Annals of the University of Bucharest (mathematical series)
2 (LX) (2011), 139–155

Existence of periodic solutions for nonautonomous second order differential systems with (p_1, p_2)–Laplacian using the duality mappings

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Communicated by George Dinca

Abstract - Some existence theorems are obtained for periodic solutions of nonautonomous second-order differential systems with (p_1, p_2)–Laplacian.

Key words and phrases : (p_1, p_2)–Laplacian, periodic solutions, duality mapping, demicontinuous operator.

Mathematics Subject Classification (2000) **:** 34C25, 49J15.

1. Introduction

In the last years many authors starting with Mawhin and Willem (see [6]) proved the existence of solutions for problem

$$\begin{aligned} &\ddot{u}(t) = \nabla F(t, u(t)) \text{ a.e. } t \in [0, T], \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{1.1}$$

under suitable conditions on the potential F (see [4], [18]-[28], [30]). Also in a series of papers (see [7]-[9]) we have generalized some of these results for the case when the potential F is just locally Lipschitz in the second variable x not continuously differentiable. Very recent (see [10] and [14]) we have considered the second order Hamiltonian inclusions systems with p–Laplacian.

In [1] the authors described a new method for proving the existence of periodic solutions for the following system

$$\begin{aligned} &\tfrac{d}{dt}\big(|\dot{u}(t)|^{p-2}\dot{u}(t)\big) = |u(t)|^{p-2}u(t) + F(t, u(t)), \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{1.2}$$

where p is a real number so that $1 < p < \infty$, $0 < T < \infty$ is a constant and $F : [0, T] \times \mathbb{R}^N \to \mathbb{R}^N$, is a measurable function in t for each $x \in \mathbb{R}^N$ and continuous in x for a.e. $t \in [0, T]$.

The aim of this paper is to show how the results obtained in [1] can be generalized. More exactly our results represent the extensions to second-order differential systems with (p_1, p_2)–Laplacian. This type of systems have been also considered in [3], [12], [13], [15]-[17].

Consider the second order system

$$\begin{cases} \frac{d}{dt}\big(|\dot{u}_1(t)|^{p_1-2}\dot{u}_1(t)\big) = |u_1(t)|^{p_1-2}u_1(t) + F_1(t, u_1(t), u_2(t)), \\ \frac{d}{dt}\big(|\dot{u}_2(t)|^{p_2-2}\dot{u}_2(t)\big) = |u_2(t)|^{p_2-2}u_2(t) + F_2(t, u_1(t), u_2(t)) \text{ a.e. } t \in [0, T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\ u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0, \end{cases} \tag{1.3}$$

where $1 < p_1, p_2 < \infty$, $T > 0$, and $F_i : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $i = 1, 2$ satisfy the following assumption (A):

- F_i is measurable in t for each $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$;
- F_i is continuous in (x_1, x_2) for a.e. $t \in [0, T]$.

2. Equivalent formulation of the problem (1.3)

Let $X = W_T^{1,p_1} \times W_T^{1,p_2}$, $X^* = (W_T^{1,p_1})^* \times (W_T^{1,p_2})^*$, $q_1 = \frac{p_1}{p_1-1}$, $q_2 = \frac{p_2}{p_2-1}$ and $J_{p_1-1,p_2-1} : X \to X^*$ defined as follows:

$$\langle J_{p_1-1,p_2-1}(u_1, u_2), (v_1, v_2)\rangle_{X^*,X} = \int_0^T \langle |u_1(t)|^{p_1-2}u_1(t), v_1(t)\rangle dt + \tag{2.1}$$

$$+ \int_0^T \langle |\dot{u}_1(t)|^{p_1-2}\dot{u}_1(t), \dot{v}_1(t)\rangle dt + \int_0^T \langle |u_2(t)|^{p_2-2}u_2(t), v_2(t)\rangle dt +$$

$$+ \int_0^T \langle |\dot{u}_2(t)|^{p_2-2}\dot{u}_2(t), \dot{v}_2(t)\rangle dt$$

for all $(v_1, v_2) \in X$.

In fact we have:

$$J_{p_1-1,p_2-1}(u_1, u_2) = (J_{p_1-1}u_1, J_{p_2-1}u_2). \tag{2.2}$$

From (2.2), following the estimates obtained in Section 2 of [1], we get:

$$\big|\langle J_{p_1-1,p_2-1}(u_1, u_2), (v_1, v_2)\rangle_{X^*,X}\big| \le$$

$$\le \big|\langle J_{p_1-1}u_1, v_1\rangle_{(W_T^{1,p_1})^*, W_T^{1,p_1}}\big| + \big|\langle J_{p_2-1}u_2, v_2\rangle_{(W_T^{1,p_2})^*, W_T^{1,p_2}}\big| \le$$

$$\le \frac{\|u_1\|^{p_1}_{W_T^{1,p_1}}}{q_1} + \frac{\|v_1\|^{p_1}_{W_T^{1,p_1}}}{p_1} + \frac{\|u_2\|^{p_2}_{W_T^{1,p_2}}}{q_2} + \frac{\|v_2\|^{p_2}_{W_T^{1,p_2}}}{p_2},$$

and

$$\langle J_{p_1-1,p_2-1}(u_1, u_2), (u_1, u_2)\rangle_{X^*,X} = \|u_1\|^{p_1}_{W_T^{1,p_1}} + \|u_2\|^{p_2}_{W_T^{1,p_2}}. \tag{2.3}$$

We known that (see Section 2 of [1]):

$$\|J_{p_1-1}u_1\|_{(W_T^{1,p_1})^*} = \|u_1\|^{p_1-1}_{W_T^{1,p_1}}, \quad \|J_{p_2-1}u_2\|_{(W_T^{1,p_2})^*} = \|u_2\|^{p_2-1}_{W_T^{1,p_2}}.$$

From (2.2) we have

$$\|J_{p_1-1,p_2-1}(u_1,u_2)\|_{X^*} = \|J_{p_1-1}u_1\|_{(W_T^{1,p_1})^*} + \|J_{p_2-1}u_2\|_{(W_T^{1,p_2})^*} = \tag{2.4}$$

$$= \|u_1\|^{p_1-1}_{W_T^{1,p_1}} + \|u_2\|^{p_2-1}_{W_T^{1,p_2}}.$$

Suppose that for any $(u_1,u_2) \in L^{p_1}(0,T;\mathbb{R}^N) \times L^{p_2}(0,T;\mathbb{R}^N)$ the map

$$t \in [0,T] \mapsto \big(F_1(t,u_1(t),u_2(t)), F_2(t,u_1(t),u_2(t))\big)$$

belongs to $L^{q_1}(0,T;\mathbb{R}^N) \times L^{q_2}(0,T;\mathbb{R}^N)$. We may consider the operator

$$A : L^{p_1}(0,T;\mathbb{R}^N) \times L^{p_2}(0,T;\mathbb{R}^N) \to L^{q_1}(0,T;\mathbb{R}^N) \times L^{q_2}(0,T;\mathbb{R}^N)$$

defined by

$$\big(A(u_1,u_2)\big)(t) = \big(F_1(t,u_1(t),u_2(t)), F_2(t,u_1(t),u_2(t))\big) \tag{2.5}$$

a.e. on $[0,T]$, and for all $(u_1,u_2) \in L^{p_1}(0,T;\mathbb{R}^N) \times L^{p_2}(0,T;\mathbb{R}^N)$.

Let i_{p_1} be the compact injection of W_T^{1,p_1} in $L^{p_1}(0,T;\mathbb{R}^N)$ and $i^*_{p_1} : L^{q_1}(0,T;\mathbb{R}^N) \to \big(W_T^{1,p_1}\big)^*$ its adjoint. Similarly, let i_{p_2} be the compact injection of W_T^{1,p_2} in $L^{p_2}(0,T;\mathbb{R}^N)$ and $i^*_{p_2} : L^{q_2}(0,T;\mathbb{R}^N) \to \big(W_T^{1,p_2}\big)^*$ its adjoint. We define

$$i : W_T^{1,p_1} \times W_T^{1,p_2} \to L^{p_1}(0,T;\mathbb{R}^N) \times L^{p_2}(0,T;\mathbb{R}^N), \quad i(x_1,x_2) = (i_{p_1}x_1, i_{p_2}x_2)$$

and

$$i^* : L^{q_1}(0,T;\mathbb{R}^N) \times L^{q_2}(0,T;\mathbb{R}^N) \to \big(W_T^{1,p_1}\big)^* \times \big(W_T^{1,p_2}\big)^*$$

$$i^*(x_1^*,x_2^*) = (i^*_{p_1}x_1^*, i^*_{p_2}x_2^*) = (x_1^* \circ i_{p_1}, x_2^* \circ i_{p_2}), \tag{2.6}$$

for all $(x_1^*,x_2^*) \in L^{q_1}(0,T;\mathbb{R}^N) \times L^{q_2}(0,T;\mathbb{R}^N)$.

Clearly, (2.6) reads as follows: for every $(v_1,v_2) \in W_T^{1,p_1} \times W_T^{1,p_2}$,

$$\langle i^*(x_1^*,x_2^*),(v_1,v_2)\rangle_{X^*,X} = \langle x_1^*, i_{p_1}(v_1)\rangle_{L^{q_1},L^{p_1}} + \langle x_2^*, i_{p_2}(v_2)\rangle_{L^{q_2},L^{p_2}} \tag{2.7}$$

Let $(u_1,u_2) \in W_T^{1,p_1} \times W_T^{1,p_2}$ be a solution of equation

$$J_{p_1-1,p_2-1}(u_1,u_2) = -(i^*Ai)(u_1,u_2). \tag{2.8}$$

Then, for every $(v_1,v_2) \in W_T^{1,p_1} \times W_T^{1,p_2}$, one has

$$\langle J_{p_1-1,p_2-1}(u_1,u_2),(v_1,v_2)\rangle_{X^*,X} = -\langle (i^*Ai)(u_1,u_2),(v_1,v_2)\rangle_{X^*,X} =$$

$$= -\langle i^*(A(i_{p_1}u_1, i_{p_2}u_2)), (v_1, v_2)\rangle_{X^*,X} =$$

$$= -\langle i^*\big(F_1(\cdot, u_1(\cdot), u_2(\cdot)), F_2(\cdot, u_1(\cdot), u_2(\cdot))\big), (v_1, v_2)\rangle_{X^*,X} =$$

$$= -\langle F_1(\cdot, u_1(\cdot), u_2(\cdot)), i_{p_1}(v_1)\rangle_{L^{q_1},L^{p_1}} - \langle F_2(\cdot, u_1(\cdot), u_2(\cdot)), i_{p_2}(v_2)\rangle_{L^{q_2},L^{p_2}} =$$

$$= -\int_0^T [\langle F_1(t, u_1(t), u_2(t)), v_1(t)\rangle + \langle F_2(t, u_1(t), u_2(t)), v_2(t)\rangle]dt.$$

Taking into account (2.1), the equality

$$\langle J_{p_1-1,p_2-1}(u_1, u_2), (v_1, v_2)\rangle_{X^*,X} = -\int_0^T [\langle F_1(t, u_1(t), u_2(t)), v_1(t)\rangle +$$

$$+\langle F_2(t, u_1(t), u_2(t)), v_2(t)\rangle]dt$$

rewrites as

$$\int_0^T \langle |\dot{u}_1(t)|^{p_1-2}\dot{u}_1(t), \dot{v}_1(t)\rangle dt + \int_0^T \langle |\dot{u}_2(t)|^{p_2-2}\dot{u}_2(t), \dot{v}_2(t)\rangle dt = \tag{2.9}$$

$$= -\int_0^T \langle |u_1(t)|^{p_1-2}u_1(t) + F_1(t, u_1(t), u_2(t)), v_1(t)\rangle dt -$$

$$-\int_0^T \langle |u_2(t)|^{p_2-2}u_2(t) + F_2(t, u_1(t), u_2(t)), v_2(t)\rangle dt$$

for all $(v_1, v_2) \in W_T^{1,p_1} \times W_T^{1,p_2}$. In particular, (2.9) is satisfied for any $(v_1, v_2) = (f_1, f_2) \in \mathcal{C}_T^\infty \times \mathcal{C}_T^\infty \subset W_T^{1,p_1} \times W_T^{1,p_2}$.

Consequently, if $(u_1, u_2) \in W_T^{1,p_1} \times W_T^{1,p_2}$ is a solution of the operator equation (2.8), then (u_1, u_2) is a solution of the problem (1.3). Thus, in order to prove the existence of a solution for the problem (1.3), it would be sufficient to prove the existence of a solution for the operator equation (2.8).

It is a simple matter to see that the operator A generated by the functions $F_1(\cdot, \cdot, \cdot)$, $F_2(\cdot, \cdot, \cdot)$ may be replaced by any operator

$$N : L^{p_1}(0, T; \mathbb{R}^N) \times L^{p_2}(0, T; \mathbb{R}^N) \to L^{q_1}(0, T; \mathbb{R}^N) \times L^{q_2}(0, T; \mathbb{R}^N)$$

defined by

$$\big(N(u_1, u_2)\big)(t) = \big(N_1(u_1(t), u_2(t)), N_2(u_1(t), u_2(t))\big). \tag{2.10}$$

Thus we obtain the following proposition:

Proposition 2.1. *Let $J_{p_1-1,p_2-1} : X \to X^*$, $1 < p_1, p_2 < \infty$ be defined by (2.1) and let $N : L^{p_1}(0, T; \mathbb{R}^N) \times L^{p_2}(0, T; \mathbb{R}^N) \to L^{q_1}(0, T; \mathbb{R}^N) \times L^{q_2}(0, T; \mathbb{R}^N)$ be given. Let $i : X \to L^{p_1}(0, T; \mathbb{R}^N) \times L^{p_2}(0, T; \mathbb{R}^N)$ and $i^* : L^{q_1}(0, T; \mathbb{R}^N) \times L^{q_2}(0, T; \mathbb{R}^N) \to X^*$ defined as above.*

If $(u_1, u_2) \in X$ *is a solution of the operator equation*

$$J_{p_1-1,p_2-1}(u_1, u_2) = -(i^*Ni)(u_1, u_2) \tag{2.11}$$

then (u_1, u_2) *is a solution for the problem*

$$\begin{cases} \frac{d}{dt}\left(|\dot{u}_1(t)|^{p_1-2}\dot{u}_1(t)\right) = |u_1(t)|^{p_1-2}u_1(t) + N_1(u_1(t), u_2(t)), \\ \frac{d}{dt}\left(|\dot{u}_2(t)|^{p_2-2}\dot{u}_2(t)\right) = |u_2(t)|^{p_2-2}u_2(t) + N_2(u_1(t), u_2(t)) \ a.e. \ t \in [0, T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\ u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0. \end{cases} \tag{2.12}$$

3. Preliminary results

In [2] (see Corollary 1) the authors have proved the following abstract result:

Theorem 3.1. *Let* X *be a reflexive real Banach space,* $T : X \to X^*$ *be a monotone, hemicontinuous, coercive operator, satisfying condition* $(S)_2$ *and let* $K : X \to X^*$ *be compact. If there is a constant* $k > 0$ *such that* $Tv = Ku$ *and* $\|u\| \leq k$ *implies* $\|v\| \leq k$*, then the equation* $Tu = Ku$ *has a solution* $u \in X$*, with* $\|u\| \leq k$*.*

We observe that in (2.11), the right-hand operator $K = -i^*Ni$ is compact and therefore equation (2.11) reduces to the case $T(u_1, u_2) = K(u_1, u_2)$ with $T(u_1, u_2) = J_{p_1-1,p_2-1}(u_1, u_2)$ and $K : X_1 \times X_2 \to X_1^* \times X_2^*$ compact.

In order to be able to apply the above abstract result to solve our problem (2.11) we start to list some definitions and useful results.

Definition 3.1. *Let* X *be a real Banach space and* X^* *denotes the dual space of* X*. An operator* $T : X \to X^*$ *is*

- *monotone if:*

$$\langle Tu - Tv, u - v\rangle \geq 0 \ for \ all \ u, v \in X,$$

- *hemicontinuous if:*

$$\langle T(u + \lambda v), w\rangle \to \langle Tu, w\rangle \ as \ \lambda \to 0 \ for \ all \ u, v, w \in X,$$

- *coercive if:*

$$\frac{\langle Tu, u\rangle}{\|u\|} \to \infty \ as \ \|u\| \to \infty,$$

- *demicontinuous if:*

$$u_n \to u \ implies \ Tu_n \rightharpoonup Tu \ as \ n \to \infty.$$

Definition 3.2. *The operator $T : X \to X^*$ is said to satisfy condition $(S)_2$ iff, as $n \to \infty$, the following holds:*

$$u_n \rightharpoonup u, Tu_n \to Tu \text{ implies } u_n \to u.$$

We have denoted by "$\rightharpoonup$" (respectively "$\to$") the convergence in the weak (respectively strong) topology.

Proposition 3.1. *If $T_1 : X_1 \to X_1^*$ and $T_2 : X_2 \to X_2^*$ are monotone, hemicontinuous, coercive operators which satisfy condition $(S)_2$ then $T : X_1 \times X_2 \to X_1^* \times X_2^*$ given by $T(u_1, u_2) = (T_1u_1, T_2u_2)$ has the same properties.*

Proof. Indeed we have

$$\langle T(u_1, u_2) - T(v_1, v_2), (u_1, u_2) - (v_1, v_2)\rangle =$$

$$= \langle (T_1u_1, T_2u_2) - (T_1v_1, T_2v_2), (u_1 - v_1, u_2 - v_2)\rangle =$$

$$= \langle (T_1u_1 - T_1v_1, T_2u_2 - T_2v_2), (u_1 - v_1, u_2 - v_2)\rangle =$$

$$= \langle T_1u_1 - T_1v_1, u_1 - v_1\rangle + \langle T_2u_2 - T_2v_2, u_2 - v_2\rangle \geq 0,$$

hence T is monotone.

If $T_1 : X_1 \to X_1^*$ and $T_2 : X_2 \to X_2^*$ are hemicontinuous operators then we have

$$\langle T((u_1, u_2) + \lambda(v_1, v_2)), (w_1, w_2)\rangle = \langle T(u_1 + \lambda v_1, u_2 + \lambda v_2), (w_1, w_2)\rangle =$$

$$= \langle (T_1(u_1 + \lambda v_1), T_2(u_2 + \lambda v_2)), (w_1, w_2)\rangle =$$

$$= \langle T_1(u_1 + \lambda v_1), w_1\rangle + \langle T_2(u_2 + \lambda v_2), w_2\rangle \longrightarrow^{\lambda\to 0} \langle T_1u_1, w_1\rangle + \langle T_2u_2, w_2\rangle =$$

$$= \langle (T_1u_1, T_2u_2), (w_1, w_2)\rangle = \langle T(u_1, u_2), (w_1, w_2)\rangle.$$

If $T_1 : X_1 \to X_1^*$ and $T_2 : X_2 \to X_2^*$ are coercive then we have:

$$\frac{\langle T(u_1, u_2), (u_1, u_2)\rangle}{\|(u_1, u_2)\|} = \frac{\langle T_1u_1, u_1\rangle + \langle T_2u_2, u_2\rangle}{\|u_1\| + \|u_2\|} =$$

$$= \frac{\langle T_1u_1, u_1\rangle}{\|u_1\|}\frac{\|u_1\|}{\|u_1\| + \|u_2\|} + \frac{\langle T_2u_2, u_2\rangle}{\|u_2\|}\frac{\|u_2\|}{\|u_1\| + \|u_2\|}.$$

If $\|u_1\| \to \infty$ and $\|u_2\|$ is bounded, then

$$\frac{\langle T_1u_1, u_1\rangle}{\|u_1\|} \to \infty, \frac{\|u_1\|}{\|u_1\| + \|u_2\|} \to 1, \frac{\langle T_2u_2, u_2\rangle}{\|u_2\|} \text{ is bounded from below,}$$

$$\frac{\|u_2\|}{\|u_1\| + \|u_2\|} \to 0, \text{and then } \frac{\langle T(u_1, u_2), (u_1, u_2)\rangle}{\|(u_1, u_2)\|} \to \infty.$$

Similar if $\|u_2\| \to \infty$ and $\|u_1\|$ is bounded. If $\|u_1\| \to \infty$ and $\|u_2\| \to \infty$ (passing to a subsequences, if necessary) we use the inequality

$$\lambda a + (1-\lambda)b \geq \min(a,b), \text{for } a,b \in \mathbb{R}, \lambda \in [0,1].$$

If $T_1 : X_1 \to X_1^*$ and $T_2 : X_2 \to X_2^*$ satisfy condition $(S)_2$ then we have:

$$(u_{1n}, u_{2n}) \rightharpoonup (u_1, u_2) \Rightarrow u_{in} \rightharpoonup u_i, i = 1, 2$$

and

$$T(u_{1n}, u_{2n}) \to T(u_1, u_2) \Rightarrow T_i u_{in} \to T_i u_i, i = 1, 2$$

and hence $u_{in} \to u_i, i = 1, 2$ which implies $(u_{1n}, u_{2n}) \to (u_1, u_2)$. □

4. Duality mappings

Let $i = 1, 2$, $(X_i, \|\cdot\|_{X_i})$ be real Banach spaces, X_i^* the corresponding dual spaces and $\langle \cdot,\cdot \rangle$ the duality between X_i^* and X_i. Let $\varphi_i : \mathbb{R}_+ \to \mathbb{R}_+$ be gauge functions, such that φ_i are continuous, strictly increasing, $\varphi_i(0) = 0$ and $\varphi_i(t) \to \infty$ as $t \to \infty$. The duality mapping corresponding to the gauge function φ_i is the set valued mapping $J_{\varphi_i} : X_i \to 2^{X_i^*}$, defined by

$$J_{\varphi_i} x = \left\{ x_i^* \in X_i^* \mid \langle x_i^*, x_i \rangle = \varphi_i(\|x_i\|_{X_i}) \|x_i\|_{X_i},\ \|x_i^*\|_{X_i^*} = \varphi_i(\|x_i\|_{X_i}) \right\}.$$

If X_i are smooth, then $J_{\varphi_i} : X_i \to X_i^*$ is defined by

$$J_{\varphi_i} 0 = 0, \qquad J_{\varphi_i} x_i = \varphi_i(\|x_i\|_{X_i}) \|\ \|'_{X_i}(x_i), \quad x_i \neq 0,$$

and the following metric properties being consequent:

$$\|J_{\varphi_i} x_i\|_{X_i^*} = \varphi_i(\|x_i\|_{X_i}), \qquad \langle J_{\varphi_i} x, x \rangle = \varphi_i(\|x_i\|_{X_i}) \|x_i\|_{X_i}. \tag{4.1}$$

Now we can define $J_{\varphi_1,\varphi_2} : X_1 \times X_2 \to 2^{X_1^*} \times 2^{X_2^*}$ by $J_{\varphi_1,\varphi_2}(x_1,x_2) = (J_{\varphi_1} x_1, J_{\varphi_2} x_2)$. From (4.1) we get

$$\|J_{\varphi_1,\varphi_2}(x_1,x_2)\|_{X_1^* \times X_2^*} = \|J_{\varphi_1} x_1\|_{X_1^*} + \|J_{\varphi_2} x_2\|_{X_2^*} = \tag{4.2}$$
$$= \varphi_1(\|x_1\|_{X_1}) + \varphi_2(\|x_2\|_{X_2}),$$
$$\langle J_{\varphi_1,\varphi_2}(x_1,x_2), (x_1,x_2) \rangle = \langle J_{\varphi_1} x_1, x_1 \rangle + \langle J_{\varphi_2} x_2, x_2 \rangle = \tag{4.3}$$
$$= \varphi_1(\|x_1\|_{X_1}) \|x_1\|_{X_1} + \varphi_2(\|x_2\|_{X_2}) \|x_2\|_{X_2}.$$

For our aim in what follows we will consider the particular case when $J_{\varphi_i} : X_i \to X_i^*$ are the duality mappings, assumed to be single-valued, corresponding to the gauge functions $\varphi_1(t) = t^{p_1-1}$, $\varphi_2(t) = t^{p_2-1}$, $1 < p_1, p_2 < \infty$. In this case we denote $J_{p_1-1,p_2-1} : X_1 \times X_2 \to X_1^* \times X_2^*$ given by $J_{p_1-1,p_2-1} = (J_{p_1-1}, J_{p_2-1})$.

Note that the hypothesis on J_{φ_i} to be single-valued mappings is satisfied iff X_i are smooth (iff X_i are with G-differentiable norms, iff X_i^* are strictly convex).

Let i_1 and i_2 the compactly embedded injections of X_1, X_2 in Z_1 and Z_2 respectively:

$$\begin{aligned} \|i_1(u_1)\|_{Z_1} &\le C_{Z_1}\|u_1\|_{X_1} \text{ for all } u_1 \in X_1, \\ \|i_2(u_2)\|_{Z_2} &\le C_{Z_2}\|u_2\|_{X_2} \text{ for all } u_2 \in X_2. \end{aligned} \tag{4.4}$$

We introduce

$$\lambda_1 = \inf\left\{ \frac{\| u_1 \|_{X_1}^q}{\| i_1(u_1) \|_{Z_1}^q} : \quad u_1 \in X_1 \setminus \{0\} \right\} > 0,$$

$$\lambda_2 = \inf\left\{ \frac{\| u_2 \|_{X_2}^p}{\| i_2(u_2) \|_{Z_2}^p} : \quad u_2 \in X_2 \setminus \{0\} \right\} > 0.$$

Proposition 4.1. *λ_1, λ_2 are attained and $\lambda_1^{-1/q}$ and $\lambda_2^{-1/p}$ are the best constants C_{Z_1} and C_{Z_2}, respectively in the writing of the embeddings of X_1 into Z_1 and X_2 into Z_2, respectively.*

Proof. See the proof of Proposition 4 in [2]. □

5. Existence result for equation $J_{p_1-1,p_2-1}(u_1,u_2) = -(i^*Ni)(u_1,u_2)$

Since J_{p_1-1,p_2-1} satisfies the metric relations (2.3), (2.4) it follows that, for any $(u_1,u_2) \in W_T^{1,p_1} \times W_T^{1,p_2}$, $J_{p_1-1,p_2-1}(u_1,u_2) \in J_{\varphi_1,\varphi_2}(u_1,u_2) = (J_{\varphi_1}u_1, J_{\varphi_2}u_2)$, where J_{φ_i}, $i = 1,2$ designates (eventually multivalued) duality mapping on W_T^{1,p_i} corresponding to the gauge function $\varphi_i(t) = t^{p_i-1}$, $1 < p_i < \infty$, $t \ge 0$. But, is well known that $W_T^{1,p}$ with $1 < p < \infty$ is a smooth Banach space (see for example Theorem 4.1 in [1]) which implies that any duality mapping on $W_T^{1,p}$, $1 < p < \infty$ is single valued. Consequently, $J_{p_i-1} : W_T^{1,p_i} \to (W_T^{1,p_i})^*$, $i = 1,2$ involved in the definition of J_{p_1-1,p_2-1} are just the duality mappings corresponding to the gauge functions $\varphi_i(t) = t^{p_i-1}$, $i = 1,2$.

Theorem 5.1. *If $1 < p_i < \infty$, $i = 1,2$ then:*

a) the spaces $(W_T^{1,p_i}, \|\cdot\|_{W_T^{1,p_i}})$, are uniformly convex and smooth;

b) the duality mappings on W_T^{1,p_i} corresponding to the gauge function $\varphi_i(t) = t^{p_i-1}$, $t \ge 0$ are single valued, $\left(J_{p_i-1} : W_T^{1,p_i} \to (W_T^{1,p_i})^*\right)$

satisfies condition $(S)_2$ *and* $J_{p_1-1,p_2-1} : X \to X^*$ *is defined as follows: if* $(u_1, u_2) \in X$*, then*

$$\langle J_{p_1-1,p_2-1}(u_1,u_2),(v_1,v_2)\rangle_{X^*,X} = \int_0^T \langle |u_1(t)|^{p_1-2}u_1(t), v_1(t)\rangle dt + \int_0^T \langle |\dot{u}_1(t)|^{p_1-2}\dot{u}_1(t), \dot{v}_1(t)\rangle dt + \int_0^T \langle |u_2(t)|^{p_2-2}u_2(t), v_2(t)\rangle dt + \int_0^T \langle |\dot{u}_2(t)|^{p_2-2}\dot{u}_2(t), \dot{v}_2(t)\rangle dt \tag{5.1}$$

for all $(v_1, v_2) \in X$.

Proof. See Theorem 4.1 in [1] and we use (4.1). □

Now, we need the following result:

Lemma 5.1. *Let* $p_1 > p_2 > 1$ *and* $a, b > 0$ *such that* $a^{p_1} + b^{p_2} \le K(a + b)$*, where* $K > 0$*. Then* $a + b \le K_1$*, where*

$$K_1 = \max\left(1 + \max\left((2K)^{\frac{1}{p_2-1}}, (2K)^{\frac{1}{p_2}}\right), 2K^{\frac{1}{p_2-1}}\right).$$

Proof. *Case 1.* If $a \ge 1$ then $a^{p_2} + b^{p_2} \le a^{p_1} + b^{p_2} \le K(a + b)$, hence $a^{p_2} + b^{p_2} \le K(a + b)$, and we get

$$(a + b)^{p_2} \le 2^{p_2-1}(a^{p_2} + b^{p_2}) \le 2^{p_2-1}K(a + b).$$

Finally $a + b \le 2K^{\frac{1}{p_2-1}}$.

Case 2. If $a < 1$ then $b^{p_2} \le a^{p_1} + b^{p_2} \le K(a + b) \le K(1 + b)$, and we get $b^{p_2} \le Kb + K$.
If $b \ge 1$ then $b^{p_2} \le 2Kb$, from where $b \le (2K)^{\frac{1}{p_2-1}}$.
If $b < 1$ then $b^{p_2} < 2K$, from where $b < (2K)^{\frac{1}{p_2}}$, and hence one has $b \le \max\left((2K)^{\frac{1}{p_2-1}}, (2K)^{\frac{1}{p_2}}\right)$. Finally we get $a+b \le 1+\max\left((2K)^{\frac{1}{p_2-1}}, (2K)^{\frac{1}{p_2}}\right)$.

Consequently $a + b \le K_1 = \max\left(1 + \max\left((2K)^{\frac{1}{p_2-1}}, (2K)^{\frac{1}{p_2}}\right), 2K^{\frac{1}{p_2-1}}\right)$. □

Remark 5.1. The case $p_2 > p_1 > 1$ can be done similarly.

Theorem 5.2. *Let* i_{p_1} *be the compact injection of* W_T^{1,p_1} *in* $L^{p_1}(0, T; \mathbb{R}^N)$ *and* $i^*_{p_1} : L^{q_1}(0, T; \mathbb{R}^N) \to \left(W_T^{1,p_1}\right)^*$ *its adjoint. Similarly, let* i_{p_2} *be the compact injection of* W_T^{1,p_2} *in* $L^{p_2}(0, T; \mathbb{R}^N)$ *and* $i^*_{p_2} : L^{q_2}(0, T; \mathbb{R}^N) \to \left(W_T^{1,p_2}\right)^*$

its adjoint. Let J_{p_1-1,p_2-1} *(given by* (2.1)*) which can be defined using the duality mappings on* $W_T^{p_i-1}$, $i=1,2$ *corresponding to the gauge functions* $\varphi_i(t)=t^{p_i-1}$, $t\geq 0$.

Suppose that

$$N: L^{p_1}(0,T;\mathbb{R}^N)\times L^{p_2}(0,T;\mathbb{R}^N)\to L^{q_1}(0,T;\mathbb{R}^N)\times L^{q_2}(0,T;\mathbb{R}^N),$$

$N=(N_1,N_2)$, *is demicontinuous operator which satisfy the growth condition*

$$\|N(u_1,u_2)\|_{L^{q_1}\times L^{q_2}}\leq c_1\|(u_1,u_2)\|^{r-1}_{L^{p_1}\times L^{p_2}}+c_2 \text{ for all } (u_1,u_2)\in L^{p_1}\times L^{p_2}, \tag{5.2}$$

where $c_1,c_2\geq 0, c_1<\min\left(\frac{1}{2},\frac{1}{2^{r-1}}\right)\min\left(\lambda_{1r},\lambda_{1R}^{\frac{r}{R}}\right)$, *with* $r=\min(p_1,p_2), R=\max(p_1,p_2)$,

$$\lambda_{1p_1}=\inf\left\{\frac{\|u_1\|^{p_1}_{W_T^{1,p_1}}}{\|i_1(u_1)\|^{p_1}_{L^{p_1}}}|u_1\neq 0\right\},\quad \lambda_{1p_2}=\inf\left\{\frac{\|u_2\|^{p_2}_{W_T^{1,p_2}}}{\|i_2(u_2)\|^{p_2}_{L^{p_2}}}|u_2\neq 0\right\}.$$

Then, the equation

$$J_{p_1-1,p_2-1}(u_1,u_2)=-(i^*Ni)(u_1,u_2) \tag{5.3}$$

has a solution in $X=W_T^{1,p_1}\times W_T^{1,p_2}$.

Consequently, the problem

$$\begin{cases} \frac{d}{dt}\left(|\dot u_1(t)|^{p_1-2}\dot u_1(t)\right)=|u_1(t)|^{p_1-2}u_1(t)+N_1(u_1(t),u_2(t)),\\ \frac{d}{dt}\left(|\dot u_2(t)|^{p_2-2}\dot u_2(t)\right)=|u_2(t)|^{p_2-2}u_2(t)+N_2(u_1(t),u_2(t)) \text{ a.e. } t\in[0,T],\\ u_1(0)-u_1(T)=\dot u_1(0)-\dot u_1(T)=0,\\ u_2(0)-u_2(T)=\dot u_2(0)-\dot u_2(T)=0.\end{cases} \tag{5.4}$$

has a solution in $X=W_T^{1,p_1}\times W_T^{1,p_2}$.

Proof. It is standard that J_{p_1-1} and J_{p_2-1} are monotone, demicontinuous (hence, hemicontinuous) and coercive. According with Proposition 2.1 J_{p_1-1,p_2-1} is monotone, hemicontinuous and coercive. Therefore, in virtue of Theorem 5.2, J_{p_1-1,p_2-1} has all properties of T in Theorem 3.1. On the other hand, $K=-i^*Ni: X_1\times X_2\to X_1^*\times X_2^*$ is compact. Let us prove that there is some $k>0$ such that $J_{p_1-1,p_2-1}(v_1,v_2)=-(i^*Ni)(u_1,u_2)$ and $\|(u_1,u_2)\|_{X_1\times X_2}\leq k$ implies $\|(v_1,v_2)\|_{X_1\times X_2}\leq k$.

For, let $(u_1,u_2),(v_1,v_2)\in X_1\times X_2$ be with

$$J_{p_1-1,p_2-1}(v_1,v_2)=-(i^*Ni)(u_1,u_2).$$

Then, by the definitions of J_{p_1-1,p_2-1} and (4.4), (5.2), we have

$$\langle J_{p_1-1,p_2-1}(v_1,v_2),(v_1,v_2)\rangle_{X_1^*\times X_2^*,X_1\times X_2}=$$

$$=\langle (J_{p_1-1}v_1,J_{p_2-1}v_2),(v_1,v_2)\rangle_{X_1^*\times X_2^*,X_1\times X_2}=$$

$$= \langle J_{p_1-1}v_1, v_1\rangle_{X_1^*\times X_1} + \langle J_{p_2-1}v_2, v_2\rangle_{X_2^*\times X_2} = \|v_1\|_{X_1}^{p_1} + \|v_2\|_{X_2}^{p_2} =$$

$$= \langle -N(i(u_1,u_2)), i(v_1,v_2)\rangle_{Z_1^*\times Z_2^*, Z_1\times Z_2} \le$$

$$\le \|N(i(u_1,u_2))\|_{Z_1^*\times Z_2^*}\|i(v_1,v_2)\|_{Z_1\times Z_2} \le$$

$$\le \Big[c_1\|i(u_1,u_2)\|_{Z_1\times Z_2}^{r-1} + c_2\Big]\|i(v_1,v_2)\|_{Z_1\times Z_2} =$$

$$= \Big[c_1\|(i_1(u_1), i_2(u_2)\|_{Z_1\times Z_2}^{r-1} + c_2\Big]\|(i_1(v_1), i_2(v_2)\|_{Z_1\times Z_2} =$$

$$= \Big[c_1\Big(\|i_1(u_1)\|_{Z_1} + \|i_2(u_2)\|_{Z_2}\Big)^{r-1} + c_2\Big]\Big[\|i_1(v_1)\|_{Z_1} + \|i_2(v_2)\|_{Z_2}\Big] \le$$

$$\le \Big[c_1\Big(C_{Z_1}\|u_1\|_{X_1} + C_{Z_2}\|u_2\|_{X_2}\Big)^{r-1} + c_2\Big]\Big[C_{Z_1}\|v_1\|_{X_1} + C_{Z_2}\|v_2\|_{X_2}\Big].$$

For the best constants $C_{Z_1} = \lambda_{1p_1}^{-1/p_1}$, $C_{Z_2} = \lambda_{1p_2}^{-1/p_2}$, we derive:

$$\|v_1\|_{X_1}^{p_1} + \|v_2\|_{X_2}^{p_2} \le \Big[c_1\Big(\lambda_{1p_1}^{-1/p_1}\|u_1\|_{X_1} + \lambda_{1p_2}^{-1/p_2}\|u_2\|_{X_2}\Big)^{r-1} + c_2\Big]$$

$$\Big[\lambda_{1p_1}^{-1/p_1}\|v_1\|_{X_1} + \lambda_{1p_2}^{-1/p_2}\|v_2\|_{X_2}\Big] \le$$

$$\le \Big[c_1\Lambda^{r-1}(\|u_1\|_{X_1} + \|u_2\|_{X_2})^{r-1} + c_2\Big]\Lambda(\|v_1\|_{X_1} + \|v_2\|_{X_2})$$

where $\Lambda = \max(\lambda_{1p_1}^{-1/p_1}, \lambda_{1p_2}^{-1/p_2})$. We get:

$$\|v_1\|_{X_1}^{p_1} + \|v_2\|_{X_2}^{p_2} \le \Big[c_1\Lambda^r\|(u_1,u_2)\|_{X_1\times X_2}^{r-1} + c_2\Lambda\Big]\|(v_1,v_2)\|_{X_1\times X_2} \le$$

$$\le \Big[c_1\Lambda^r k^{r-1} + c_2\Lambda\Big]\|(v_1,v_2)\|_{X_1\times X_2}.$$

With $K = c_1\Lambda^r k^{r-1} + c_2\Lambda$ we can apply Lemma 5.1 and we get (if $p_1 > p_2 > 1$):

$$\|(v_1,v_2)\|_{X_1\times X_2} = \|v_1\|_{X_1} + \|v_2\|_{X_2} \le$$

$$\le \max\Big(1 + \max\big((2K)^{\frac{1}{p_2-1}}, (2K)^{\frac{1}{p_2}}\big), 2K^{\frac{1}{p_2-1}}\Big).$$

Taking into account that $r = \min(p_1, p_2)$, it is easy to see that we can choose $k > 0$ such that

$$K_1 = \max\Big(1 + \max\big((2K)^{\frac{1}{p_2-1}}, (2K)^{\frac{1}{p_2}}\big), 2K^{\frac{1}{p_2-1}}\Big) \le k.$$

Indeed we have the following cases:

(a) $K_1 = 2K^{\frac{1}{p_2-1}}$. Now, because $p_1 > p_2$ and

$$c_1 < \min\big(\frac{1}{2}, \frac{1}{2^{r-1}}\big) \min\big(\lambda_{1r}, \lambda_{1R}^{\frac{r}{R}}\big)$$

we have

$$c_1 < \frac{1}{2^{p_2-1}} \min(\lambda_{1p_2}, \lambda_{1p_1}^{p_2/p_1}).$$

Furthermore $c_1 2^{p_2-1} \frac{1}{\min(\lambda_{1p_2}, \lambda_{1p_1}^{p_2/p_1})} < 1$ that is

$$c_1 2^{p_2-1} \max(\lambda_{1p_2}^{-1}, \lambda_{1p_1}^{-p_2/p_1}) < 1 \text{ so that } c_1 2^{p_2-1} \Lambda^{p_2} < 1.$$

Consequently

$$t^{p_2-1} - 2^{p_2-1}(c_1 \Lambda^{p_2} t^{p_2-1} + c_2\Lambda) \to \infty \text{ as } t \to \infty.$$

Hence, there is some $k > 0$ such that

$$k^{p_2-1} - 2^{p_2-1}(c_1 \Lambda^{p_2} k^{p_2-1} + c_2\Lambda) \geq 0$$

which implies $2(c_1\Lambda^{p_2} k^{p_2-1} + c_2\Lambda)^{\frac{1}{p_2-1}} \leq k$, so that $K_1 \leq k$, and then $\|(v_1, v_2)\|_{X_1 \times X_2} \leq k$.

(b) $K_1 = 1 + (2K)^{\frac{1}{p_2-1}}$. In this case, because $c_1 < \frac{1}{2}\min(\lambda_{1p_2}, \lambda_{1p_1}^{p_2/p_1})$, we have $2c_1\Lambda^{p_2} < 1$ and then

$$t - \Big(2(c_1\Lambda^{p_2} t^{p_2-1} + c_2\Lambda)\Big)^{\frac{1}{p_2-1}} - 1 \to \infty \text{ as } t \to \infty.$$

Hence, there is some $k > 0$ such that

$$k - \Big(2(c_1\Lambda^{r} k^{p_2-1} + c_2\Lambda)\Big)^{\frac{1}{p_2-1}} - 1 \geq 0$$

which implies

$$1 + \Big(2(c_1\Lambda^{r} k^{p_2-1} + c_2\Lambda)\Big)^{\frac{1}{p_2-1}} \leq k$$

so that $K_1 \leq k$, and then $\|(v_1, v_2)\|_{X_1 \times X_2} \leq k$.

(c) $K_1 = 1 + (2K)^{\frac{1}{p_2}}$. In this case we have

$$t - \Big(2(c_1\Lambda^{p_2} t^{p_2-1} + c_2\Lambda)\Big)^{\frac{1}{p_2}} - 1 \to \infty \text{ as } t \to \infty$$

because $\frac{p_2-1}{p_2} < 1$, and we conclude as in (b).

The case $p_2 > p_1 > 1$ can be done similarly.

Theorem 3.1 now applies by considering $X = X_1 \times X_2$, $T = J_{p_1-1,p_2-1}$ and $K = -i^*Ni$. □

Taking into account Theorem 5.2 we obtain

Corollary 5.1. *Assume*

(*i*) J_{p_1-1} *and* J_{p_2-1} *satisfy condition* $(S)_2$ *(which implies according with Proposition* 2.1 *that* J_{p_1-1,p_2-1} *satisfies condition* $(S)_2$*);*

(*ii*) $N : Z_1 \times Z_2 \to Z_1^* \times Z_2^*$ *is a demicontinuous operator satisfying the growth condition*

$$\|N(v_1, v_2)\|_{Z_1^* \times Z_2^*} \le c_1 \|(v_1, v_2)\|_{Z_1 \times Z_2}^{s-1} + c_2 \text{ for all } (v_1, v_2) \in i(X_1 \times X_2) \tag{5.5}$$

where $s < \min(p_1, p_2)$ *and* $c_1, c_2 \ge 0$.

Then the equation $J_{p_1-1,p_2-1}(u_1, u_2) = N(u_1, u_2)$ *has a solution in* $X_1 \times X_2$.

We need the following result:

Lemma 5.2. *Let* $r_1, r_2, k_1, k_2 > 0$. *Then there are the constants* $k_3, k_4 > 0$ *such that*

$$k_1 a^{r_1} + k_2 b^{r_2} \le k_3 (a + b)^{\max(r_1, r_2)} + k_4, \quad \text{for all } a, b > 0.$$

Proof. If $a, b \ge 1$ we have

$$k_1 a^{r_1} + k_2 b^{r_2} \le k_1 a^{\max(r_1,r_2)} + k_2 b^{\max(r_1,r_2)} \le$$

$$\le \max(k_1, k_2)(a^{\max(r_1,r_2)} + b^{\max(r_1,r_2)}) \le \max(k_1, k_2)(a + b)^{\max(r_1,r_2)},$$

and the proof is ready with $k_3 = \max(k_1, k_2)$ and $k_4 > 0$, arbitrary.

If $a, b < 1$ then

$$k_1 a^{r_1} + k_2 b^{r_2} \le k_1 + k_2$$

and we may take $k_4 = k_1 + k_2, k_3 > 0$, arbitrary.

If $a \ge 1, b < 1$,

$$k_1 a^{r_1} + k_2 b^{r_2} \le k_1 a^{r_1} + k_2 \le k_1 (a + b)^{r_1} + k_2 \le k_1 (a + b)^{\max(r_1,r_2)} + k_2,$$

and similarly if $a < 1, b \ge 1$. □

Proposition 5.1. *Let* $r_1, r_2 > 1$, $F_i : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $(t, x_1, x_2) \mapsto F_i(t, x_1, x_2)$ $i = 1, 2$ *be two functions measurable in* t *for each* $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ *and continuous in* (x_1, x_2) *for a.e.* $t \in [0, T]$. *Assume that:*

$$\|F_1(t, x_1, x_2)\| \le c_1 \|x_1\|^{r_1 - 1} + c_2 \|x_2\|^{(r_1-1)\frac{r_2}{r_1}} + b_1(t), \tag{5.6}$$

$$\textit{for } x_1, x_2 \in \mathbb{R}^{\mathbb{N}}, t \in [0,T],$$

$$\|F_2(t,x_1,x_2)\| \le c_3\|x_1\|^{(r_2-1)\frac{r_1}{r_2}} + c_4\|x_2\|^{r_2-1} + b_2(t), \tag{5.7}$$

$$\textit{for } x_1, x_2 \in \mathbb{R}^{\mathbb{N}}, t \in [0,T],$$

where $c_1, c_2, c_3, c_4 > 0$ *are constants,* $b_1 \in L^{r_1'}(0,T;\mathbb{R}_+), b_2 \in L^{r_2'}(0,T;\mathbb{R}_+)$, $\frac{1}{r_1} + \frac{1}{r_1'} = 1, \frac{1}{r_2} + \frac{1}{r_2'} = 1$. *Then the operator defined by*

$$\big(N(u_1,u_2)\big)(t) = \big(F_1(t,u_1(t),u_2(t)), F_2(t,u_1(t),u_2(t))\big)$$

is continuous from
$L^{r_1}(0,T;\mathbb{R}^N) \times L^{r_2}(0,T;\mathbb{R}^N)$ *into* $L^{r_1'}(0,T;\mathbb{R}^N) \times L^{r_2'}(0,T;\mathbb{R}^N)$ *and*

$$\|N(v_1,v_2)\|_{L^{r_1'}(0,T;\mathbb{R}^N)\times L^{r_2'}(0,T;\mathbb{R}^N)} \le c_8\|(v_1,v_2)\|^{R_1-1}_{L^{r_1}(0,T;\mathbb{R}^N)\times L^{r_2}(0,T;\mathbb{R}^N)} + c_9, \tag{5.8}$$

for all $(v_1,v_2) \in L^{r_1}(0,T;\mathbb{R}^N)\times L^{r_2}(0,T;\mathbb{R}^N)$, *where* $c_8, c_9 > 0$ *are constants and* $R_1 = \max(r_1, r_2)$.

Proof. From (5.6) and (5.7), for $(v_1,v_2) \in L^{r_1}(0,T;\mathbb{R}^N) \times L^{r_2}(0,T;\mathbb{R}^N))$ we have

$$\|N(v_1,v_2)\|_{L^{r_1'}(0,T;\mathbb{R}^N)\times L^{r_2'}(0,T;\mathbb{R}^N)} =$$
$$= \|N_1(v_1,v_2)\|_{L^{r_1'}(0,T;\mathbb{R}^N)} + \|N_2(v_1,v_2)\|_{L^{r_2'}(0,T;\mathbb{R}^N)} \le$$
$$\le c_1\||v_1|^{r_1-1}\|_{L^{r_1'}} + c_2\Big\||v_2|^{(r_1-1)\frac{r_2}{r_1}}\Big\|_{L^{r_1'}} + \|b_1\|_{L^{r_1'}} +$$
$$+c_3\Big\||v_1|^{(r_2-1)\frac{r_1}{r_2}}\Big\|_{L^{r_2'}} + c_4\||v_2|^{r_2-1}\|_{L^{r_2'}} + \|b_2\|_{L^{r_2'}} =$$
$$= c_1\|v_1\|^{r_1-1}_{L^{r_1}} + c_2\|v_2\|^{(r_1-1)\frac{r_2}{r_1}}_{L^{r_2}} + K_1 + c_3\|v_1\|^{(r_2-1)\frac{r_1}{r_2}}_{L^{r_1}} + c_4\|v_2\|^{r_2-1}_{L^{r_2}} + K_2.$$

By Lemma 5.2 there are the constants $c_5, c_6, c_7 > 0$, such that

$$\|N(v_1,v_2)\|_{L^{r_1'}(0,T;\mathbb{R}^N)\times L^{r_2'}(0,T;\mathbb{R}^N)} \le c_5(\|v_1\|_{L^{r_1}} + \|v_2\|_{L^{r_2}})^{\max(r_2-1,r_1-1)} +$$
$$+c_6(\|v_1\|_{L^{r_1}} + \|v_2\|_{L^{r_2}})^{\max\left((r_1-1)\frac{r_2}{r_1},(r_2-1)\frac{r_1}{r_2}\right)} + c_7 =$$
$$= c_5\|(v_1,v_2)\|^{\max(r_2-1,r_1-1)}_{L^{r_1}\times L^{r_2})} + c_6\|(v_1,v_2)\|^{\max\left((r_1-1)\frac{r_2}{r_1},(r_2-1)\frac{r_1}{r_2}\right)}_{L^{r_1}\times L^{r_2}} + c_7.$$

Since

$$\max\left((r_1-1)\frac{r_2}{r_1},(r_2-1)\frac{r_1}{r_2}\right) \le \max(r_2-1, r_1-1)$$

we obtain

$$\|N(v_1,v_2)\|_{L^{r_1'}(0,T;\mathbb{R}^N)\times L^{r_2'}(0,T;\mathbb{R}^N)} \le c_8\|(v_1,v_2)\|^{R_1-1}_{L^{r_1}(0,T;\mathbb{R}^N)\times L^{r_2}(0,T;\mathbb{R}^N)} + c_9,$$

for all $(v_1,v_2) \in L^{r_1}(0,T;\mathbb{R}^N)\times L^{r_2}(0,T;\mathbb{R}^N)$, where $c_8, c_9 > 0$ are constants and $R_1 = \max(r_1, r_2)$. □

Remark 5.2. If we choose $r_1, r_2 > 1$ be such that $R_1 = \max(r_1, r_2) < r = \min(p_1, p_2)$, then $r_1 < p_1, r_2 < p_2$ and then $q_1 < r_1', q_2 < r_2'$. So we have the embeddings

$$L^{p_1}(0,T;\mathbb{R}^N) \times L^{p_2}(0,T;\mathbb{R}^N) \to L^{r_1}(0,T;\mathbb{R}^N) \times L^{r_2}(0,T;\mathbb{R}^N),$$

$$L^{r_1'}(0,T;\mathbb{R}^N) \times L^{r_2'}(0,T;\mathbb{R}^N) \to L^{q_1}(0,T;\mathbb{R}^N) \times L^{q_2}(0,T;\mathbb{R}^N),$$

and then there are the constants $c_{10}, c_{11}, c_{12} > 0$ such that

$$\|N(v_1,v_2)\|_{L^{q_1}(0,T;\mathbb{R}^N)\times L^{q_2}(0,T;\mathbb{R}^N)} \le c_{10}\|N(v_1,v_2)\|_{L^{r_1'}(0,T;\mathbb{R}^N)\times L^{r_2'}(0,T;\mathbb{R}^N)} \le$$

$$\le c_{10}\Big(c_8\|(v_1,v_2)\|^{R_1-1}_{L^{r_1}(0,T;\mathbb{R}^N)\times L^{r_2}(0,T;\mathbb{R}^N)} + c_9\Big) \le$$

$$\le c_{11}\|(v_1,v_2)\|^{R_1-1}_{L^{p_1}(0,T;\mathbb{R}^N)\times L^{p_2}(0,T;\mathbb{R}^N)} + c_{12},$$

for all $(v_1, v_2) \in L^{p_1}(0,T;\mathbb{R}^N) \times L^{p_2}(0,T;\mathbb{R}^N)$.

Let us remark, too, that if $R_1 = \max(r_1, r_2) = r = \min(p_1, p_2)$ we can choose the constants $c_1, c_2, c_3, c_4 > 0$, small enough, such that $c_{11} < \min\left(\frac{1}{2}, \frac{1}{2^{r-1}}\right)\min\left(\lambda_{1r}, \lambda_{1R}^{\frac{r}{R}}\right)$.

As an application of Theorem 5.2 we give an existence result for problem (1.3). This result is contained in the following theorem:

Theorem 5.3. *Let $r_1, r_2 > 1$, $F_i : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$, $(t, x_1, x_2) \mapsto F_i(t, x_1, x_2)$ $i = 1, 2$ be two functions measurable in t for each $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and continuous in (x_1, x_2) for a.e. $t \in [0,T]$, satisfy conditions* (5.6) *and* (5.7) *with either*
(i) $R_1 < r$ and $c_1, c_2, c_3, c_4 > 0$, or
(ii) $R_1 = r$ and $c_1, c_2, c_3, c_4 > 0$, small enough, such that
$c_{11} < \min\left(\frac{1}{2}, \frac{1}{2^{r-1}}\right)\min\left(\lambda_{1r}, \lambda_{1R}^{\frac{r}{R}}\right)$.
Then, problem (1.3) *has a solution in $X = W_T^{1,p_1} \times W_T^{1,p_2}$.*

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