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Some remarks on a class of nonconvex second-order differential inclusions

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Abstract - We prove a Filippov type existence theorem for a class of a nonconvex second-order differential inclusions by applying the contraction principle in the space of selections of the multifunction instead of the space of solutions. This approach allows to obtain also the Lipschitz dependence on the initial condition of the solution set of the problem considered.

Key words and phrases : fixed point, contractive set-valued map, solution set.

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1. Introduction

In this paper we study second-order differential inclusions of the form

$$(p(t)x'(t))' \in F(t, x(t)) \quad a.e.\ ([0,T]), \quad x(0) = x_0, \quad x'(0) = x_1, \qquad (1.1)$$

where $F : [0,T] \times X \to \mathcal{P}(X)$ is a set-valued map, X is a separable Banach space, $x_0, x_1 \in X$ and $p(.) : [0,T] \to (0,\infty)$ is continuous.

The present paper is motivated by a recent paper of Chang and Li ([10]) in which several existence results concerning problem (1.1) are obtained via fixed point techniques. Even if we deal with an initial value problem instead of a boundary value problem as usual, the differential inclusion (1.1) may be regarded as an extension to the set-valued framework of the classical Sturm-Liouville differential equation. Some existence and qualitative results for problem (1.1) may be found in [7-10,16] etc.

The aim of our paper is to provide several additional results for problem (1.1). More exactly, we prove a Filippov type result concerning the existence of solutions to problem (1.1). We recall that for a differential inclusion defined by a lipschitzian set-valued map with nonconvex values, Filippov's theorem (see [13]) consists in proving the existence of a solution starting from a given "quasi" or "almost" solution. Moreover, the result provides an estimate between the "quasi" solution and the solution obtained.

Our approach is different from the one in [10] and consists in applying the contraction principle in the space of selections of the multifunction instead of the space of solutions. At the same time, using the same idea we prove that the map that associates to a given initial condition $(x_0, x_1) \in X \times X$ the set of solutions of problem (1.1) starting from (x_0, x_1) depends Lipschitz-continuously on the initial condition.

The Filppov type result we propose in the present paper is an alternative to the one in [8]. The two results are not comparable since the methods used in their proofs are also different: the proof of the result in [8] follows Filippov's construction, while in our approach we obtain a "pointwise" estimate from a norm estimate.

We note that the idea of applying the set-valued contraction principle due to Covitz and Nadler ([11]) in the space of derivatives of the solutions belongs to Kannai and Tallos ([14], [17]) and it was already used for similar results obtained for other classes of differential inclusions ([2-6]).

The paper is organized as follows: in Section 2 we present the notations, definitions and the preliminary results to be used in the sequel and in Section 3 we prove the main results.

2. Preliminaries

Let denote by I the interval $[0, T]$, $T > 0$ and let X be a real separable Banach space with the norm $|.|$ and with the corresponding metric $d(.,.)$.

Consider $F : I \times X \to \mathcal{P}(X)$ a set-valued map, $x_0, x_1 \in X$ and $p(.) : I \to (0, \infty)$ a continuous mapping that defined the Cauchy problem (1.1).

A continuous mapping $x(.) \in C(I, X)$ is called a solution of problem (1.1) if there exists a (Bochner) integrable function $f(.) \in L^1(I, X)$ such that:

$$f(t) \in F(t, x(t)) \quad a.e.\,(I), \tag{2.1}$$

$$x(t) = x_0 + p(0)x_1 \int_0^t \frac{1}{p(s)} ds + \int_0^t \frac{1}{p(s)} \int_0^s f(u) du ds \quad \forall t \in I. \tag{2.2}$$

Note that, if we denote $S(t, u) := \int_u^t \frac{1}{p(s)}$, $t \in I$, then (2.2) may be rewrite as

$$x(t) = x_0 + p(0)x_1 S(t, 0) + \int_0^t S(t, u) f(u) du \quad \forall t \in I, \tag{2.3}$$

We shall call $(x(.), f(.))$ a *trajectory-selection pair* of (1.1) if (2.1) and (2.2) are satisfied.

We shall use the following notations for the solution sets of (1.1).

$$\mathcal{S}(x_0, x_1) = \{(x(.), f(.)) \,|\, (x(.), f(.)) \text{ is a trajectory-selection pair of (1.1)}\} \tag{2.4}$$

$$\mathcal{S}_1(x_0, x_1) = \{x(.) \,|\, x(.) \text{ is a solution of (1.1)}\}. \tag{2.5}$$

In the sequel the following conditions are satisfied.

Hypothesis 2.1. (i) $F(.,.) : I \times X \to \mathcal{P}(X)$ has nonempty closed values and for every $x \in X$, $F(.,x)$ is measurable.

(ii) There exists $L(.) \in L^1(I,(0,\infty))$ such that for almost all $t \in I$, $F(t,.)$ is $L(t)$-Lipschitz in the sense that

$$d_H(F(t,x), F(t,y)) \leq L(t)|x-y| \quad \forall \ x,y \in X,$$

where $d_H(A,B)$ is the Pompeiu-Hausdorff distance between $A, B \subset X$

$$d_H(A,B) = \max\{d^*(A,B), d^*(B,A)\}, \quad d^*(A,B) = \sup\{d(a,B) \,|\, a \in A\}.$$

(iii) $d(0, F(t,0)) \leq L(t) \quad a.e.(I)$

(iv) $p(.) : I \to (0,\infty)$ is a continuous function.

Let $m(t) = \int_0^t L(u)du$ and $M := \sup_{t\in I} \frac{1}{p(t)}$. Note that $|S(t,u)| \leq Mt$ $\forall t,u \in I, u \leq t$.

Given $\alpha \in \mathbf{R}$ we consider on $L^1(I,X)$ the following norm

$$|f|_1 = \int_0^T e^{-\alpha m(t)}|f(t)|dt, \quad f \in L^1(I,X),$$

which is equivalent with the usual norm on $L^1(I,X)$.

Consider the following norm on $C(I,X) \times L^1(I,X)$

$$|(x,f)|_{C\times L} = |x|_C + |f|_1 \quad \forall \ (x,f) \in C(I,X) \times L^1(I,X),$$

where, as usual, $|x|_C = \sup_{t\in I} |x(t)| \ \forall \ x \in C(I,X)$.

Finally we recall some basic results concerning set-valued contractions that we shall use in what follows.

Let (Z,d) be a metric space and consider a set-valued map T on Z with nonempty closed values in Z. T is said to be a λ-contraction if there exists $0 < \lambda < 1$ such that:

$$d_H(T(x), T(y)) \leq \lambda d(x,y) \quad \forall x,y \in Z.$$

If Z is complete, then every set-valued contraction has a fixed point, i.e. a point $z \in Z$ such that $z \in T(z)$ (see [11]).

We denote by $Fix(T)$ the set of all fixed point of the multifunction T. Obviously, $Fix(T)$ is closed.

Proposition 2.1. (see [15]) *Let Z be a complete metric space and suppose that T_1, T_2 are λ-contractions with closed values in Z. Then*

$$d_H(Fix(T_1), Fix(T_2)) \leq \frac{1}{1-\lambda} \sup_{z\in Z} d_H(T_1(z), T_2(z)).$$

3. The main results

We are ready now to present an existence theorem concerning solutions for the Cauchy problem (1.1).

Theorem 3.1. *Let Hypothesis* 2.1 *be satisfied and let* $\alpha > MT$ *and let* $y(.)$ *be a solution of the problem*

$$(p(t)y'(t))' = g(t) \quad a.e.\ ([0,T]), \quad y(0) = y_0, \quad y'(0) = y_1,$$

where $g(.) \in L^1(I,X)$ *and there exists* $q(.) \in L^1(I,\mathbf{R})$ *such that*

$$d(g(t), F(t, y(t))) \leq q(t), \quad a.e.\ (I).$$

Then for every $\varepsilon > 0$ *there exists* $x(.)$ *a solution of* (1.1) *satisfying for all* $t \in I$

$$|x(t) - y(t)| \leq (1 + \frac{MT}{\alpha - MT} e^{\alpha m(t)})|x_0 - y_0| + p(0)MT(1+ \\ \frac{MT}{\alpha - MT} e^{\alpha m(t)})|x_1 - y_1| + \frac{\alpha MT e^{\alpha m(t)}}{\alpha - MT} \int_0^T e^{-\alpha m(s)} q(s) ds + \varepsilon. \tag{3.1}$$

Proof. Let us consider $x_0, x_1 \in X$, $f(.) \in L^1(I,X)$ and define the following set-valued maps

$$M_{x_0,x_1,f}(t) = F(t, x_0 + S(t,0)p(0)x_1 + \int_0^t S(t,u)f(u)du), \quad t \geq 0, \tag{3.2}$$

$$T_{x_0,x_1}(f) = \{\phi(.) \in L^1(I,X) \,|\, \phi(t) \in M_{x_0,x_1,f}(t) \quad a.e.\ (I)\}. \tag{3.3}$$

We shall prove first that $T_{x_0,x_1}(f)$ is nonempty and closed for every $f \in L^1(I,X)$. The fact that that the set-valued map $M_{x_0,x_1,f}(.)$ is measurable is well known. For example, the map $t \to x_0 + S(t,0)p(0)x_1 + \int_0^t S(t,u)f(u)du$ can be approximated by step functions and we can apply Theorem III. 40 in [1]. Since the values of F are closed and X is separable with the measurable selection theorem (Theorem III.6 in [1]) we infer that $M_{x_0,x_1,f}(.)$ admits a measurable selection ϕ. According to Hypothesis 2.1 one has

$$|\phi(t)| \leq d(0, F(t,0)) + d_H(F(t,0), F(t,x(t))) \leq L(t)(1 + |x(t)|)$$

$$\leq L(t)(1 + |x_0| + p(0)Mt|x_1| + \int_0^t M(t-s)|f(s)|ds).$$

Thus integrating by parts we obtain

$$\int_0^T e^{-\alpha m(t)} |\phi(t)| dt \leq \int_0^T e^{-\alpha m(t)} L(t)(1 + |x_0| + p(0)Mt|x_1| +$$

$$\int_0^t M(t-s)|f(s)|ds)dt \leq \frac{1+|x_0|}{\alpha} + \frac{MTp(0)|x_1|}{\alpha} + \frac{MT|f|_1}{\alpha}.$$

Hence, if $\phi(.)$ is a measurable selection of $M_{x_0,x_1,f}(.)$, then $\phi(.) \in L^1(I,X)$ and thus $T_{x_0,x_1}(f) \neq \emptyset$.

The set $T_{x_0,x_1}(f)$ is closed. Indeed, if $\phi_n \in T_{x_0,x_1}(f)$ and $|\phi_n - \phi|_1 \to 0$ then we can pass to a subsequence ϕ_{n_k} such that $\phi_{n_k}(t) \to \phi(t)$ for a.e. $t \in I$, and we find that $\phi \in T_{x_0,x_1}(f)$.

The next step of the proof will show that $T_{x_0,x_1}(.)$ is a contraction on $L^1(I,X)$.

Let $f, g \in L^1(I,X)$ be given, $\phi \in T_{x_0,x_1}(f)$ and let $\delta > 0$. Consider the following set-valued map

$$G(t) = M_{x_0,x_1,g}(t) \cap \{x \in X \,|\, |\phi(t) - x| \leq L(t)|\int_0^t S(t,s)(f(s)-g(s))ds| + \delta\}.$$

Since

$$d(\phi(t), M_{x_0,x_1,g}(t)) \leq d(F(t, x_0 + S(t,0)p(0)x_1 + \int_0^t S(t,u)f(u)du),$$

$$F(t, x_0 + S(t,0)p(0)x_1 + \int_0^t S(t,u)g(u)du)) \leq L(t)|\int_0^t S(t,u)(f(u)-g(u))du|$$

we deduce that $G(.)$ has nonempty closed values. Moreover, according to Proposition III.4 in [1], $G(.)$ is measurable. Let $\psi(.)$ be a measurable selection of $G(.)$. It follows that $\psi \in T_{x_0,x_1}(g)$ and

$$|\phi - \psi|_1 = \int_0^T e^{-\alpha m(t)}|\phi(t) - \psi(t)|dt \leq \int_0^T e^{-\alpha m(t)} L(t)(\int_0^t M(t-s)|f(s)-$$

$$g(s)|ds)dt + \int_0^T \delta e^{-\alpha m(t)}dt \leq \frac{MT}{\alpha}|f-g|_1 + \delta \int_0^T e^{-\alpha m(t)}dt.$$

Since δ was arbitrary, we deduce that

$$d(\phi, T_{x_0,x_1}(g)) \leq \frac{MT}{\alpha}|f-g|_1.$$

Replacing f by g we obtain

$$d(T_{x_0,x_1}(f), T_{x_0,x_1}(g)) \leq \frac{MT}{\alpha}|f-g|_1,$$

hence $T_{x_0,x_1}(.)$ is a contraction on $L^1(I,X)$.

We consider next the following set-valued maps

$$\tilde{F}(t,x) = F(t,x) + q(t)B, \quad (t,x) \in I \times X,$$

$$\tilde{M}_{y_0,y_1,f}(t) = \tilde{F}(t, y_0 + S(t,0)p(0)y_1 + \int_0^t S(t,u)f(u)du),\ t \in I, y_0, y_1 \in X,$$

$$\tilde{T}_{y_0,y_1}(f) = \{\phi(.) \in L^1(I,X) \,|\, \phi(t) \in \tilde{M}_{y_0,y_1,f}(t) \quad a.e.\ (I)\},\ f \in L^1(I,X),$$

where B denotes the closed unit ball in X. Obviously, $\tilde{F}(.,.)$ satisfies Hypothesis 2.1.

Repeating the previous step of the proof we obtain that $\tilde{T}_{y_0,y_1}(.)$ is also a $\frac{MT}{\alpha}$-contraction on $L^1(I,X)$ with closed nonempty values.

We prove next the following estimate

$$d_H(T_{x_0,x_1}(f), \tilde{T}_{y_0,y_1}(f)) \leq \frac{1}{\alpha}|x_0 - y_0| + \frac{MTp(0)}{\alpha}|x_1 - y_1| + \int_0^T e^{-\alpha m(t)}q(t)dt \tag{3.4}$$

$\forall\ f(.) \in L^1(I,X)$.

Let $\phi \in T_{x_0,x_1}(f)$, $\delta > 0$ and, for $t \in I$, define

$$G_1(t) = \tilde{M}_{y_0,y_1,f}(t)\ \cap\ \{z \in X \,|\, |\phi(t) - z| \leq L(t)(|x_0 - y_0| + p(0)|S(t,0)| \cdot |x_1 - y_1|) + q(t) + \delta\}$$

With the same arguments used for the set-valued map $G(.)$, we deduce that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.)$ be a measurable selection of $G_1(.)$. It follows that $\psi(.) \in \tilde{T}_{y_0,y_1}(f)$ and one has

$$|\phi - \psi|_1 = \int_0^T e^{-\alpha m(t)}|\phi(t) - \psi(t)|dt \leq$$

$$\int_0^T e^{-\alpha m(t)}[L(t)(|x_0 - y_0| + p(0)|S(t)| \cdot |x_1 - y_1|) + q(t) + \delta]dt \leq$$

$$\frac{1}{\alpha}|x_0 - y_0| + \frac{p(0)MT}{\alpha}|x_1 - y_1| + \int_0^T e^{-\alpha m(t)}q(t)dt + \delta\int_0^T e^{-\alpha m(t)}q(t)dt.$$

Since $\delta > 0$ was arbitrary, as above, we obtain (3.4). Applying Proposition 2.1 we get

$$d_H(Fix(T_{x_0,x_1}), Fix(\tilde{T}_{y_0,y_1})) \leq \frac{1}{\alpha - MT}|x_0 - y_0|$$

$$+\frac{p(0)MT}{\alpha - MT}|x_1 - y_1| + \frac{\alpha}{\alpha - MT}\int_0^T e^{-\alpha m(t)}q(t)dt.$$

Since $g(.) \in Fix(\tilde{T}_{y_0,y_1})$ it follows that there exists $f(.) \in Fix(T_{x_0,x_1})$ such that for any $\varepsilon > 0$

$$|g - f|_1 \leq \frac{1}{\alpha - MT}|x_0 - y_0| + \frac{p(0)MT}{\alpha - MT}|x_1 - y_1| + \frac{\alpha}{\alpha - MT}\int_0^T e^{-\alpha m(t)}q(t)dt + \frac{\varepsilon}{MTe^{\alpha m(T)}}. \tag{3.5}$$

We define $x(t)=x_0+S(t,0)p(0)x_1+\int_0^t S(t,u)f(u)du$, $t\in I$ and we have

$$|x(t)-y(t)|\leq|x_0-y_0|+p(0)MT|x_1-y_1|+MTe^{\alpha m(t)}|f-g|_1.$$

Combining the last inequality with (3.5) we obtain (3.1).

As we already pointed out the idea in the proof of Theorem 3.1 can be adapted in order to prove that the set of all trajectory-selection pairs of (1.1) depends Lipschitz-continuously on the initial condition.

Theorem 3.2. *Let Hypothesis* 2.1 *be satisfied and let* $\alpha>MT$.

Then the map $(x_0,x_1)\mapsto\mathcal{S}(x_0,x_1)$ *is Lipschitz-continuous on* $X\times X$ *with nonempty closed values in* $C(I,X)\times L^1(I,X)$.

Proof. For $x_0,y_0\in X, f(.)\in L^1(I,X)$ we consider the set valued maps $M_{x_0,x_1,f}(.)$ and $T_{x_0,x_1}(.)$ defined in (3.2) and (3.3), respectively. We have already proved that $T_{x_0,x_1}(.)$ is a $\frac{MT}{\alpha}$-contraction on $L^1(I,X)$.

Consequently $T_{x_0,x_1}(.)$ admits a fixed point $f(.)\in L^1(I,X)$. We define $x(t)=x_0+S(t,0)p(0)x_1+\int_0^t S(t,u)f(u)du$.

We have that $\mathcal{S}(x_0,x_1)\subset C(I,X)\times L^1(I,X)$ is a closed subset. Let $(x_n,f_n)\in\mathcal{S}(x_0,x_1)$, $|(x_n,f_n)-(x,f)|_{C\times L}\to 0$. In particular, we have $f_n\in Fix(T_{x_0,x_1})$ which is a closed set, and thus $f(.)\in Fix(T_{x_0,x_1})$. We define $y(t)=x_0+S(t,0)p(0)x_1+\int_0^t S(t,u)f(u)du$ and we prove that $y(.)=x(.)$. One may write

$$|y-x_n|_C=\sup_{t\in I}|y(t)-x_n(t)|\leq$$

$$\leq\sup_{t\in I}M\int_0^t(t-u)|f_n(u)-f(u)|du\leq MTe^{\alpha m(T)}|f_n-f|_1$$

and finally we get that $y(.)=x(.)$.

We prove next the following inequality

$$d_H(T_{x_0,x_1}(f),T_{\xi_0,\xi_1}(f))\leq\frac{1}{\alpha}(|x_0-\xi_0|+p(0)MT|x_1-\xi_1|)\tag{3.6}$$

$\forall f\in L^1(I,X), x_0,x_1,\xi_0,\xi_1\in X$. Let us consider the set-valued map $G_1(t)=$

$$M_{\xi_0,\xi_1,f}(t)\cap\{z\in X\,|\,|\phi(t)-z|\leq L(t)(|x_0-\xi_0|+p(0)|S(t,0)|\cdot|x_1-\xi_1|)+\varepsilon\},$$

$t\in I$, where $\phi(.)$ is a measurable selection of $M_{x_0,x_1,f}(.)$ and $\varepsilon>0$.

With the same arguments used for the set valued map $G(.)$, we deduce that $G_1(.)$ is measurable with nonempty closed values. Let $\psi(.)$ be a measurable selection of $G_1(.)$. It follows that $\psi(.)\in T_{\xi_0,\xi_1}(f)$ and

$$|\phi-\psi|_1=\int_0^T e^{-\alpha m(t)}|\phi(t)-\psi(t)|dt\leq$$

$$\int_0^T e^{-\alpha m(t)} L(t)(|x_0-\xi_0|+p(0)|S(t)|\cdot|x_1-\xi_1|)dt+\varepsilon\int_0^T e^{-\alpha m(t)}dt$$

$$\leq \frac{1}{\alpha}|x_0-\xi_0|+\frac{MTp(0)}{\alpha}|x_1-\xi_1|+\varepsilon\int_0^T e^{-\alpha m(t)}dt.$$

Since ε was arbitrary, we deduce that

$$d(\phi,T_{x_2,y_2}(f))\leq\frac{1}{\alpha}(|x_0-\xi_0|+MTp(0)|\xi_1-\xi_1|).$$

Replacing (x_0,x_1) by (ξ_0,ξ_1) we obtain (3.6).

From (3.6) and Proposition 2.1 we obtain

$$d_H(Fix(T_{x_0,x_1}),Fix(T_{\xi_0,\xi_1}))\leq\frac{1}{\alpha-MT}(|x_0-\xi_0|+p(0)MT|x_1-\xi_1|).$$

Let $x_0,x_1,\xi_0,\xi_1\in X$ and $(x(.),f(.))\in\mathcal{S}(x_0,x_1)$. In particular, $f(.)\in Fix(T_{x_0,x_1})$ and thus, for every $\varepsilon>0$ there exists $g(.)\in Fix(T_{\xi_0,\xi_1})$ such that

$$|f-g|_1\leq\frac{1}{\alpha-MT}(|x_0-\xi_0|+MTp(0)|x_1-\xi_1|)+\varepsilon.$$

Put $z(t)=\xi_0+S(t,0)p(0)\xi_1+\int_0^t S(t,u)g(u)du$. One has

$$|x-z|_C=\sup_{t\in I}|x(t)-z(t)|\leq|x_0-\xi_0|+MTp(0)|x_1-\xi_1|+$$

$$\sup_{t\in I}\int_0^t M(t-s)|f(s)-g(s)|ds\leq|x_0-\xi_0|+MTp(0)|x_1-\xi_1|+MTe^{\alpha m(t)}|f-g|_1$$

$$\leq(1+\frac{MTe^{\alpha m(t)}}{\alpha-MT})(|x_0-\xi_1|+MTp(0)|x_1-\xi_1|)+\frac{MTe^{\alpha m(t)}}{\alpha-MT}\varepsilon.$$

If we denote $k=\max\{1+\frac{MTe^{\alpha m(T)}}{\alpha-MT},p(0)MT(1+\frac{MTe^{\alpha m(T)}}{\alpha-MT}\}$ we deduce first that

$$d((x,f),\mathcal{S}(\xi_0,\xi_1))\leq k[|x_0-\xi_0|+|x_1-\xi_1|]$$

and by interchanging (x_0,x_1) and (ξ_0,ξ_1) we obtain

$$d_H(\mathcal{S}(x_0,x_1),\mathcal{S}(\xi_0,\xi_1))\leq k[|x_0-\xi_0|+|x_1-\xi_1|]$$

and the proof is complete.

Obviously, from Theorem 3.2 we also obtain

Corollary 3.1. *Let Hypothesis* 2.1 *be satisfied and let* $\alpha>MT$. *Then the map* $(x_0,x_1)\to\mathcal{S}_1(x_0,x_1)$ *is Lipschitz continuous on* $X\times X$ *with nonempty values in* $C(I,X)$.

In general, under the hypothesis of Theorem 3.2 the solution set $\mathcal{S}_1$ is not closed in $C(I,X)$. The next result shows that if X is reflexive and the multifunction $F(.,.)$ is convex valued and integrably bounded then $\mathcal{S}_1(x_0,x_1) \subset C(I,X)$ is closed.

Let B be the closed unit ball in X.

Theorem 3.3. *Assume that X is reflexive, $\alpha > MT$ and let $F(.,.) : I \times X \to \mathcal{P}(X)$ be a convex valued set valued map that satisfies Hypothesis* 2.1. *Assume that there exists $k(.) \in L^1(I,X)$ such that for almost all $t \in I$ and for all $x \in X$, $F(t,x) \subset k(t)B$.*

Then for every $x_0, x_1 \in X$, the set $\mathcal{S}_1(x_0,x_1) \subset C(I,X)$ is closed.

Proof. Let $x_n(.) \in \mathcal{S}_1(x_0,x_1)$ such that $|x_n - x|_C \to 0$. There exists $h_n(.) \in L^1(I,X)$ such that $(x_n(.), h_n(.))$ is a trajectory-selection pair of (1.1) for all $n \in \mathbf{N}$. We define $f_n(t) = e^{-\alpha m(t)} h_n(t),\ t \in I$.

The set valued map $F(.,.)$ being integrably bounded, we have that $f_n(.)$ is bounded in $L^1(I,X)$ and for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any $E \subset I,\ \mu(E) < \delta,\ \ |\int_E f_n(s)ds| < \varepsilon$ uniformly with respect to n. Moreover, X is reflexive and so by the Dunford-Pettis criterion (see [12]), taking a subsequence and keeping the same notations, we may assume that $f_n(.)$ converges weakly in $L^1(I,X)$ to some $f(.) \in L^1(I,X)$.

We recall that for convex subsets of a Banach space the strong closure coincides with the weak closure. We apply this result. Since $f_n(.)$ converges weakly in $L^1(I,X)$ to $f(.) \in L^1(I,X)$ for all $h \geq 0$, $f(.)$ belongs to the weak closure of the convex hull $co\{f_n(.)\}_{n\geq h}$ of the subset $\{f_n(.)\}_{n\geq h}$. It coincides with the strong closure of $co\{f_n(.)\}_{n\geq h}$. Hence there exist $\lambda_i^n > 0, i = n, \dots k(n)$ such that

$$\sum_{i=1}^{k(n)} \lambda_i^n = 1, \quad g_n(.) = \sum_{i=n}^{k(n)} \lambda_i^n f_i(.) \in co\{f_n(.)\}_{n\geq h}$$

and such that $g_n(.)$ converges strongly to $f(.)$ in $L^1(I,X)$. Let

$$l_n(.) = \sum_{i=n}^{k(n)} \lambda_i^n h_i(.)$$

Then there exists a subsequence $g_{n_j}(.)$ that converges to $f(.)$ almost everywhere. In particular, $l_{n_j}(.)$ converges almost everywhere to $l(.) = e^{\alpha m(.)} f(.)$ $\in L^1(I,X)$. Hence using the Lebesgue dominated convergence theorem, for every $t \in I$ we obtain

$$\lim_{j\to\infty} \int_0^t S(t,u) l_{n_j}(u)du = \int_0^t S(t,u) l(u)du$$

We define

$$y(t) = x_0 + S(t,0)p(0)x_1 + \int_0^t S(t,u)l(u)du, \quad t \in I$$

and observe that

$$|x(t) - y(t)| \le |x(.) - x_{n_j}(.)|_C + |\int_0^t S(t,u)l_{n_j}(u)du - \int_0^t S(t,u)l(u)du|,$$

which yields $x(t) = y(t)\ \forall t \in I$.

Let us observe now that for almost every $t \in I$

$$l_{n_j}(t) \in \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} F(t, x_i(t)) \subset F(t, x(t)) + L(t) \sum_{i=n_j}^{k(n_j)} \lambda_i^{n_j} |x(t) - x_i(t)| B.$$

Since $\lim_{i\to\infty} |x(t) - x_i(t)| = 0$, we deduce that $f(t) \in F(t, x(t))\ a.e.(I)$ and the proof is complete.

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