3) was first established in the classical quasiconformal theory. Namely, it was obtained in [13], p. 221, for quasiconformal mappings in the complex plane that

$$M(f\Gamma)\ \leq\ \int\limits_{\mathbb{C}} K(z)\cdot\rho^2(z)\ dxdy \tag{1.4}$$

where

$$K(z)\ =\ \frac{|f_z|+|f_{\overline{z}}|}{|f_z|-|f_{\overline{z}}|} \tag{1.5}$$

is a (local) maximal dilatation of the mapping f at a point z. Next, it was obtain in [2], Lemma 2.1, for quasiconformal mappings in space, $n\geq 2$, that

$$M(f\Gamma)\ \leq\ \int\limits_{D} K_I(x,f)\,\rho^n(x)\ dm(x) \tag{1.6}$$

where $K_I(x,f)$ stands for the inner dilatation of f at x, see (1.8) below.

Given a mapping $f:G\to\mathbb{R}^n$ with partial derivatives a.e., $f'(x)$ denotes the Jacobian matrix of f at $x\in G$ if it exists, $J(x)=J(x,f)=\det f'(x)$ the Jacobian of f at x, and $|f'(x)|$ the operator norm of $f'(x)$, i.e., $|f'(x)|=\max\{|f'(x)h|\,|\,h\in\mathbb{R}^n,|h|=1\}$. We also let $l(f'(x))=\min\{|f'(x)h|\,|\,h\in\mathbb{R}^n,|h|=1\}$. The *outer dilatation* of f at x is defined by

$$K_O(x,f)=\begin{cases}\frac{|f'(x)|^n}{|J(x,f)|}, & \text{if } J(x,f)\neq 0\\ 1, & \text{if } f'(x)=0\\ \infty, & \text{otherwise,}\end{cases} \tag{1.7}$$

the *inner dilatation* of f at x by

$$K_I(x,f)=\begin{cases}\frac{|J(x,f)|}{l(f'(x))^n}, & \text{if } J(x,f)\neq 0\\ 1, & \text{if } f'(x)=0\\ \infty, & \text{otherwise}\end{cases}\ . \tag{1.8}$$

2. Preliminaries

Let $E,\ F\subset\mathbb{R}^n$ be arbitrary sets. Denote by $\Delta(E,F,G)$ a family of all curves $\gamma:[a,b]\to\overline{\mathbb{R}^n}$ joining E and F in G, i.e., $\gamma(a)\in E,\gamma(b)\in F$ and $\gamma(t)\in G$ for $t\in(a,b)$.

Here a *condenser* is a pair $E=(A,C)$ where $A\subset\mathbb{R}^n$ is open and C is a non–empty compact set contained in A . E is a *ringlike condenser* if $B=A\backslash C$ is a ring, i.e., if B is a domain whose complement $\overline{\mathbb{R}^n}\backslash B$ has exactly two components where $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$ is the one point compactification of $\mathbb{R}^n$. E is a *bounded condenser* if A is bounded. A condenser $E=(A,C)$ is said to be in a domain G if $A\subset G$.

The following proposition is immediate.

Proposition 2.1. *If $f:G\to\mathbb{R}^n$ is open and $E=(A,C)$ is a condenser in G, then $(fA,\ fC)$ is a condenser in fG.*

In the above situation we denote $fE=(fA,\ fC)$.

Let $E=(A,\ C)$ be a condenser. Then $W_0(E)=W_0(A,\ C)$ denotes the family of non–negative functions $u:A\to R^1$ such that (1) $u\in C_0(A)$, (2) $u(x)\geq 1$ for $x\in C$, and (3) u is ACL. We set

$$cap_p\,E\ =\ cap_p\ (A,\ C)\ =\ \inf_{u\,\in\,W_0(E)}\int\limits_A|\nabla u|^p\ dm(x) \tag{2.1}$$

where

$$|\nabla u|\ =\ \left(\sum_{i=1}^{n}\ (\partial_i u)^2\right)^{1/2}$$

and call the quantity (2.1) the *p-capacity* of the condenser E.

For the next statement, see, e.g., [6], [9] and [19].

Proposition 2.2. *Suppose $E=(A,C)$ is a condenser such that C is connected. Then*

$$cap_p\,E=M_p(\Delta(\partial A,\partial C;A\setminus C))\,.$$

We give here also the following two useful statements (see Proposition 6 in [12]).

Proposition 2.3. *Let $E=(A,C)$ be a condenser such that C is connected. Then for $n-1<p\leq n$*

$$(cap_p\ E)^{n-1}\ \geq\ \gamma_{n,p}\,\frac{d(C)^p}{m(A)^{1-n+p}},$$

where $\gamma_{n,p}$ is a positive constant that depends only on n and p, $d(A)$ is a diameter and $m(A)$ is the Lebesgue measure of A in $\mathbb{R}^n$.

Proposition 2.4. (see [18]) *Let $E=(A,C)$ be a condenser such that C is connected. Then for $1\le p<n$*

$$cap_p\, E \ge n\Omega_n^{\frac{p}{n}}\left(\frac{n-p}{p-1}\right)^{p-1}[mC]^{\frac{n-p}{n}}\,, \tag{2.2}$$

where Ω_n denotes the volume of the unit ball in $\mathbb{R}^n$, and mC is the n-dimensional Lebesgue measure of C.

3. On finite by Lipschitz Q-homeomorphisms with respect to the p-modulus

Given a mapping $\varphi : E\to\mathbb{R}^n$ and a point $x\in E\subseteq\mathbb{R}^n$, set

$$L(x,\varphi)\;=\;\limsup_{y\to x\; y\in E}\frac{|\varphi(y)-\varphi(x)|}{|y-x|}. \tag{3.1}$$

Given a set $A\subseteq\mathbb{R}^n$, $n\ge 1$, we say that a mapping $f : A\to\mathbb{R}^n$ is called *Lipschitz* if there is number $L>0$ such that the inequality

$$|\,f(x)-f(y)|\,\le L\,|\,x-y| \tag{3.2}$$

holds for all x and y in A. Given an open set $\Omega\subseteq\mathbb{R}^n$, we say that a mapping $f:\Omega\to\mathbb{R}^n$ is *finitely Lipschitz* if $L(x,f)<\infty$ for all $x\in\Omega$.

Theorem 3.1. *Let G and G' be domains in $\mathbb{R}^n$, $n\ge 2$, and $Q : G\to[0,\infty]$ be a measurable function such that*

$$Q_0=\limsup_{r\to 0}\ \frac{1}{\Omega_n\varepsilon^n}\int\limits_{B(x_0,\varepsilon)}Q(x)\,dm(x)<\infty. \tag{3.3}$$

Then for every $Q-$homeomorphism $f : G\to G'$ with respect to the p-modulus, $n-1<p<n$,

$$L(x_0,f)=\limsup_{x\to x_0}\frac{|f(x)-f(x_0)|}{|x-x_0|}\le C_{n,p}\,Q_0^{\frac{1}{n-p}} \tag{3.4}$$

where $C_{n,p}$ is a positive constant that depends only on n and p.

Proof. Let us consider the spherical ring $A=A(x_0,\varepsilon_1,\varepsilon_2)=\{x\,|\,\varepsilon_1<|x-x_0|<\varepsilon_2\}$, $x\in G$, $\varepsilon_1,\varepsilon_2>0$ such that $A(x_0,\varepsilon_1,\varepsilon_2)\subset G$. Since $\left(fB\,(x_0,\varepsilon_2)\,,\overline{fB\,(x_0,\varepsilon_1)}\right)=\left(fB\,(x_0,\varepsilon_2)\,,f\overline{B\,(x_0,\varepsilon_1)}\right)$ are ringlike condensers in G', according to Proposition 2.2, we obtain

$$cap_p\ (fB(x_0,\varepsilon_2),\overline{fB(x_0,\varepsilon_1)})=M_p(\triangle(\partial fB(x_0,\varepsilon_2),\partial fB(x_0,\varepsilon_1);fA))\,.$$

Note that, since f is homeomorphism, we have

$$\triangle\left(\partial fB\left(x_0,\varepsilon_2\right),\partial fB\left(x_0,\varepsilon_1\right);fA\right)=f\left(\triangle\left(\partial B(x_0,\varepsilon_2),\partial B(x_0,\varepsilon_1);A\right)\right).$$

By the definition of $Q-$homeomorphisms with respect to the p-modulus

$$cap_p\ (fB(x_0,\varepsilon_2),\overline{fB(x_0,\varepsilon_1)})\leq\int\limits_{A(x_0,\varepsilon_1,\varepsilon_2)}Q(x)\ \rho^p(x)\,dm(x)\tag{3.5}$$

for every admissible function ρ of $\triangle\left(\partial B(x_0,\varepsilon_2),\partial B(x_0,\varepsilon_1);A(x_0,\varepsilon_1,\varepsilon_2)\right)$.

The function

$$\rho(x)\ =\left\{\begin{array}{rl}\frac{1}{\varepsilon_2-\varepsilon_1}, & \text{if } x\in A(x_0,\varepsilon_1,\varepsilon_2),\\ 0, & \text{if } x\in G\setminus A(x_0,\varepsilon_1,\varepsilon_2)\end{array}\right.$$

is admissible and, thus,

$$cap_p\ (fB(x_0,\varepsilon_2),\overline{fB(x_0,\varepsilon_1)})\leq\frac{1}{(\varepsilon_2-\varepsilon_1)^p}\int\limits_{B(x_0,\varepsilon_2)}Q(x)\,dm(x)\,.\tag{3.6}$$

By choosing $\varepsilon_1=\varepsilon$ and $\varepsilon_2=2\varepsilon$, we have

$$cap_p\ (fB(x_0,2\varepsilon),\overline{fB(x_0,\varepsilon)})\leq\frac{1}{\varepsilon^p}\int\limits_{B(x_0,2\varepsilon)}Q(x)\,dm(x).\tag{3.7}$$

On the other hand, by Proposition 2.3

$$cap_p\ (fB(x_0,2\varepsilon),\overline{fB(x_0,\varepsilon)})\geq\left(\gamma_{n,p}\frac{d^p(fB(x_0,\varepsilon))}{m^{1-n+p}(fB(x_0,2\varepsilon))}\right)^{\frac{1}{n-1}},\tag{3.8}$$

where $\gamma_{n,p}$ is a positive constant that depends only on n and p.

Combining (3.7) and (3.8) we obtain

$$\left(\gamma_{n,p}\frac{d^p(fB(x_0,\varepsilon))}{m^{1-n+p}(fB(x_0,2\varepsilon))}\right)^{\frac{1}{n-1}}\leq\frac{1}{\varepsilon^p}\int\limits_{B(x_0,2\varepsilon)}Q(x)\,dm(x)\,.\tag{3.9}$$

Next, by choosing $\varepsilon_1=2\varepsilon$ and $\varepsilon_2=4\varepsilon$, we have that

$$cap_p(fB(x_0,4\varepsilon),\overline{fB(x_0,2\varepsilon)})\leq\frac{1}{(2\varepsilon)^p}\int\limits_{B(x_0,4\varepsilon)}Q(x)dm(x)\,.\tag{3.10}$$

By Proposition 2.4

$$cap_p\ (fB(x_0,4\varepsilon),f\overline{B(x_0,2\varepsilon)})\geq\alpha_{n,p}\left[m(fB(x_0,2\varepsilon))\right]^{\frac{n-p}{n}},\tag{3.11}$$

where $\alpha_{n,p}$ is a positive constant that depends only on n and p.

Combining (3.10) and (3.11) we obtain

$$m(fB(x_0, 2\varepsilon)) \leqslant \beta_{n,p}\, \varepsilon^n \left[\frac{1}{m(B(x, 4\varepsilon))} \int\limits_{B(x_0, 4\varepsilon)} Q(x)\, dm(x) \right]^{\frac{n}{n-p}}, \quad (3.12)$$

where $\beta_{n,p}$ is a positive constant that depends only on n and p.

Combining (3.9) and (3.12), we obtain

$$\frac{d(fB(x_0, \varepsilon))}{\varepsilon} \leq C_{n,p} \left(\frac{\int\limits_{B(x_0, 4\varepsilon)} Q(x)\, dm(x)}{m(B(x_0, 4\varepsilon))} \right)^{\frac{1}{n-p}}$$

and hence

$$L(x_0, f) = \limsup_{x \to x_0} \frac{|f(x) - f(x_0)|}{|x - x_0|} \leq \limsup_{\varepsilon \to 0} \frac{d(fB(x_0, \varepsilon))}{\varepsilon} \leq C_{n,p}\, Q_0^{\frac{1}{n-p}},$$

where $C_{n,p}$ is a positive constant that depends only on n and p. □

Corollary 3.1. *Let G and G' be domains in $\mathbb{R}^n$, $n \geq 2$, $f : G \to G'$ be a $Q-$homeomorphism with respect to the p-modulus, $n - 1 < p < n$, such that*

$$\limsup_{\varepsilon \to 0} \frac{1}{\Omega_n \varepsilon^n} \int\limits_{B(x_0, \varepsilon)} Q(x)\, dm(x) < \infty \quad \forall\ x_0 \in G. \quad (3.13)$$

Then f is finitely Lipschitz.

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