

Again on regularity conditions in differentiable vector optimization

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Abstract - This paper is concerned with regularity conditions for a differentiable vector optimization problem with inequality and equality constraints, where the terms “regularity conditions” are intended both in the usual sense of “constraint qualifications” and in the meanings considered by Bigi and Pappalardo (1999).

Key words and phrases : constraint qualifications, regularity conditions, vector optimization, optimality conditions.

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1. Introduction

This paper is concerned with some basic questions on regularity conditions for a differentiable vector optimization problem. We prefer the terms “regularity condition”, as these conditions involve both the (vector) objective function and the constraints, and shall use the terms “constraint qualification” for those conditions involving only the constraints.

The paper is organized as follows.

In Section 2 we investigate various regularity conditions for a differentiable Pareto optimization problem, with both inequality and equality constraints, generalizing some results of Maeda (see [24]) and specifying, for the case under consideration, some corresponding results of Giorgi, Jiménez and Novo (see [11]).

In Section 3 we shall be concerned, always for the problem introduced in the previous sections, with various regularity conditions of the type considered by Bigi and Pappalardo (see [4]), Castellani, Mastroeni and Pappalardo (see [6]), and recently by Maciel, Santos and Sottosanto (see [23]). We shall give shorter and more compact proofs of some results obtained by the said authors and we shall point out some new results.

We first fix some notations and the terminology used in this paper.

If x and y are two vectors of \mathbb{R}^n , we write $x \leq y$ if $x_i \leq y_i$, $i = 1, 2, \dots, n$, and $x < y$ if $x_i < y_i$, $i = 1, 2, \dots, n$.

If S is a subset of \mathbb{R}^n , by clS , coS , $coneS$ we denote, respectively, the closure of S , the convex hull of S , and the cone generated by S .

We denote by $B(x^0, \delta)$ the ball centered at x^0 , with radius δ .

Given $S \subset \mathbb{R}^n$ and $f : \mathbb{R}^n \rightarrow \mathbb{R}^p$, consider the following vector optimization problem:

$$(VOP) \quad \min \{f(x) \mid x \in S\}.$$

A point $x^0 \in S$ is said to be a *Pareto minimum* of f on S or an *efficient solution* to problem (VOP) if there is no $x \in S$ such that $f(x) \leq f(x^0)$ and $f(x) \neq f(x^0)$; the point x^0 is a *local Pareto minimum* if there exists $B(x^0, \delta)$ such that

$$S_f \cap S \cap B(x^0, \delta) = \emptyset$$

being

$$S_f = \{x \in \mathbb{R}^n \mid f(x) \leq f(x^0), f(x) \neq f(x^0)\}.$$

A point $x^0 \in S$ is said to be a *weak Pareto minimum* of f on S if there is no $x \in S$ such that $f(x) < f(x^0)$.

It is evident the notion of *local weak Pareto minimum* of f on S .

Obviously every Pareto minimum point is also a weak Pareto minimum point and it can be proved (see [27]) that if S is convex and if the objective functions are quasiconvex with at least one strictly quasiconvex, the set of local Pareto minimum points is a subset of the set of weak Pareto minimum points: indeed, under the above-mentioned assumptions, all the local Pareto minimum points are also global.

Definition 1.1. Let $S \subset \mathbb{R}^n$ and $x^0 \in clS$.

(1.1a) The set

$$T(S, x^0) = \left\{ v \in \mathbb{R}^n \mid \exists t_k > 0, \exists x^k \in S, x^k \rightarrow x^0 \text{ s.t. } t_k(x^k - x^0) \rightarrow v \right\}$$

is the **Bouligand tangent cone to S at x^0 or contingent cone to S at x^0** .

(1.1b) The set

$$A(S, x^0) = \left\{ \begin{array}{l} v \in \mathbb{R}^n \mid \exists \delta > 0, \exists \gamma : [0, \delta] \rightarrow \mathbb{R}^n \text{ such that} \\ \gamma(0) = x^0, \gamma(t) \in S, \forall t \in (0, \delta], \gamma'(0) = v \end{array} \right\}$$

is the **cone of the attainable directions to S at x^0** .

(1.1c) The set

$$Z(S, x^0) = \{v \in \mathbb{R}^n \mid \exists \delta > 0 \text{ such that } x^0 + tv \in S, \forall t \in (0, \delta)\}$$

is the **cone of the feasible directions to S at x^0** .

It is well known that $T(S, x^0)$ is a nonempty closed cone and that, if S is convex, then so is $T(S, x^0)$ (see [1], [2], [10], [9]). For these cones we have the following inclusions:

$$Z(S, x^0) \subset A(S, x^0) \subset T(S, x^0).$$

Now we introduce a generalization of the concept of linear independence for vectors of \mathbb{R}^n .

Given a set of vectors of \mathbb{R}^n , $A = \{a^1, a^2, \dots, a^p\}$, we define the following sets:

$$A^- = \{v \in \mathbb{R}^n \mid a^i v < 0, \forall i\};$$

$$A^* = \{v \in \mathbb{R}^n \mid a^i v \leq 0, \forall i\};$$

$$\ker(A) = \{v \in \mathbb{R}^n \mid a^i v = 0, \forall i\};$$

$\text{lin}(A)$ is the linear subspace generated by A .

We say that the set of vectors of A are *positively linearly independent* or that A is *positively linearly independent* (pli) if

$$\left\{ \sum_{i=1}^p \lambda_i a^i = 0, \lambda \geq 0 \right\} \Rightarrow \lambda = 0.$$

Otherwise we say that A is *positively linearly dependent* (pld).

Definition 1.2. (see [29]) Let $A = \{a^1, a^2, \dots, a^p\}$ and $B = \{b^1, b^2, \dots, b^q\}$ two finite sets of vectors of \mathbb{R}^n such that $A \cup B \neq \emptyset$. We say that (A, B) is **positive linearly independent** if there is no $(\lambda, \mu) \neq 0$, $\lambda \in \mathbb{R}^p$, $\mu \in \mathbb{R}^q$, $\lambda \geq 0$, such that

$$\sum_{i=1}^p \lambda_i a^i + \sum_{j=1}^q \mu_j b^j = 0,$$

or, equivalently, if

$$\left\{ \sum_{i=1}^p \lambda_i a^i + \sum_{j=1}^q \mu_j b^j = 0, \lambda \geq 0 \right\} \Rightarrow (\lambda, \mu) = 0. \quad (1.1)$$

Otherwise we say, that (A, B) is **positively linearly dependent** (pld).

Of course if $B = \emptyset$ we get the previous definitions of A pli or pld.

Let us suppose that $A \neq \emptyset$ (B can be also empty). We have the following results.

Lemma 1.1. *The following equivalence holds:*

$$\{A \text{ is pli}\} \Leftrightarrow \left\{ \begin{array}{l} A^- \neq \emptyset, \text{ i.e. there exists } v \in \mathbb{R}^n \\ \text{such that } a^i v < 0, \forall i \end{array} \right\}.$$

This lemma is nothing but the Gordan theorem of the alternative (see, e.g., [10], [25]).

Lemma 1.2. *The following equivalences hold:*

$$\begin{aligned} & \{(A, B) \text{ is pli}\} \Leftrightarrow \\ & \Leftrightarrow \left\{ \begin{array}{l} B \text{ is linearly independent and } A^- \cap \ker(B) \neq \emptyset, \\ \text{i.e. there exists } v \in \mathbb{R}^n \\ \text{such that } a^i v < 0, \forall i \text{ and } b^j v = 0, \forall j \end{array} \right\} \Leftrightarrow \\ & \Leftrightarrow \{A \text{ is pli, } B \text{ is linearly independent and } (\text{cone } coA) \cap \text{lin}(B) = \{0\}\}. \end{aligned}$$

Proof. For the proof of the first two equivalences see, e.g. [15] or [29, Proposition 2.3]. The third equivalence is proved as follows. Let us denote by (a), (b) and (c) respectively, the first, second and third proposition between brackets.

(b) \Rightarrow (c) Being $A^- \neq \emptyset$, for the Motzkin theorem of the alternative (see, e.g., [25]), there exists no $\lambda \geq 0$, $\lambda \neq 0$, such that $\sum_{i=1}^p \lambda_i a^i = 0$, and therefore A is pli.

Let $u \in (\text{cone } coA) \cap \text{lin}B$, therefore $u = \sum_{i=1}^p \lambda_i a^i = \sum_{j=1}^q \mu_j b^j$, with $\lambda_i \geq 0$, so we have $\sum_{i=1}^p \lambda_i a^i + \sum_{j=1}^q (-\mu_j) b^j = 0$. From (a) $\lambda = \mu = 0$, therefore $u = 0$.

(c) \Rightarrow (a) Let us suppose that (1.1) holds. We have

$$u = \sum_{i=1}^p \lambda_i a^i = - \sum_{j=1}^q \mu_j b^j \in (\text{cone } coA) \cap \text{lin}B = \{0\},$$

therefore $u = 0$ and, being B linearly independent, $\mu = 0$.

Finally, being A pli, we have $\lambda = 0$. □

2. Regularity conditions for a differentiable vector optimization problem

A classical paper on regularity conditions for a vector optimization problem, under differentiability assumptions of the functions involved, is the one by Maeda (see [24]). This author however, treats a problem with only inequality constraints, whereas in the present paper we are concerned with a problem with both inequality and equality constraints. Moreover, we shall consider some regularity conditions not examined in [24].

The analysis of the regularity conditions for a vector optimization problem has been performed also by other authors, under various differentiability assumptions (see, e.g., the papers by Cambini, Carosi and Martein [5], Giorgi, Jiménez and Novo [11, 12, 13], Jiménez and Novo [18, 19, 20], Ishizuka [16], Preda and Chiteşcu [28]).

Here we follow the approach of Giorgi, Jiménez and Novo (see [11]), adapted to the simpler case of differentiability of the functions involved. See also the paper by Jiménez and Novo [17] for a similar approach.

Let us consider the problem (VOP) with

$$S = \{x \in \mathbb{R}^n \mid g(x) \leq 0, h(x) = 0\}, \quad (2.1)$$

being $f: \mathbb{R}^n \rightarrow \mathbb{R}^p$, $g: \mathbb{R}^n \rightarrow \mathbb{R}^m$, $h: \mathbb{R}^n \rightarrow \mathbb{R}^r$, with component functions, respectively, f_i , $i \in I = \{1, 2, \dots, p\}$, g_j , $j \in J = \{1, 2, \dots, m\}$, and h_k , $k \in K = \{1, 2, \dots, r\}$.

Let $G = \{x \mid g(x) \leq 0\}$, $H = \{x \mid h(x) = 0\}$, so $S = G \cap H$.

If $x^0 \in S$, we denote by $J(x^0)$ the subset of J defined by

$$J(x^0) = \{j \in J \mid g_j(x^0) = 0\},$$

i.e. $J(x^0)$ is the set of the *active indices* at x^0 . We define then the following sets

$$\begin{aligned} F &= \{x \mid f(x) \leq f(x^0)\}; \\ S^0 &= F \cap S; \\ F^i &= \{x \mid f_j(x) \leq f_j(x^0), \forall j \in I - \{i\}, i \in I\}; \\ S^i &= F^i \cap S, i \in I. \end{aligned}$$

Obviously we have $F = \bigcap_{i \in I} F^i$ and $S^0 = \bigcap_{i \in I} S^i$.

We suppose that all functions are Fréchet differentiable at the point taken into consideration, i.e. $x^0 \in S$.

We consider also the following *linearizing cones* to S at x^0 :

$$C_0(S) = \{v \in \mathbb{R}^n \mid \nabla g_j(x^0)v < 0, \forall j \in J(x^0), \nabla h_k(x^0)v = 0, \forall k \in K\};$$

$$C(S) = \{v \in \mathbb{R}^n \mid \nabla g_j(x^0)v \leq 0, \forall j \in J(x^0), \nabla h_k(x^0)v = 0, \forall k \in K\}.$$

Likewise we define these cones with reference to the previous sets given by restrictions, such as G , H , F , F^i , S^i , S^0 . We remark that for the set F all functions f_i , $i \in I$, are active at x^0 and for the set F^i the same is true for the functions f_j , $j \in I - \{i\}$.

We put $K(H) = \ker(\nabla h_k(x^0))$.

It is obvious that $C_0(S) = C_0(G) \cap K(H)$ and $C(S) = C(G) \cap K(H)$. We have also $C_0(S^i) = C_0(F^i) \cap C_0(S)$.

Similar expressions hold for $C(S^i)$, $C_0(S_0)$, $C(S^0)$, etc.

Lemma 2.1. *We have the following inclusions*

$$\begin{aligned} Z(S^0, x^0) \subset A(S^0, x^0) \subset T(S^0, x^0) \subset \\ \subset \left\{ \begin{array}{l} \text{cl co}T(S^0, x^0) \\ \bigcap_{i \in I} T(S^i, x^0) \end{array} \right\} \subset \bigcap_{i \in I} \text{cl co}T(S^i, x^0) \subset C(S^0). \end{aligned}$$

If, moreover, $\nabla h_k(x^0)$, $k \in K$, are linearly independent, we have $C_0(S^0) \subset A(S^0, x^0)$.

Proof. The inclusions

$$Z(S^0, x^0) \subset A(S^0, x^0) \subset T(S^0, x^0) \subset C(S^0)$$

are proved in [2] as well as the second part of the lemma.

The inclusion

$$T(S^0, x^0) \subset \text{cl co}T(S, x^0)$$

is obvious, and, being $S^0 \subset S^i$, $\forall i \in I$, it follows,

$$T(S^0, x^0) \subset T(S^i, x^0) \subset \text{cl co}T(S^i, x^0) \subset C(S^i),$$

taking into account that $C(S^i)$ is closed and convex and contains $T(S^i, x^0)$. Therefore

$$T(S^0, x^0) \subset \bigcap_{i \in I} T(S^i, x^0) \subset \bigcap_{i \in I} \text{cl co}T(S^i, x^0) \subset \bigcap_{i \in I} C(S^i) = C(S^0).$$

Finally,

$$\text{cl co}T(S^0, x^0) \subset \text{cl co}T(S^i, x^0), \forall i \in I,$$

and therefore

$$\text{cl co}T(S^0, x^0) \subset \bigcap_{i \in I} \text{cl co}T(S^i, x^0). \quad \square$$

Definition 2.1. Let $\Gamma \subset \mathbb{R}^n$ and $F : \Gamma \rightarrow \mathbb{R}$, f differentiable on Γ . Then: f is said to be **pseudoconvex at x^0** if $\forall x \in \Gamma$,

$$\{f(x) < f(x^0)\} \Rightarrow \{\nabla f(x^0)(x - x^0) < 0\};$$

f is **pseudoconcave at x^0** if and only if $-f$ is pseudoconvex at x^0 ;

f is **pseudolinear at x^0** if and only if f is both pseudoconvex and pseudoconcave at x^0 .

Lemma 2.2.

(2.2a) *If every g_j , $j \in J(x^0)$, is pseudoconcave at x^0 , then*

$$Z(G, x^0) = C(G).$$

(2.2b) *If every h_k , $k \in K$, is pseudolinear at x^0 , then:*

(i) $Z(H, x^0) = T(H, x^0) = K(H);$

(ii) $H = x^0 + K(H);$

(iii) $C_0(S^0) \subset Z(S^0, x^0).$

Proof. For a) and the points (i) and (ii) of b), see [11, Lemma 3.1]. For the proof of (iii), take (i) into account and the inclusion $C_0(F \cap G) \subset Z(F \cap G, x^0)$, which always hold, if there are no equality constraints. Then we have

$$C_0(S^0) \subset C_0(F \cap G) \cap K(H) \subset Z(F \cap G, x^0) \cap Z(H, x^0) = Z(S^0, x^0). \quad \square$$

Lemma 2.3. *We have the following inclusions*

$$\begin{aligned} C_0(S^0) = C_0(F) \cap C_0(G) \cap K(H) &\subset \left\{ \begin{array}{l} C(F) \cap C_0(G) \cap K(H) \\ C_0(F) \cap C(G) \cap K(H) \end{array} \right\} \subset \\ &\subset C(F) \cap C(G) \cap K(H) = C(S^0). \end{aligned}$$

If some of the sets $C_0(S^0)$, $C(F) \cap C_0(G) \cap K(H)$ and $C_0(F) \cap C(G) \cap K(H)$ is nonempty, then its closure is $C(S^0)$.

Proof. See [11, Lemma 3.2] (the proof is an easy exercise). □

We now consider the following regularity conditions (r.c.) for (VOP).

Definition 2.2. *We say that for problem (VOP), with S given by (2.1), it holds:*

(1) *the Generalized Guignard r.c., (GGRC), if*

$$C(S^0) = \bigcap_{i \in I} \text{cl co} T(S^i, x^0);$$

(2) *the Abadie r.c., (ARC), if*

$$C(S^0) = T(S^0, x^0);$$

(3) *the Generalized Abadie r.c., (GARC) if*

$$C(S^0) = \bigcap_{i \in I} T(S^i, x^0);$$

(4) *the Global Cottle r.c., (GCRC), if*

$$C_0(F) \cap C_0(S) \neq \emptyset \text{ and the set } \{\nabla h_k(x^0) \mid k \in K\} \text{ is linearly independent};$$

- (5) *the Cottle r.c., (CRC), if,*
for each $i \in I$, $C_0(S^i) \neq \emptyset$ and the set $\{\nabla h_k(x^0) \mid k \in K\}$
is linearly independent;
- (6) *the Slater r.c., (SRC), if*
 f_i , $i \in I$, g_j , $j \in J(x^0)$, are pseudonvex at x^0 ;
 h_k , $k \in K$, are pseudolinear at x^0 ;
 $\{\nabla h_k(x^0) \mid k \in K\}$ is linearly independent and, for each
 $i = 1, 2, \dots, p$, there exists $x^i \in \mathbb{R}^n$ such that $f_j(x^i) < f_j(x^0)$,
 $\forall j \neq i$, $g_j(x^i) < 0$, $\forall j \in J(x^0)$, and $h_k(x^i) = 0$, $\forall k \in K$;
- (7) *the Linear r.c., (LRC), if*
 $f_j, g_j, h_k, i \in I, j \in J(x^0)$ and $k \in K$, are all linear (affine);
- (8) *the Linear objectives r.c., (LORC), if*
 $f_i, i \in I, h_k, k \in K$, are linear and $C(F) \cap C_0(G) \neq \emptyset$;
- (9) *the Mangasarian-Fromovitz r.c., further specified under*
the following sets of assumptions:
- (9a) *with linearly independent objective functions, (LIOMFRC), if*
 $C(F) \cap C_0(S) \neq \emptyset$ and $\{\nabla f_i(x^0) \mid i \in I\} \cup \{\nabla h_k(x^0) \mid k \in K\}$
is linearly independent;
- (9b) *with positively linearly independent objective functions*
(PLIOMFRC), if $C(F) \cap C_0(S) \neq \emptyset$ and the set
 $\left(\{\nabla f_i(x^0) \mid i \in I\}, \{\nabla h_k(x^0) \mid k \in K\}\right)$ *is pli;*
- (9c) *with positively linearly independent constraints, (PLICMFRC),*
if $C_0(F) \cap C(S) \neq \emptyset$ and the set
 $\left(\{\nabla g_j(x^0) \mid j \in J(x^0)\}, \{\nabla h_k(x^0) \mid k \in K\}\right)$ *is pli;*
- (9d) *the Maeda-Mangasarian-Fromovitz r.c., (MMFRC), if*
 $K(F) \cap C_0(S) \neq \emptyset$ and $\{\nabla f_i(x^0) \mid i \in I\} \cup \{\nabla h_k(x^0) \mid k \in K\}$
is linearly independent;
- (10) *the Zangwill r.c., (ZRC), if*

$$\text{cl}Z(S^0, x^0) = C(S^0);$$
- (11) *the Kuhn-Tucker r.c., (KTRC), if*

$$A(S^0, x^0) = C(S^0);$$
- (12) *the Reverse r.c., (RRC), if*
 $f_i, i \in I, g_j, j \in J(x^0)$, are pseudoconcave at x^0
and every $h_k, k \in K$, is pseudolinear at x^0 .

Theorem 2.1. *The following implications are verified:*

- (1) *Linear* \Rightarrow *Reverse* \Rightarrow *Zangwill* \Rightarrow *Kuhn-Tucker*;
- (2) *Linear objectives* \Rightarrow *Zangwill*;
- (3) *Slater* \Rightarrow *Cottle*;
- (4) (MMFRC) \Rightarrow (LIOMFRC) \Rightarrow (PLIOMFRC);
- (5) (PLICMFRC) \Leftrightarrow *Global Cottle* \Leftrightarrow (PLIOMFRC);
- (6) *Global Cottle* \Rightarrow *Cottle*;
- (7) *Cottle* \Rightarrow *Generalized Abadie*;
- (8) *Global Cottle* \Rightarrow *Kuhn-Tucker* \Rightarrow *Abadie* \Rightarrow
 \Rightarrow *Generalized Abadie* \Rightarrow *Generalized Guignard*.

Proof. The proof follows the same lines of the proof of [11, Theorem 3.1] with the obvious modifications for the case under examination (differentiability assumptions). In particular, Lemmas 1.2, 2.1, 2.2 and 2.3 must be taken into account. However, for the reader's convenience, we give the complete proof.

(1a) Linear \Rightarrow Reverse

It is trivial.

(1b) Reverse \Rightarrow Zangwill

Thanks to Lemma 2.2 we have $Z(F \cap G, x^0) = C(F \cap G)$ and, always for the same Lemma, we have $Z(H, x^0) = K(H)$. Therefore, $Z(S^0, x^0) = Z(F \cap G \cap H, x^0) = C(S^0)$.

(1c) Zangwill \Rightarrow Kuhn-Tucker

It follows from Lemma 2.1.

(2) Linear objectives \Rightarrow Zangwill

Thanks to Lemma 2.2 we have $Z(F, x^0) = C(F)$ and $Z(H, x^0) = K(H)$. The inclusions $C_0(G) \subset Z(G, x^0) \subset C(G)$ are always true; therefore we get

$$\begin{aligned} C(F) \cap C_0(G) \cap K(H) &\subset Z(F; x^0) \cap Z(G, x^0) \cap Z(H, x^0) = \\ &= Z(S^0, x^0) \subset C(F) \cap C(G) \cap K(H) = C(S^0). \end{aligned}$$

Applying Lemma 2.3 we conclude that $cl Z(S^0, x^0) = C(S^0)$. We point out that in Linear objectives r.c. we could substitute “ f_i, h_k linear” with “ f_i pseudoconcave and h_k pseudolinear”.

(3) Slater \Rightarrow Cottle

It is sufficient to apply to each scalar problem

$$(P_j) \quad \min \{ f_j(x) \mid x \in S^j \}$$

the implication Slater \Rightarrow Cottle, proved in [2, Theorem 6.2.3, *ii*]

(4a) (MMFRC) \Rightarrow (LIOMFRC)

It is sufficient to note that $K(F) \cap C_0(S) \subset C(F) \cap C_0(S)$.

(4b) (LIOMFRC) \Rightarrow (PLIOMFRC)

It is trivial, as if $A \cup B$ is linearly independent, then (A, B) is positively linearly independent.

(5a) (PLICMFRC) \Rightarrow Global Cottle

If $(\nabla g_j(x^0), j \in J(x^0)), (\nabla h_k(x^0), k \in K)$ is positively linearly independent, by Lemma 1.2, this is equivalent to $C_0(G) \cap K(H) \neq \emptyset$ and $(\nabla h_k(x^0), k \in K)$ is linearly independent. Therefore

$$cl C_0(G) \cap K(H) = C(G) \cap K(H). \quad (2.2)$$

Let $u \in C_0(F)$ and $u \in C(G) \cap K(H)$ (u exists by assumptions). Being $C_0(F)$ open, there exists a neighborhood $B(u)$ of u such that $B(u) \subset C_0(F)$, and, thanks to (2.2), $B(u) \cap (C_0(G) \cap K(H)) \neq \emptyset$. Therefore, $C_0(F) \cap C_0(G) \cap K(H) \neq \emptyset$, so Global Cottle r.c. is verified.

(5b) Global Cottle \Rightarrow (PLICMFRC)

By assumptions we have

$$C_0(F) \cap C_0(G) \cap K(H) \neq \emptyset. \quad (2.3)$$

Therefore, $C_0(G) \cap K(H) \neq \emptyset$ and, being $(\nabla h_k(x^0), k \in K)$ linearly independent, by Lemma 1.2,

$$\left((\nabla g_j(x^0), j \in J(x^0)), (\nabla h_k(x^0), k \in K) \right)$$

is positively linearly independent. From (2.3) we get

$$C_0(F) \cap C(G) \cap K(H) \neq \emptyset,$$

that is (PLICMFRC) is verified.

(5c) Global Cottle \Leftrightarrow (PLIOMFRC)

It is sufficient to note that the role played f_i and g_j in PLIOMFRC and in PLICMFRC is symmetric, therefore also the said equivalence is true.

(6) Global Cottle \Rightarrow Cottle

It is trivial.

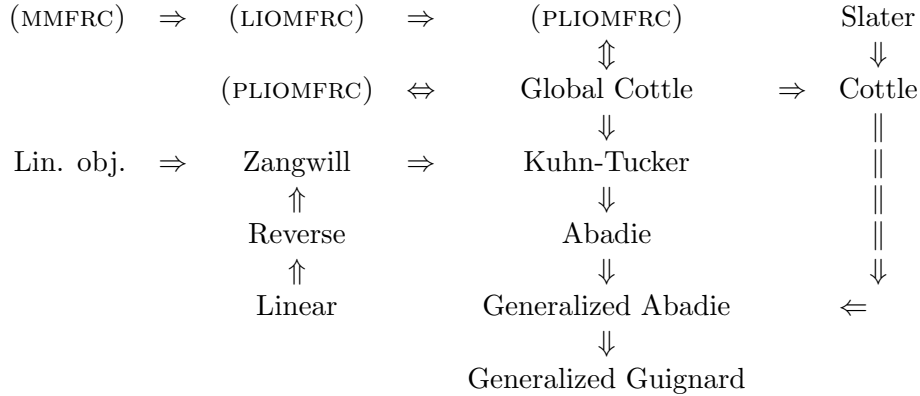
(7) Cottle \Rightarrow Generalized Abadie

For each $i = 1, 2, \dots, p$, the condition $C_0(S^i) \neq \emptyset$ implies $T(S^i, x^0) = C(S^i)$. Therefore, $\bigcap_{i \in I} T(S^i, x^0) = \bigcap_{i \in I} C(S^i) = C(S^0)$.

$$(8) \quad \text{Global Cottle} \Rightarrow \text{Kuhn-Tucker} \Rightarrow \text{Abadie} \Rightarrow \\ \Rightarrow \text{Generalized Abadie} \Rightarrow \text{Generalized Guignard}$$

The implications follow trivially from Lemma 2.1 □

The following scheme shows the various relationships pointed out in the previous theorem.



Remark 2.1. It is known that if x^0 is a local efficient point for (VOP), then

$$T(S, x^0) \cap C_0(F) = \emptyset.$$

If, moreover, the set $\{\nabla h_k(x^0) \mid k \in K\}$ is l.i., from Lemma 1.1 we have $C_0(S) \cap C_0(F) = \emptyset$, and consequently at x^0 the Global Cottle r.c. cannot occur and obviously cannot occur any r.c. which implies the Global Cottle r.c. . For example the (MMFRC) and the (LIOMFRC).

For this reason the Global Cottle r.c. cannot be considered a true regularity condition, i.e. a condition assuring the existence of not all zero multipliers in the usual Kuhn-Tucker condition for a local efficient point x^0 of (VOP), with S given by (2.1).

Remark 2.2. We have said that, at least for vector optimization problems, it is better to use the term “constraint qualifications” only for those conditions involving only the constraints. With reference to (VOP), where S is given by (2.1) and all the functions are differentiable on a common open set, a constraint qualification condition has been considered, e.g. by Lin (see [22]), Marusciac (see [26]), Singh (see [31, 32]), Wang ([33]). These authors consider the following *Abadie constraint qualification* (Abadie c.q.):

$$C(S) = T(S, x^0). \tag{2.4}$$

Contrary to the scalar case, where the Abadie c.q. assures the *positivity* of the multiplier associated to $\nabla f(x^0)$, in the Fritz John necessary optimality conditions, for (VOP) this *does not happen*.

Obviously, this holds true also for all other constraint qualifications which imply the Abadie c.q.

Consider, e.g., the following example.

Example 2.1. Consider the problem

$$\min f(x_1, x_2) = \left((x_1 - x_2), -(x_1 - x_2)^3 \right)$$

subject to

$$S = \{(x_1 - x_2) \in \mathbb{R}^2 \mid x_1 \leq 0\}.$$

Every feasible point is an efficient solution. In particular $x^0 = (0, 0) \in S$ is efficient, it holds $C(S) = T(S, x^0)$, but we have

$$\lambda_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 0 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

for $\lambda_1 = 0$, $\lambda_2 > 0$, $\mu_1 = 0$, so $\lambda = (0, \lambda_2) \notin \text{int}\mathbb{R}_+^2$.

Indeed, the authors quoted above obtain the following result.

If x^0 is a local efficient point for (VOP), with S given by (2.1), or also a local weak efficient point, and the Abadie c.q. (2.4) is satisfied, then there exist $\lambda \in \mathbb{R}^p$, $\lambda \geq 0$, $\lambda \neq 0$, $\mu \in \mathbb{R}^m$, $\mu \geq 0$, and $\nu \in \mathbb{R}^r$ such that

$$\begin{aligned} \sum_{i \in I} \lambda_i \nabla f_i(x^0) + \sum_{j \in J} \mu_j \nabla g_j(x^0) + \sum_{k \in K} \nu_k \nabla h_k(x^0) &= 0 \\ \sum_{j \in J} \mu_j g_j(x^0) &= 0. \end{aligned}$$

In other words, it is possible to obtain, under the Abadie c.q., only a necessary optimality condition “halfway” between a Fritz John type condition and a Kuhn-Tucker type condition.

We have to note that the above “weak” Kuhn-Tucker conditions are useful, together with appropriate (generalized) convexity assumptions, to obtain sufficient conditions for the weak efficiency (see, e.g., the paper by Singh [31]). However, the “strong” Kuhn-Tucker conditions, i.e. where $\lambda > 0$ (i.e. $\lambda \in \text{int}\mathbb{R}_+^p$) are useful to obtain sufficient conditions for the efficiency and, above all, are connected with the notion of *proper efficiency*, a notion which goes back to the basic paper of Kuhn and Tucker [21]. See also the paper by Geoffrion [8] and the next Remark 2.4.

We remark that the original proofs of Lin (see [22]), Marusciac (see [26]) and Singh (see [31]) work only if x^0 is a global solution or a global weak solution.

An error in the proof of Lin has been corrected by Marusciac in [26] and in a more precise way by Wang ([33]). See also the Errata Corrige of

Singh [32], where, however, no justifying reason is given. For more general treatments see, e.g., [12] and [18]. It is well known that the Mangasarian-Fromovitz c.q., (MFCQ), expressed by:

$$C_0(S) \neq \emptyset \text{ and } \{\nabla f_i(x^0), i \in I\} \text{ is linearly independent}$$

is a sufficient condition for the Abadie c.q. (2.4).

A point $x^0 \in S$ satisfying the previous necessary optimality conditions (i.e. with $\lambda \geq 0, \lambda \neq 0$.) is called *weak regular* in the sense of Bigi and Pappalardo (see [4]). See the next Section 3.

Remark 2.3. Those regularity conditions which involve generalized convexity assumptions could be further generalized, as pointed out in [5], by means of classes of vector generalized convex functions, broader than the corresponding classes of functions, where the generalized convexity is required for the various components of the vector functions involved. More precisely, we consider the following definitions.

Definition 2.3. Let $F : \mathbb{R}^n \rightarrow \mathbb{R}^p$ defined on an open convex set $X \subseteq \mathbb{R}^n$ and differentiable at $x^0 \in X$.

(i) F is said to be *int \mathbb{R}_-^p -pseudoconvex* at x^0 if the following implication holds

$$x \in X, F(x) \in F(x^0) + \text{int}\mathbb{R}_-^p \Rightarrow \nabla F(x^0)(x - x^0) \in \text{int}\mathbb{R}_-^p$$

(ii) F is said to be *reverse pseudoconcave* at x^0 if the following implication holds

$$x \in X, \nabla F(x^0)(x - x^0) \in \mathbb{R}_-^p \Rightarrow F(x) \in F(x^0) + \mathbb{R}_-^p.$$

When $p = 1$, (i) and (ii) in Definition 2.3 collapse to the usual definition of pseudoconvexity at x^0 and pseudoncavity at x^0 , respectively.

It can be proved that the class of componentwise pseudoconvex (pseudoncave) vector-valued functions is *strictly* contained in the class of *int \mathbb{R}_-^p -pseudoconvex* (reverse pseudoncave) functions.

Finally, we obtain, under the Generalized Guignard r.c., the Kuhn-Tucker type necessary conditions for (VOP), with S given by (2.1).

Theorem 2.2. Let x^0 be a local efficient point for (VOP), with S given by (2.1). Let the Generalized Guignard r.c. be verified at x^0 .

Then the system

$$\begin{cases} \nabla f_j(x^0)v \leq 0, & \forall j \in I, \\ \nabla f_i(x^0)v < 0, & \text{for at least one } i \in I, \\ \nabla g_j(x^0)v \leq 0, & \forall j \in J(x^0), \\ \nabla h_k(x^0)v = 0, & \forall k \in K, \end{cases} \quad (2.5)$$

has no solution $v \in \mathbb{R}^n$.

Proof. Let us suppose that there exists $v \in \mathbb{R}^n$ which verifies the said system.

If x^0 is a local efficient point for (VOP), with S given by (2.1), then x^0 is a local solution of each scalar problem

$$(P_j) \quad \min \{f_j(x) \mid x \in S^j\}.$$

Therefore, we have $\nabla f_i(x^0)u \geq 0$, $\forall u \in T(S^i, x^0)$ and also, for the linearity and continuity of the differential,

$$\nabla f_i(x^0)u \geq 0, \quad \forall u \in cl\,coT(S^i, x^0). \quad (2.6)$$

We have that $v \in C(S^0)$ and for the Generalized Guignard r.c.,

$$v \in \bigcap_{j \in I} cl\,coT(S^j, x^0).$$

Therefore, in particular, $v \in cl\,coT(S^i, x^0)$ and, taking relation (2.6) into account, $\nabla f_i(x^0)v \geq 0$, in contradiction with the first two conditions of system (2.5). \square

Theorem 2.3. *Let the same assumptions of Theorem 2.2 hold. Then there exist $\lambda > 0$, $\mu \geq 0$ and $\nu \in \mathbb{R}^r$ such that*

$$\sum_{i \in I} \lambda_i \nabla f_i(x^0) + \sum_{j \in J(x^0)} \mu_j \nabla g_j(x^0) + \sum_{k \in K} \nu_k \nabla h_k(x^0) = 0.$$

Proof. Apply to system (2.5) the Tucker theorem of the alternative (see, e.g., [25]). \square

Remark 2.4. An efficient solution of (VOP), with S given by (2.1), is said to be a *properly efficient solution*, in the sense of Kuhn and Tucker, if the system (2.5) has no solution in $v \in \mathbb{R}^n$. This concept was introduced by Kuhn and Tucker (see [21]), in order to avoid some undesirable situations. Thus, any “true” regularity condition which holds at an efficient point x^0 is also a condition assuring that x^0 is a properly efficient point, in the sense of Kuhn and Tucker. For the various notions of proper efficiency proposed in the literature and for their relationships, see the survey paper [14].

3. The approach of Bigi and Pappalardo

We recall the following Fritz John necessary optimality conditions for (VOP), with S given by (2.1).

Theorem 3.1. *Let us consider (VOP), with S given by (2.1) and where the functions f_i , $i \in I = \{1, 2, \dots, p\}$, g_j , $j \in J = \{1, 2, \dots, m\}$ and h_k , $k \in K = \{1, 2, \dots, r\}$ are continuously differentiable in a neighbourhood of $x^0 \in S$. A necessary condition for x^0 to be a local weak efficient point is that there exist vectors $\lambda \in \mathbb{R}_+^p$, $\mu \in \mathbb{R}_+^m$ and $\nu \in \mathbb{R}^r$ such that*

$$\sum_{i \in I} \lambda_i \nabla f_i(x^0) + \sum_{j \in J} \mu_j \nabla g_j(x^0) + \sum_{k \in K} \nu_k \nabla h_k(x^0) = 0; \quad (3.1)$$

$$\mu_j g_j(x^0) = 0, \quad j \in J; \quad (3.2)$$

$$(\lambda, \mu, \nu) \neq (0, 0, 0). \quad (3.3)$$

See, e.g., [7], [27], [30] and, for further insights and generalizations, [12, 18, 19, 20].

Let $M(x^0)$ denote the set of Fritz John multipliers (λ, μ, ν) satisfying (3.1)-(3.3), associated to x^0 . Let us introduce the following notations

$$a^i = \nabla f_i(x^0), \quad i \in I; \quad b^j = \nabla g_j(x^0), \quad j \in J; \quad c^k = \nabla h_k(x^0), \quad k \in K;$$

$$A = \{a^i \mid i \in I\}; \quad B = \{b^j \mid j \in J\}; \quad C = \{c^k \mid k \in K\}.$$

The subsets of vectors of these three sets, associated respectively to the subsets of indices I_0 of I , J_0 of J and K_0 of K , are denoted $A_0(\subset A)$; $B_0(\subset B)$; $C_0(\subset C)$.

We note that the following equivalences hold:

$$\begin{aligned} & \left\{ M(x^0) \neq \emptyset \right\} \Leftrightarrow \left\{ (A \cup B, C) \text{ is pld} \right\} \Leftrightarrow \\ & \Leftrightarrow \left\{ \exists (\lambda, \mu, \nu) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^r \text{ such that} \right. \\ & \quad \left. \sum_{i \in I} \lambda_i a^i + \sum_{j \in J} \mu_j b^j + \sum_{k \in K} \nu_k c^k = 0, \right. \end{aligned} \quad (3.4)$$

$$\left. (\lambda, \mu) \geq 0, \quad (\lambda, \mu, \nu) \neq (0, 0, 0) \right\}. \quad (3.5)$$

Following [4] and [23], we state the following notions of regularity for (VOP), with S given by (2.1).

Definition 3.1. *Given $x \in S$ such that $M(x) \neq \emptyset$, we say that:*

- (a) x is **weak-regular** if there exists $(\lambda, \mu, \nu) \in M(x)$ with $\lambda \neq 0$;

- (b) x is **totally weak-regular** if, for all $(\lambda, \mu, \nu) \in M(x)$, there exists $i \in I$ such that $\lambda_i \neq 0$;
- (c) x is **regular** if there exists $(\lambda, \mu, \nu) \in M(x)$, with $\lambda_i > 0$ for all $i \in I$;
- (d) x is **totally regular** if, for all $(\lambda, \mu, \nu) \in M(x)$, one has $\lambda_i > 0$ for all $i \in I$.

We note that (d) \Rightarrow (c), (d) \Rightarrow (b), and (b) and (c) imply (a).

The notions (b) and (c) are not related, as shown by examples in [4].

We now prove, in a more direct and synthetic way, some results of Maciel, Santos and Sottosanto (see [23]), adding some new results and remarks. First of all, we note that, without loss of generality, we can always suppose that all the inequality constraints are active at x^0 , i.e. $J = J(x^0)$.

Theorem 3.2. *The following equivalence holds:*

$$\begin{aligned} \{M(x^0) \neq \emptyset \text{ and } x^0 \text{ is totally weak regular}\} &\Leftrightarrow \\ &\Leftrightarrow \{(A \cup B, C) \text{ is pld and } (B, C) \text{ is pli}\}. \end{aligned}$$

Proof. (\Rightarrow)

The first part is clear. Assume that (B, C) is pli. Then there exists $(\mu, \nu) \in \mathbb{R}^m \times \mathbb{R}^r$ such that $\mu \geq 0$, $(\mu, \nu) \neq 0$ and

$$\sum_{j \in J} \mu_j b^j + \sum_{k \in K} \nu_k c^k = 0. \quad (3.6)$$

Choosing $\lambda = 0$, we have that $(0, \mu, \nu) \in M(x^0)$, which contradicts that x^0 is totally weak-regular.

(\Leftarrow)

It is clear that $M(x^0) \neq \emptyset$, since $(A \cup B, C)$ is pld. Choose $(\lambda, \mu, \nu) \in M(x^0)$, then (3.4)-(3.5) hold. If we assume that $\lambda = 0$, then (3.6) holds with $(\mu, \nu) \neq 0$, but this contradicts the assumption that (B, C) is pli. \square

Remark 3.1. We note that if we *assume* that $M(x^0) \neq \emptyset$, then we can easily prove that (MFCQ) is both necessary and sufficient for x^0 to be totally weak regular (Maciel, Santos and Sottosanto are not clear on this point, in Theorem 3.2 and Remark 3.2 of [23]).

Theorem 3.3. *Assume that there exist $B_0 \subset B$ and $C_0 \subset C$ such that (B_0, C_0) is pli and $(A \cup B_0, C_0)$ is pld. Then $M(x^0) \neq \emptyset$ and x^0 is weak-regular.*

Proof. Similar to the proof of the previous theorem. \square

Remark 3.2. We point out that the condition $(A \cup B_0, C_0)$ is pld is weaker than $(A_0 \cup B_0, C_0)$ is pld for some $A_0 \subset A$, i.e. the condition $(A_0 \cup B_0, C_0)$ is pld for some $A_0 \subset A$ implies that $(A \cup B_0, C_0)$ is pld, but the converse is false, if $A_0 \neq A$. Consider, e.g.,

$$A = \{(-1, 0, 0), (0, -1, 0)\}, B = \{(1, 1, 0)\} \text{ and } C = \{(0, 0, 1)\}.$$

Theorem 3.4. *If x^0 is weak-regular, then A is pld or there exist $B_0 \subset B$ and $C_0 \subset C$ such that (B_0, C_0) is pli and $(A \cup B_0, C_0)$ is pld.*

Proof. The result is a consequence of Carathéodory's lemma (see, e.g., [3]) and can be proved also following a scheme similar to the one used by Qi and Wei (see [29]) in their Proposition 2.2. \square

We note that of course the converse of the first part of the theorem is true because if A is pld, then x^0 is weak-regular.

Definition 3.2. *A vector $d \in \mathbb{R}^n$ is said to be a **positive linear combination** (plc) of (B, C) if there exists (μ, ν) with $\mu \geq 0$ such that*

$$d = \sum_{j \in J} \mu_j b^j + \sum_{k \in K} \nu_k c^k.$$

Definition 3.3. *The triplet (A, B, C) satisfies the **strict positive linear dependence** (SPLD) if for every $s \in I$, $-a^s$ is a plc of $(A_s \cup B, C)$, where $A_s = A - \{a^s\}$. Equivalently: for every $s \in I$ there exists $(\alpha^s, \beta^s, \gamma^s) \in \mathbb{R}^{\text{card}I - \{s\}} \times \mathbb{R}^m \times \mathbb{R}^r$ such that $(\alpha^s, \beta^s) \geq 0$ and*

$$a^s + \sum_{i \neq s} \alpha_i^s a^i + \sum_{j \in J} \beta_j^s b^j + \sum_{k \in K} \gamma_k^s c^k = 0. \quad (3.7)$$

Theorem 3.5. *The following equivalence holds:*

$$\{(\text{SPLD}) \text{ holds}\} \Leftrightarrow \{M(x^0) \neq \emptyset \text{ and } x^0 \text{ is regular}\}.$$

Proof.

(\implies)

It is sufficient to sum up in $s \in I$ the equations (3.7) to obtain $(\lambda, \mu, \nu) \in M(x^0)$ with $\lambda > 0$.

(\impliedby)

It is obvious, since if (3.2) holds with $\lambda_s > 0, \forall s \in I$, we can find $-a^s$ for each $s \in I$, satisfying Definition 3.3. \square

We now recall the *Cottle regularity condition* (Cottle r.c.): see Section 2 of the present paper.

Definition 3.4. We say that Cottle r.c. holds at $x^0 \in S$ if C is linearly independent and for each $s \in I$,

$$A_s^- \cap B^- \cap \ker(C) \neq \emptyset, \text{ where } A_s = A - \{a^s\}.$$

The condition above is called in [23] “Mangasarian-Fromovitz regularity condition” (MFRC), but we prefer to use the term “Cottle r.c.”, as the Maeda-Mangasarian-Fromovitz r.c. is another type of regularity condition (see Section 2 of the present paper).

According to Lemma 1.2 we have the following result.

Theorem 3.6. The following equivalence holds:

$$\{(Cottle \text{ r.c.}) \text{ holds}\} \Leftrightarrow \{\text{For every } s \in I \text{ the set } (A_s \cup B, C) \text{ is pli}\}.$$

The following result connects Cottle r.c. and totally regular points.

Theorem 3.7. Let $M(x^0) \neq \emptyset$. Then:

$$\{(Cottle \text{ r.c.}) \text{ holds}\} \Leftrightarrow \{x^0 \text{ is totally regular}\}.$$

Proof.

(\implies)

Take $(\lambda, \mu, \nu) \in M(x^0)$ then (3.2)-(3.3) hold. Assume that $\lambda_s = 0$ for some $s \in I$. This implies that $(A_s \cup B, C)$ is pld, which is a contradiction to the fact that $(A_s \cup B, C)$ is pli, which holds, thanks to Theorem 3.6.

(\impliedby)

If for some $s \in I$, $(A_s \cup B, C)$ is pld, then (3.2)-(3.3) are satisfied with $\lambda_s = 0$, but this contradicts the assumption that x^0 is totally regular. \square

Definition 3.5. We say that (VOP), with S given by (2.1), satisfies the positive linear independence regularity condition (PLIRC) at $x^0 \in S$ if:

- (i) (B, C) is pli;
- (ii) for every $s \in I$ there does not exist $(\alpha, \beta, \gamma) \in \mathbb{R}^{\text{card}I - \{s\}} \times \mathbb{R}^m \times \mathbb{R}^r$ such that $(\alpha, \beta) \geq 0$, $\alpha \neq 0$, and

$$\sum_{i \neq s} \alpha_i a^i + \sum_{j \in J} \beta_j b^j + \sum_{k \in K} \gamma_k c^k = 0.$$

Remark 3.3.

- (a) According to Lemma 1.2, the condition (i) of the previous Definition is equivalent to: C is linearly independent and $B^- \cap \ker(C) \neq \emptyset$.

- (b) By Motzkin alternative theorem (see, e.g., the papers [10], [25]), the condition (ii) of the previous Definition is equivalent to:
for each $s \in I$ it holds $A_s^- \cap B^* \cap \ker(C) \neq \emptyset$.

Theorem 3.8. *The following equivalence holds:*

$$\{(\text{PLIRC}) \text{ holds}\} \Leftrightarrow \{(\text{Cottle r.c.}) \text{ holds}\}.$$

Proof.

(\implies)

Let us prove that for each $s \in I$ we have $A_s^- \cap B^- \cap \ker(C) \neq \emptyset$. Indeed, by Remark 3.3 (a), there exists $u \in B^- \cap \ker(C)$ and by Remark 3.3 (b), there exists $v \in A_s \cap B^* \cap \ker(C)$. For each $\alpha \in (0, 1)$ let $\omega_\alpha = \alpha u + (1 - \alpha)v$. It is easy to check that $\omega_\alpha \in B^- \cap \ker(C)$, $\forall \alpha \in (0, 1)$. As $\lim_{\alpha \rightarrow 0^+} \omega_\alpha = v \in A_s^-$ and A_s^- is an open set, it follows that $\omega_\alpha \in A_s^- \cap B^- \cap \ker(C)$, for $\alpha > 0$ small enough.

Now, as C is linearly independent (see Remark 3.3 (a)), we have that (Cottle r.c.) holds.

(\impliedby)

The proof is obtained from the definition of (Cottle r.c.), taking Remark 3.3 in to account and observing that

$$\{A_s^- \cap B^- \cap \ker(C) \neq \emptyset\} \Rightarrow \{B^- \cap \ker(C) \neq \emptyset\}. \quad \square$$

An immediate consequence of Theorem 3.7 is the following result.

Theorem 3.9. *Let $M(x^*) \neq \emptyset$. Then:*

$$\{(\text{PLIRC}) \text{ holds}\} \Leftrightarrow \{x^* \text{ is totally regular}\}.$$

Both Theorems 3.7 and 3.8 are present in [23], but our proofs are more direct and stringent.

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