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On the exact solution of certain generic nonlinear polynomial systems with cyclic symmetry

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Abstract - Solving exactly nonlinear systems of polynomial equations, especially those involving parameters, is notoriously hard, in spite of the availability of some general methods, like the generalized resultant method or the Gröbner basis method. When a system presents symmetries, as most systems naturally occurring in mathematical physics often do, an effective use of substitutions involving the generators of the invariants of the system may alleviate the problem. In this paper we reiterate this point of view in the case of certain generic systems in three and four variables, with cyclic symmetry, and then exemplify it for the Swift-Soward convection system and the Noonburg neural network system.

Key words and phrases : polynomial system, exact solution, cyclic symmetry, invariant ring, Swift-Soward convection system, Noonburg neural network system.

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1. Introduction

Nonlinear systems of polynomial equations appear naturally in mathematical physics. Often their solutions represent equilibrium points of evolution processes modeled by ordinary differential equations [2, 4, 5]. Such processes usually depend on various parameters and often present symmetries. In view of the fundamental theorem of algebra and basic Galois theory solving them exactly means in general reducing them to one single equation in one variable. We proceed now to present the abstract set-up of our problem.

Let $P(X_1, X_2, \ldots, X_n)$ be an arbitrary polynomial of degree $m \geq 2$ in $n \geq 2$ indeterminates $(X_1, X_2, \ldots, X_n) = \boldsymbol{X}$ with *generic* complex coefficients. Here and in the future generic will mean that the coefficients belong to the complement of some finite union of algebraic varieties of positive co-dimension in the coefficient space. P can then be represented as

$$P(X_1, X_2, \ldots, X_n) = \sum_{I, |I| \leq m} a_I X^I, \quad a_I \in \mathbb{C}, \tag{1.1}$$

where $I=(i_1,i_2,\dots,i_n)$ runs through the set of all multi-indeces such that $i_1\geq 0$, $i_2\geq 0$, $\dots$, $i_n\geq 0$, $|I|:=i_1+i_2+\cdots+i_n\leq m$, and X^I stands for $X_1^{i_1}X_2^{i_2}\dots X_n^{i_n}$. Then the ideal of $\mathbb{C}[X_1,X_2,\dots,X_n]$ generated by the n polynomials $P(X_1,X_2,\dots,X_n)$, $P(X_2,X_3,\dots,X_1)$, $P(X_3,\dots,X_1,X_2)$, $\dots$, $P(X_n,X_1,\dots,X_{n-1})$ generates a 0-dimensional algebraic variety. As such, the system of n equations in n complex variables $(z_1,z_2,\dots,z_n)=\boldsymbol{z}$,

$$\begin{cases} P(z_1,z_2,\dots,z_n)=0 \\ P(z_2,\dots,z_n,z_1)=0 \\ \qquad\vdots \\ P(z_n,z_1,\dots,z_{n-1})=0 \end{cases} \tag{1.2}$$

admits finitely many solutions. In fact, fundamental results in algebraic geometry due to van der Waerden [6, 7] show that the system (1.2) admits exactly m^n solutions for generic values of the coefficients a_I. Typically, only the solutions $(z_1,z_2,\dots,z_n)$ of (1.2) with $z_i\neq z_j$ for $i\neq j$ present physical interest. Such solutions will be called *nontrivial*. Any nontrivial solution generates n such, by permuting circularly its components.

Notice that each polynomial system (1.2) has at least m *trivial* solutions, in the generic case, all of whose components are equal, $z_1=z_2=\cdots=z_n$. In fact, we conjecture (motivated by all our subsequent results and some simple combinatorial arguments) that all the solutions of (1.2) could potentially be obtained in the following manner: If d is a divisor of n, $n=dq$, then replacing $(z_1,z_2,\dots,z_n)$ by $\underbrace{(z_1,z_2,\dots,z_d,z_1,z_2,\dots,z_d,\dots,z_1,z_2,\dots,z_d)}_{q\text{ times}}$ reduces the system (1.2) to a subsystem of d polynomial equations in d variables $(z_1,z_2,\dots,z_d)$, and so all the solutions of (1.2) would eventually be obtained from the nontrivial solutions of these associated d-subsystems.

Ideally, solving the system (1.2) will require finding a polynomial $p\in\mathbb{C}[x]$ in one indeterminate x whose roots supply all the components of all solutions of (1.2), nontrivial or trivial, and eventually finding a factorization of this polynomial in factors whose roots correspond to nontrivial, respectively trivial, solutions. For instance, generically

$$\sum_{k=0}^{m}\left(\sum_{I,|I|=k}a_I\right)x^k \tag{1.3}$$

will always be a factor of $p(x)$ yielding trivial solutions with all components equal.

If $(z_1,z_2,\dots,z_n)$ is a solution of (1.2) then $(x-z_1)(x-z_2)\dots(x-z_n)=x^n-\sigma_1x^{n-1}+\sigma_2x^{n-2}+\cdots+(-1)^{n-1}\sigma_{n-1}x+(-1)^n\sigma_n$ will certainly be a factor of $p(x)$, where σ_1, σ_2, $\dots,\sigma_n$ are the elementary symmetric expressions associated to $z_1,z_2,\dots,z_n$, namely, $\sigma_1=z_1+z_2+\cdots+z_n$, $\sigma_2=z_1z_2+$

$z_1z_3 + \cdots + z_{n-1}z_n, \ldots, \sigma_n = z_1z_2 \ldots z_n$. Now, solving for $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ instead of $(z_1, z_2, \ldots, z_n)$ represents a major simplification, as the symmetry of the system will be increased from circular to permutational. The loss of component order can be restored by considering secondary invariants [1], as described below.

The first step in passing from solutions $(z_1, z_2, \ldots, z_n)$ to $(\sigma_1, \sigma_2, \ldots, \sigma_n)$ is to replace the system (1.2) by the following more elaborate one:

$$\begin{cases} z_1P(z_1,\ldots,z_n) + z_2P(z_2,\ldots,z_n,z_1) + \cdots + z_nP(z_n,z_1,\ldots,z_{n-1}) = 0 \\ z_2P(z_1,\ldots,z_n) + z_3P(z_2,\ldots,z_n,z_1) + \cdots + z_1P(z_n,z_1,\ldots,z_{n-1}) = 0 \\ \vdots \\ z_nP(z_1,\ldots,z_n) + z_1P(z_2,\ldots,z_n,z_1) + \cdots + z_{n-1}P(z_n,\ldots,z_{n-1}) = 0 \end{cases} \tag{1.4}$$

Notice that $(0, 0, \ldots, 0)$ is always a solution of (1.4), but generically not of (1.2). It is obvious that all the solutions of (1.2) are also solutions of (1.4), and that any solution $(z_1, z_2, \ldots, z_n)$ of (1.4) with $\det(Z) \neq 0$, where

$$Z = \begin{bmatrix} z_1 & z_2 & \ldots & z_{n-1} & z_n \\ z_2 & z_3 & \ldots & z_n & z_1 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ z_n & z_1 & \ldots & z_{n-2} & z_{n-1} \end{bmatrix}, \tag{1.5}$$

is also solution of (1.2).

With the warning that a solution $(z_1, z_2, \ldots, z_n)$ of (1.4) with $\det(Z) = 0$ may not be solution of (1.2), as the case of $(0, 0, \ldots, 0)$ is, we shift our analysis from system (1.2) to system (1.4).

In (1.4), unlike (1.2), cyclic permutations leave invariant *each* equation of the system. Of course, there are other ways of associating to (1.2) systems with this property, some involving the simpler equation $P(z_1, z_2, \ldots, z_n) + P(z_2, z_3, \ldots, z_n, z_1) + \cdots + P(z_n, z_1, \ldots, z_{n-1}) = 0$, but (1.4) has the benefit of also preserving the *degree* of each equation.

At any rate, the use of (1.4) paves the way for the implementation of some powerful polynomial invariant theory [1], which we proceed to explain.

The polynomial ring in n indeterminates $\mathbb{C}[X_1, \ldots, X_n]$ and its associated fraction field $\mathbb{C}(X_1, \ldots, X_n)$ are acted upon by the symmetric group $\mathfrak{S}_n$ and its subgroup $\mathfrak{C}_n$ of cyclic permutations in the obvious way, by permuting the indeterminates. The associated invariant rings/fields, $\mathbb{C}[X_1, \ldots, X_n]^{\mathfrak{S}_n}/$ $\mathbb{C}(X_1, \ldots, X_n)^{\mathfrak{S}_n}$ and $\mathbb{C}[X_1, \ldots, X_n]^{\mathfrak{C}_n}$ / $\mathbb{C}(X_1, \ldots, X_n)^{\mathfrak{C}_n}$, that is the ring of polynomials/fractions left invariant by $\mathfrak{S}_n/\mathfrak{C}_n$ obviously satisfy

$$\mathbb{C}[X_1, \ldots, X_n]^{\mathfrak{S}_n} \subset \mathbb{C}[X_1, \ldots, X_n]^{\mathfrak{C}_n},$$
$$\mathbb{C}(X_1, \ldots, X_n)^{\mathfrak{S}_n} \subset \mathbb{C}(X_1, \ldots, X_n)^{\mathfrak{C}_n}.$$

By the fundamental theorem of symmetric polynomials, the elementary symmetric polynomials $\Sigma_1 = X_1 + X_2 + \cdots + X_n$, $\Sigma_2 = X_1X_2 + X_1X_3 + \cdots + X_{n-1}X_n$, $\ldots$, $\Sigma_n = X_1X_2 \ldots X_n$, are polynomially independent over $\mathbb{C}$ and,

$$\mathbb{C}[X_1, \ldots, X_n]^{\mathfrak{S}_n} = \mathbb{C}[\Sigma_1, \ldots, \Sigma_n],$$
$$\mathbb{C}(X_1, \ldots, X_n)^{\mathfrak{S}_n} = \mathbb{C}(\Sigma_1, \ldots, \Sigma_n).$$

Then the Hironaka decomposition [1] says that $\mathbb{C}[X_1, \ldots, X_n]^{\mathfrak{C}_n}$ is a free module of rank $(n-1)!$ over $\mathbb{C}[\Sigma_1, \ldots, \Sigma_n]$, and if Θ is a primitive invariant polynomial of $\mathfrak{C}_n$ with respect to $\mathfrak{S}_n$ in the sense that Θ is $\mathfrak{C}_n$-invariant and the only elements in $\mathfrak{S}_n$ leaving Θ invariant belong to $\mathfrak{C}_n$, then Θ is algebraic over $\mathbb{C}[\Sigma_1, \ldots, \Sigma_n]$ with minimal polynomial of degree $(n-1)!$ and,

$$\mathbb{C}(X_1, \ldots, X_n)^{\mathfrak{C}_n} = \bigoplus_{i=0}^{(n-1)!-1} \mathbb{C}(\Sigma_1, \ldots, \Sigma_n)\Theta^i. \tag{1.6}$$

The minimal (monic) polynomial in $\mathbb{C}[\Sigma_1, \Sigma_2, \ldots, \Sigma_n][T]$ whose root Θ is, equals $\prod_{\hat{g} \in \mathfrak{S}_n/\mathfrak{C}_n} (T - \hat{g} \cdot \Theta)$.

The polynomials $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ are then called primary invariants, and those from a basis of $\mathbb{C}[X_1, \ldots, X_n]^{\mathfrak{C}_n}$ as a free module over $\mathbb{C}[\Sigma_1, \ldots, \Sigma_n]$, or the relevant powers of Θ, secondary invariants.

An example of such a primitive element Θ is the polynomial $X_1X_2^2 + X_2X_3^2 + \cdots + X_{n-1}X_n^2 + X_nX_1^2$, which will be our standard Θ from now on.

When applying the Hironaka decomposition (1.6) to the system (1.4), all of whose equations are derived from invariant polynomials belonging to $\mathbb{C}(X_1, \ldots, X_n)^{\mathfrak{C}_n}$, we obtain a new nonlinear system of $n+1$ polynomial equations in the $n+1$ variables $\sigma_1, \sigma_2, \ldots, \sigma_n, \theta$, corresponding to the primary and secondary invariants $\Sigma_1, \Sigma_2, \ldots, \Sigma_n, \Theta$. The first n equations of this new system correspond directly to system (1.4) and can be viewed as providing sets containing the components of solutions of (1.2), while the last one is derived from the minimal polynomial of the secondary invariant Θ and has the role of putting the solution components in the proper order.

This last system is itself formidable enough, in general, and the best one can hope for is reducing it to a single equation in σ_1, with the added possibility of solving for $\sigma_2, \sigma_3, \ldots, \sigma_n$, and θ in terms of σ_1.

2. Systems in Three Variables

In this section we specialize the abstract discourse of Section 1 to systems in three indeterminates z_1, z_2, z_3, with a detailed analysis of those associated to quadratic and cubic polynomials $P(X_1, X_2, X_3)$.

For starters, for $n = 3$, $\Sigma_1 = X_1 + X_2 + X_3$, $\Sigma_2 = X_1X_2 + X_2X_3 + X_3X_1$, $\Sigma_3 = X_1X_2X_3$, $\Theta = X_1X_2^2 + X_2X_3^2 + X_3X_1^2$, and the Hironaka decomposition

(1.6) reduces to

$$\mathbb{C}[X_1, X_2, X_3]^{\mathfrak{C}_3} = \mathbb{C}[\Sigma_1, \Sigma_2, \Sigma_3] \oplus \mathbb{C}[\Sigma_1, \Sigma_2, \Sigma_3]\Theta$$

The minimal polynomial of Θ gives then

$$\Theta^2 - (\Sigma_1\Sigma_2 - 3\Sigma_3)\Theta + (\Sigma_1^3\Sigma_3 + \Sigma_2^3 + 9\Sigma_3^2 - 6\Sigma_1\Sigma_2\Sigma_3) = 0. \tag{2.1}$$

Case A). $P(X_1, X_2, X_3)$ has degree $m = 2$. It is convenient to represent $P(X_1, X_2, X_3)$ as

$$\begin{aligned} P(\boldsymbol{X}) = {} & a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + \\ & b_1X_2X_3 + b_2X_3X_1 + b_3X_1X_2 + \\ & c_1X_1 + c_2X_2 + c_3X_3 + d, \end{aligned} \tag{2.2}$$

where a_i, b_i, c_i, $i = 1, 2, 3$, d, are arbitrary coefficients, packaged, excepting d, as vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in $\mathbb{C}^3$.

The system (1.2) associated to this $P(X_1, X_2, X_3)$ must have eight solutions in the generic case, of whose two are trivial with identical components and six nontrivial, in two sets of three distinct components. This predicts that the desired polynomial equation in σ_1 resulting from solving the associated system (1.4) should have the factor $(a + b)\sigma_1^2 + 3c\sigma_1 + 9d$, and a more complicated factor, quadratic in σ_1, corresponding to the nontrivial solutions. (Here and in what follows in this section, $a := a_1 + a_2 + a_3$, $b := b_1 + b_2 + b_3$, and $c := c_1 + c_2 + c_3$).

A simple analysis based on degree consideration shows that the Hironaka representation of the polynomials building up system (1.4),

$$X_iP(X_1, X_2, X_3) + X_{i+1}P(X_2, X_3, X_1) + X_{i+2}P(X_1, X_2, X_3),$$

$i = 1, 2, 3$, $X_4 = X_1$, $X_5 = X_2$, must have the type

$$A\Sigma_1^3 + B\Sigma_1^2 + C\Sigma_1 + (D\Sigma_1 + E)\Sigma_2 + F\Sigma_3 + G\Theta,$$

where $A, B, \ldots, F, G$ are suitable coefficients, depending on the coefficients of $P(X_1, X_2, X_3)$. Actually, the system (1.4) to which we add the equation generated by (2.1), becomes in this case

$$\begin{cases} a_1\sigma_1^3 + c_1\sigma_1^2 + d\sigma_1 + [(e_3 - 3a_1)\sigma_1 + c - 3c_1]\sigma_2 + \\ 3(e_1 - e_3)\sigma_3 + (e_2 - e_3)\theta = 0 \\ a_2\sigma_1^3 + c_2\sigma_1^2 + d\sigma_1 + [(e_1 - 3a_2)\sigma_1 + c - 3c_2]\sigma_2 + \\ 3(e_2 - e_1)\sigma_3 + (e_3 - e_1)\theta = 0 \\ a_3\sigma_1^3 + c_3\sigma_1^2 + d\sigma_1 + [(e_2 - 3a_3)\sigma_1 + c - 3c_3]\sigma_2 + \\ 3(e_3 - e_2)\sigma_3 + (e_1 - e_2)\theta = 0 \\ \theta^2 - (\sigma_1\sigma_2 - 3\sigma_3)\theta + (\sigma_1^3\sigma_3 + \sigma_2^3 + 9\sigma_3^2 - 6\sigma_1\sigma_2\sigma_3) = 0, \end{cases} \tag{2.3}$$

where $e_i := a_i + b_i$, $i = 1, 2, 3$, or $\boldsymbol{e}=\boldsymbol{a}+\boldsymbol{b}$.

The first three equations of (2.3) constitute a linear system in σ_2, σ_3, and θ. Its solution naturally reflects the symmetries of this system, albeit in complicated ways, so in order to express it in some reasonable fashion we need additional notation. If $\boldsymbol{u}= (u_1, u_2, u_3)$, $\boldsymbol{v}= (v_1, v_2, v_3)$, $\boldsymbol{w}= (w_1, w_2, w_3)$ are vectors in $\mathbb{C}^3$ we will denote the vectors obtained from $\boldsymbol{u}$ by circularly permuting its components by $\boldsymbol{u}^{+1}$ and $\boldsymbol{u}^{+2}$, namely $\boldsymbol{u}^{+1}:=(u_2, u_3, u_1)$ and $\boldsymbol{u}^{+2}:=(u_3, u_1, u_2)$. Also, $\boldsymbol{u} \cdot \boldsymbol{v}$ will mean the expression $u_1v_1 + u_2v_2 + u_3v_3$, while $\begin{vmatrix} \boldsymbol{u} \\ \boldsymbol{v} \\ \boldsymbol{w} \end{vmatrix}$ will stand for the determinant of the 3×3-matrix whose rows are $\boldsymbol{u}$, $\boldsymbol{v}$, and $\boldsymbol{w}$, in this order. Finally, the elementary symmetric expressions in the components of $\boldsymbol{u}$ will be denoted by $u := u_1 + u_2 + u_3$, as before, $s_2(\boldsymbol{u}):=u_1u_2 + u_2u_3 + u_3u_1$, and $s_3(\boldsymbol{u}):=u_1u_2u_3$.

With the above notations we have, in the generic case,

$$
\begin{aligned}
\sigma_2 &= \frac{a\sigma_1^2 + c\sigma_1 + 3d}{3a - e} \\
\sigma_3 &= \frac{1}{3(3a-e)\,(s_2(\boldsymbol{e}) - \boldsymbol{e}\cdot\boldsymbol{e})} \Bigg[\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{e}^{+1} \\ \boldsymbol{e}^{+2} \end{vmatrix} \sigma_1^3 + \left(\begin{vmatrix} \boldsymbol{c} \\ \boldsymbol{e}^{+1} \\ \boldsymbol{e}^{+2} \end{vmatrix} + ace - 3c\boldsymbol{a}\cdot\boldsymbol{e} \right) \sigma_1^2 + \\
&\quad \left(3\begin{vmatrix} \boldsymbol{e} \\ \boldsymbol{c}^{+1} \\ \boldsymbol{c}^{+2} \end{vmatrix} - 3es_2(\boldsymbol{c}) + c^2e + d\left(3s_2(\boldsymbol{e}) - e^2\right) - 9d\boldsymbol{a}\cdot\boldsymbol{e} + 3dae \right) \sigma_1 + \\
&\quad 3d(ce - 3\boldsymbol{c}\cdot\boldsymbol{e}) \Bigg] \\
\theta &= \frac{1}{(3a-e)\,(s_2(\boldsymbol{e}) - \boldsymbol{e}\cdot\boldsymbol{e})} \Bigg[\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{e}^{+2} \\ \boldsymbol{e} \end{vmatrix} \sigma_1^3 + \left(\begin{vmatrix} \boldsymbol{c} \\ \boldsymbol{e}^{+2} \\ \boldsymbol{e} \end{vmatrix} + ace - 3c\boldsymbol{a}\cdot\boldsymbol{e}^{+1} \right) \sigma_1^2 + \\
&\quad \left(d\left(3s_2(\boldsymbol{e}) - e^2\right) + 3d\left(ae - 3\boldsymbol{a}\cdot\boldsymbol{e}^{+1}\right) + c^2e - 3c\boldsymbol{c}\cdot\boldsymbol{e}^{+1} \right) \sigma_1 + \\
&\quad 3d(ce - 3\boldsymbol{c}\cdot\boldsymbol{e}^{+1}) \Bigg]
\end{aligned} \tag{2.4}
$$

With the values (2.4) of σ_2, σ_3, and θ, the fourth equation of (2.3) becomes, after factoring,

$$
\begin{aligned}
&\frac{-1}{3(3a-e)^3\,(s_2(\boldsymbol{e}) - \boldsymbol{e}\cdot\boldsymbol{e})} (e\sigma_1^2 + 3c\sigma_1 + 9d)^2 \times \\
&\Bigg[\left(3\begin{vmatrix} \boldsymbol{e} \\ \boldsymbol{a}^{+1} \\ \boldsymbol{a}^{+2} \end{vmatrix} - \begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{e}^{+1} \\ \boldsymbol{e}^{+2} \end{vmatrix} + 3a^3 - 9as_2(\boldsymbol{a}) \right) \sigma_1^2 + \\
&\left(3\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{c}^{+1} \\ \boldsymbol{e}^{+2} \end{vmatrix} + 3\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{e}^{+1} \\ \boldsymbol{c}^{+2} \end{vmatrix} - \begin{vmatrix} \boldsymbol{c} \\ \boldsymbol{e}^{+1} \\ \boldsymbol{e}^{+2} \end{vmatrix} + 3c\boldsymbol{a}\cdot\boldsymbol{e} - ace + 9a\boldsymbol{a}\cdot\boldsymbol{c} - 3a^2c \right) \sigma_1 + \\
&d(e^2 - 3s_2(\boldsymbol{e})) + (3a-e)(c^2 - 3s_2(\boldsymbol{c})) \Bigg] = 0
\end{aligned}
$$

So generically, either $e\sigma_1^2 + 3c\sigma_1 + 9d = 0$, or

$$
\begin{aligned}
&\left(3\begin{vmatrix} \boldsymbol{e} \\ \boldsymbol{a}^{+1} \\ \boldsymbol{a}^{+2} \end{vmatrix} - \begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{e}^{+1} \\ \boldsymbol{e}^{+2} \end{vmatrix} + 3a^3 - 9as_2(\boldsymbol{a}) \right) \sigma_1^2 + \\
&\left(3\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{c}^{+1} \\ \boldsymbol{e}^{+2} \end{vmatrix} + 3\begin{vmatrix} \boldsymbol{a} \\ \boldsymbol{e}^{+1} \\ \boldsymbol{c}^{+2} \end{vmatrix} - \begin{vmatrix} \boldsymbol{c} \\ \boldsymbol{e}^{+1} \\ \boldsymbol{e}^{+2} \end{vmatrix} + 3c\boldsymbol{a}\cdot\boldsymbol{e} - ace \right) \sigma_1 + 9a\boldsymbol{a}\cdot\boldsymbol{c} - 3a^2c + \\
&d(e^2 - 3s_2(\boldsymbol{e})) + (3a-e)(c^2 - 3s_2(\boldsymbol{c})) = 0,
\end{aligned} \tag{2.5}
$$

as predicted.

For either one of the two expressions of σ_1 given by (2.5) the triple $(\sigma_1, \sigma_2, \sigma_3)$ with σ_2 and σ_3 as in (2.4) yields the unordered components of a nontrivial solution (z_1, z_2, z_3) of system (1.2) and the value of θ given by (2.4) puts them in the proper order, since $\theta = z_1 z_2^2 + z_2 z_3^2 + z_3 z_1^2$.

Remark 2.1. Equation (2.5) allows one to express σ_2, σ_3, and θ given by (2.4) linearly in σ_1, in the generic case, but the resulting formulas are too complicated to present interest, in general. However, in particular situations this will bring about considerable simplification, as it will be seen in the Applications section.

Case B). $P(X_1, X_2, X_3)$ has degree $m = 3$. It is now desirable to represent $P(X_1, X_2, X_3)$ as

$$\begin{aligned} P(\boldsymbol{X}) = {} & \alpha_1 X_1^3 + \alpha_2 X_2^3 + \alpha_3 X_3^3 + \\ & \beta_1 X_2 X_3^2 + \beta_2 X_3 X_1^2 + \beta_3 X_1 X_2^2 + \\ & \gamma_1 X_2^2 X_3 + \gamma_2 X_3^2 X_1 + \gamma_3 X_1^2 X_2 + \delta X_1 X_2 X_3 + \\ & a_1 X_1^2 + a_2 X_2^2 + a_3 X_3^2 + b_1 X_2 X_3 + b_2 X_3 X_1 + b_3 X_1 X_2 + \\ & c_1 X_1 + c_2 X_2 + c_3 X_3 + d, \end{aligned} \tag{2.6}$$

where $\alpha_i, \beta_i, \gamma_i, a_i, b_i, c_i$, $i = 1, 2, 3$, δ, d, are arbitrary coefficients, packaged, excepting δ and d, as vectors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$, $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in $\mathbb{C}^3$.

The system (1.2) associated to this $P(X_1, X_2, X_3)$ must have twenty seven solutions in the generic case, of whose three are trivial with identical components and twenty four nontrivial, in eight sets of three distinct components. This predicts that the desired polynomial equation in σ_1 resulting from solving the associated system (1.4) should have the factor $(\alpha + \beta + \gamma + \delta)\sigma_1^3 + 3(a + b)\sigma_1^2 + 9c\sigma_1 + 27d$, and a complicated factor of degree eight in σ_1, corresponding to the nontrivial solutions. (As before, we use the notations $\alpha := \alpha_1 + \alpha_2 + \alpha_3$, $\beta := \beta_1 + \beta_2 + \beta_3$, $\gamma := \gamma_1 + \gamma_2 + \gamma_3$, $a := a_1 + a_2 + a_3$, $b := b_1 + b_2 + b_3$, and $c := c_1 + c_2 + c_3$).

A degree analysis shows that the Hironaka representation of the polynomials building up system (1.4) has typical form

$$\begin{aligned} & A\Sigma_1^4 + B\Sigma_1^3 + C\Sigma_1^2 + D\Sigma_1 + E\Sigma_2^2 + (F\Sigma_1^2 + G\Sigma_1 + H)\Sigma_2 + \\ & (I\Sigma_1 + J)\Sigma_3 + (K\Sigma_1 + L)\Theta, \end{aligned}$$

where $A, B, \ldots, K, L$ are suitable coefficients, depending on the coefficients of $P(X_1, X_2, X_3)$.

The actual system in σ_1, σ_2, σ_3, and θ is

$$
\begin{cases}
\alpha_1\sigma_1^4 + a_1\sigma_1^3 + c_1\sigma_1^2 + d\sigma_1 + (3\alpha_1 - \alpha + \beta_3 - \beta_2 + \gamma_2 - \gamma_3)\sigma_2^2 + \\
[(\alpha_3 - 4\alpha_1 + \gamma_3)\sigma_1^2 + (e_3 - 3a_1)\sigma_1 + c - 3c_1]\sigma_2 + [(\alpha + 3(\alpha_1 - \alpha_3) + \\
\beta - 3\beta_3 + 3\gamma_1 - 2\gamma + \delta)\sigma_1 + 3(e_1 - e_3)]\sigma_3 + [(\alpha_2 - \alpha_3 + \beta_2 - \gamma_3)\sigma_1 + \\
e_2 - e_3]\theta = 0 \\
\alpha_2\sigma_1^4 + a_2\sigma_1^3 + c_2\sigma_1^2 + d\sigma_1 + (3\alpha_2 - \alpha + \beta_1 - \beta_3 + \gamma_3 - \gamma_1)\sigma_2^2 + \\
[(\alpha_1 - 4\alpha_2 + \gamma_1)\sigma_1^2 + (e_1 - 3a_2)\sigma_1 + c - 3c_2]\sigma_2 + [(\alpha + 3(\alpha_2 - \alpha_1) + \\
\beta - 3\beta_1 + 3\gamma_2 - 2\gamma + \delta)\sigma_1 + 3(e_2 - e_1)]\sigma_3 + [(\alpha_3 - \alpha_1 + \beta_3 - \gamma_1)\sigma_1 + \\
e_3 - e_1]\theta = 0 \\
\alpha_3\sigma_1^4 + a_3\sigma_1^3 + c_3\sigma_1^2 + d\sigma_1 + (3\alpha_3 - \alpha + \beta_2 - \beta_1 + \gamma_1 - \gamma_2)\sigma_2^2 + \\
[(\alpha_2 - 4\alpha_3 + \gamma_2)\sigma_1^2 + (e_2 - 3a_3)\sigma_1 + c - 3c_3]\sigma_2 + [(\alpha + 3(\alpha_3 - \alpha_2) + \\
\beta - 3\beta_2 + 3\gamma_3 - 2\gamma + \delta)\sigma_1 + 3(e_3 - e_2)]\sigma_3 + [(\alpha_1 - \alpha_2 + \beta_1 - \gamma_2)\sigma_1 + \\
e_1 - e_2]\theta = 0 \\
\theta^2 - (\sigma_1\sigma_2 - 3\sigma_3)\theta + (\sigma_1^3\sigma_3 + \sigma_2^3 + 9\sigma_3^2 - 6\sigma_1\sigma_2\sigma_3) = 0
\end{cases}
\tag{2.7}
$$

where, as before, $\boldsymbol{e}=\boldsymbol{a}+\boldsymbol{b}$.

By adding up the first three equations of system (2.7) we obtain a simpler equation satisfied by σ_1, σ_2, σ_3, and θ, namely

$$
\begin{aligned}
&\alpha\sigma_1^3 + a\sigma_1^2 + c\sigma_1 + 3d + [(\gamma - 3\alpha)\sigma_1 + (b - 2a)]\sigma_2 + \\
&3(\alpha - \gamma + \delta)\sigma_3 + (\beta - \gamma)\theta = 0.
\end{aligned}
\tag{2.8}
$$

Now system (2.7) is too complex even for the most potent computer algebra systems (CAS). Maple 13 produced the outcome 'too large to handle'! One way of obtaining, in principle, the desired equation in σ_1 is the following: After solving for σ_3 and θ (in terms of σ_1 and σ_2) the linear system obtained from the first two equations of (2.7), substituting these values in (2.8) and the fourth equation of (2.7) produces two complicated polynomial equations in σ_1 and σ_2. The resultant of these two equations with respect to σ_2 is the desired equation in σ_1.

This plan will work for *particular* systems of cubic equations, as it will be seen in the Applications section.

3. Systems in Four Variables

When $n = 4$, $\Sigma_1 = X_1 + X_2 + X_3 + X_4$, $\Sigma_2 = X_1X_2 + X_1X_3 + X_1X_4 + X_2X_3 + X_2X_4 + X_3X_4$, $\Sigma_3 = X_1X_2X_3 + X_1X_2X_4 + X_1X_3X_4 + X_2X_3X_4$, $\Sigma_4 = X_1X_2X_3X_4$, and $\Theta = X_1X_2^2 + X_2X_3^2 + X_3X_4^2 + X_4X_1^2$. Since $(n-1)! = 6$, the Hironaka decomposition (1.6) contains a direct sum of six terms, or equivalently the minimal polynomial in Θ over $\mathbb{C}[\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4]$ has degree six. Rather than calculating it we prefer to work, in addition, with another secondary invariant, namely $\Omega := X_1X_3 + X_2X_4$, which is better suited for the purpose of solving system (1.2).

The following two equations involving Θ and Ω hold over $\mathbb{C}[\Sigma_1, \Sigma_2, \Sigma_3, \Sigma_4]$,

$$\begin{aligned}
&\Omega^3 - \Sigma_2\Omega^2 + (\Sigma_1\Sigma_3 - 4\Sigma_4)\Omega - (\Sigma_1^2\Sigma_4 + \Sigma_3^2 - 4\Sigma_2\Sigma_4) = 0,\\
&\Theta^2 - (\Sigma_1\Sigma_2 - 2\Sigma_3 - \Sigma_1\Omega)\Theta + [\Omega^3 + (\Sigma_1^2 - 3\Sigma_2)\Omega^2 +\\
&(\Sigma_2^2 - \Sigma_1^2\Sigma_2 + 3\Sigma_1\Sigma_3 - 8\Sigma_4)\Omega +\\
&(\Sigma_1^3\Sigma_3 + \Sigma_2^3 + 3\Sigma_3^2 - 2\Sigma_1^2\Sigma_4 + 8\Sigma_2\Sigma_4 - 5\Sigma_1\Sigma_2\Sigma_3)] = 0.
\end{aligned} \tag{3.1}$$

Case C). $P(X_1, X_2, X_3, X_4)$ has degree $m = 2$. It is convenient to represent now $P(X_1, X_2, X_3, X_4)$ as

$$\begin{aligned}
P(\boldsymbol{X}) = {} & a_1X_1^2 + a_2X_2^2 + a_3X_3^2 + a_4X_4^2 +\\
& b_1X_2X_3 + b_2X_3X_4 + b_3X_4X_1 +\\
& b_4X_1X_2 + b_5X_1X_3 + b_6X_2X_4 +\\
& c_1X_1 + c_2X_2 + c_3X_3 + c_4X_4 + d,
\end{aligned} \tag{3.2}$$

where a_i, c_i, $i = 1, 2, 3, 4$, b_i, $i = 1, 2, \ldots, 6$, d, are arbitrary coefficients, now conveniently packaged, excepting d, b_5, and b_6, as vectors $\boldsymbol{a}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ in $\mathbb{C}^4$.

The system (1.2) associated to this $P(X_1, X_2, X_3, X_4)$ must have sixteen solutions in the generic case, of whose four are trivial, two with identical components (z, z, z, z) where $(a + b + b_5 + b_6)\sigma_1^2 + 4c\sigma_1 + 16d$, $\sigma_1 = 4z$, and two with components (z_1, z_2, z_1, z_2), $z_1 \neq z_2$ with $(a_1 + a_3 + b_5 - a_2 - a_4 - b_6)\sigma_1 + 2(c_1 + c_3 - c_2 - c_4) = 0$, where $\sigma_1 = 2(z_1 + z_2)$. This last equation is a particular instance of what one obtains when treating the much simpler case $n = 2, m = 2$. The remaining twelve solutions belong to three sets with four distinct components. (Here and in what follows in Case *C*), $a := a_1 + a_2 + a_3 + a_4$, $b := b_1 + b_2 + b_3 + b_4$, and $c := c_1 + c_2 + c_3 + c_4$).

A degree consideration shows that the Hironaka representation of the polynomials building up system (1.4), must have the type

$$\begin{aligned}
&A\Sigma_1^3 + B\Sigma_1^2 + C\Sigma_1 + (D\Sigma_1 + E)\Sigma_2 + F\Sigma_3 +\\
&(G\Sigma_1 + H)\Omega + I\Theta
\end{aligned}$$

where $A, B, \ldots, H, I$ are suitable coefficients, depending on the coefficients of $P(X_1, X_2, X_3, X_4)$. Actually, the system (1.4) to which we add the two equations generated by (3.1), becomes in this case

$$\begin{cases} a_1\sigma_1^3 + c_1\sigma_1^2 + d\sigma_1 + [(e_4 - 3a_1)\sigma_1 + c - 3c_1 - c_3]\sigma_2 + \\ (2a_1 - a_3 + b_2 + b_6 - b_5 + e_1 - 2e_4)\sigma_3 + [(a_3 + b_5 - e_4)\sigma_1 + \\ c_1 + 3c_3 - c]\omega + (a_2 + b_3 - e_4)\theta = 0 \\ a_2\sigma_1^3 + c_2\sigma_1^2 + d\sigma_1 + [(e_1 - 3a_2)\sigma_1 + c - 3c_2 - c_4]\sigma_2 + \\ (2a_2 - a_4 + b_3 + b_5 - b_6 + e_2 - 2e_1)\sigma_3 + [(a_4 + b_6 - e_1)\sigma_1 + \\ c_2 + 3c_4 - c]\omega + (a_3 + b_4 - e_1)\theta = 0 \\ a_3\sigma_1^3 + c_3\sigma_1^2 + d\sigma_1 + [(e_2 - 3a_3)\sigma_1 + c - 3c_3 - c_1]\sigma_2 + \\ (2a_3 - a_1 + b_4 + b_6 - b_5 + e_3 - 2e_2)\sigma_3 + [(a_1 + b_5 - e_2)\sigma_1 + \\ c_3 + 3c_1 - c]\omega + (a_4 + b_1 - e_2)\theta = 0 \\ a_4\sigma_1^3 + c_4\sigma_1^2 + d\sigma_1 + [(e_3 - 3a_4)\sigma_1 + c - 3c_4 - c_2]\sigma_2 + \\ (2a_4 - a_2 + b_1 + b_5 - b_6 + e_4 - 2e_3)\sigma_3 + [(a_2 + b_6 - e_3)\sigma_1 + \\ c_4 + 3c_2 - c]\omega + (a_1 - a_2 + e_2 - e_3)\theta = 0 \\ \omega^3 - \sigma_2\omega^2 + (\sigma_1\sigma_3 - 4\sigma_4)\omega - (\sigma_1^2\sigma_4 + \sigma_3^2 - 4\sigma_2\sigma_4) = 0 \\ \theta^2 - (\sigma_1\sigma_2 - 2\sigma_3 - \sigma_1\omega)\theta + [\omega^3 + (\sigma_1^2 - 3\sigma_2)\omega^2 + \\ (\sigma_2^2 - \sigma_1^2\sigma_2 + 3\sigma_1\sigma_3 - 8\sigma_4)\omega + \\ (\sigma_1^3\sigma_3 + \sigma_2^3 + 3\sigma_3^2 - 2\sigma_1^2\sigma_4 + 8\sigma_2\sigma_4 - 5\sigma_1\sigma_2\sigma_3] = 0, \end{cases} \tag{3.3}$$

where $e_i := a_i + b_i$, $i = 1, 2, 3, 4$, or $\boldsymbol{e}=\boldsymbol{a}+\boldsymbol{b}$.

By adding up the first three equations of system (3.3) we obtain a simpler equation satisfied by σ_1, σ_2, σ_3, σ_4, ω, and θ, namely

$$a\sigma_1^2 + c\sigma_1 + 4d + (e - 3a)\sigma_2 + [2(b_5 + b_6) - b]\omega = 0. \tag{3.4}$$

Notice also that σ_4 can be eliminated between the last two equations of (3.3) to yield

$$\begin{gathered} \theta^2 + (\sigma_1\omega - \sigma_1\sigma_2 + 2\sigma_3)\theta + [-\omega^3 + (\sigma_1^2 - \sigma_2)\omega^2 + \\ (-\sigma_1^2\sigma_2 + \sigma_1\sigma_3 + \sigma_2^2)\omega + \sigma_1^3\sigma_3 + \sigma_2^3 + 5\sigma_3^2 - 5\sigma_1\sigma_2\sigma_3] = 0. \end{gathered} \tag{3.5}$$

Again, there is a definite path of reducing system (3.3) to one single equation in σ_1, but the calculations are too involved even for the most potent machines: The first four equations of (3.3) form a linear system in σ_2, σ_3, ω, and θ, which allows one to express these quantities in terms of σ_1. Then the equation (3.5) yields the desired polynomial equation in σ_1 alone, after substituting in it the σ_1-values of σ_2, σ_3, ω, and θ.

4. Applications

In the previous two sections we treated in some degree of detail certain fully generic systems of polynomial equations with cyclic symmetry. However, it is clear, by a continuity argument of the solutions with respect to the parameters [7], that one can study this way *any* system of polynomial equations, by assigning specific values to some of the parameters. The only difference is that certain divisions by zero have to be interpreted properly as yielding solutions of higher multiplicity or solutions at infinity [7]. In particular, many partly generic systems continue to behave as if they were fully generic.

In this section we present (three) applications to systems relevant to mathematical physics, the Swift-Soward convection system [5, 4] and the Noonburg neural network system [2]. They all are partly generic cubic systems which can be embedded in a quadratic system framework via suitable substitutions. This is also in line with the general philosophy that any nonlinear polynomial system can be treated as a (larger) quadratic system by means of adequate substitutions.

Example 1. The Swift-Soward convection system. The solutions of this polynomial system describe the stationary points of a system of three ordinary differential equations associated to the motion of a fluid layer, rotating about a vertical axis and being heated from below, at the thermal convection onset [3, 4, 5].

In our terminology, the Swift-Soward system is the cyclic system of three cubic equations associated to the polynomial $P(X_1, X_2, X_3) = X_1(\gamma X_1^2 + \alpha X_2^2 + \beta X_3^2) - \epsilon X_2 X_3 - \lambda X_1$, where α, β, γ, ϵ, and λ are real parameters. It is easy to see that in addition to the circular symmetry its solution set is also symmetric with respect to transformations of type

$$X_i \to -X_i, X_{i+1} \to -X_{i+1}, X_{i+2} \to X_{i+2}, \quad i = 1, 2, 3, \quad X_4 = X_1, X_5 = X_2,$$

which should make for a better understanding of the solution, which is actually the case [3].

For the length of this example we will denote by s_1, s_2, and s_3 the elementary symmetric expressions associated to α, β, and γ. It is rather easy to describe the trivial solutions of system (1.2) in this case: $\left(\frac{\lambda}{\gamma}, 0, 0\right)$ and its symmetries (a total of six), and (z, z, z), z root of $s_1 z^3 - \epsilon z^2 - \lambda z = 0$, and its symmetries (a total of nine). The remaining twelve solutions are nontrivial, and due to the symmetries it suffices to find one of them, or even the fundamental symmetric expressions of the squares of its components, and then the proper ordering of these components. Replacing $P(X_1, X_2, X_3)$ by $X_1 P(X_1, X_2, X_3)$ will not change the nontrivial solutions. Notice now that each equation of the cyclic system (1.2)associated to $X_1 P(X_1, X_2, X_3)$ contains the invariant monomial $-\epsilon X_1 X_2 X_3$, so making the substitutions

$X_1X_2X_3 = \Delta$, and $X_1^2 \to X_1$, $X_2^2 \to X_2$, $X_3^2 \to X_3$ leads to a cyclic system associated to a new (quadratic) polynomial

$$P(X_1, X_2, X_3) = X_1(\gamma X_1 + \alpha X_2 + \beta X_3) - \lambda X_1 - \delta\epsilon, \tag{4.1}$$

where δ is an extra parameter corresponding to Δ. So, when solving for σ_1, σ_2, σ_3, and θ the system (2.3) associated to (4.1) we just have to remember that for (z_1, z_2, z_3) to be a (nontrivial) solution of the Swift-Soward system we must have $\sigma_1 = z_1^2 + z_2^2 + z_3^2$, $\sigma_2 = z_1^2z_2^2 + z_2^2z_3^2 + z_3^2z_1^2$, $\sigma_3 = z_1^2z_2^2z_3^2$, $\theta = z_1^2z_2^4 + z_2^2z_3^4 + z_3^2z_1^4$, and in addition $\sigma_3 = \delta^2$.

For our new system Equation (2.5) becomes, in the generic case,

$$\gamma(\alpha\beta - \gamma^2)\sigma_1^2 - \lambda(s_2 - 3\gamma^2)\sigma_1 + [\lambda^2(s_1 - 3\gamma) + \delta\epsilon(s_1^2 - 3s_2)] = 0. \tag{4.2}$$

Now, in the presence of (4.2) the formulae (2.4), excepting θ which is too complicated to warrant inclusion here, become (see also the Remark following (2.5))

$$\begin{aligned} \sigma_2 &= \frac{\lambda^2 + \delta\epsilon s_1 - \gamma\lambda\sigma_1}{\alpha\beta - \gamma^2} \\ \sigma_3 &= -\frac{(\gamma\sigma_1 - \lambda)\delta\epsilon}{s_1^2 - 3s_2} \end{aligned} \tag{4.3}$$

Since $\sigma_3 = \delta^2$, the expression of σ_3 in (4.3) yields

$$\delta = -\frac{(\gamma\sigma_1 - \lambda)\epsilon}{s_1^2 - 3s_2}. \tag{4.4}$$

When substituting the expression (4.4) of δ in (4.2) this factors to give

$$(\gamma\sigma_1 - \lambda)[(\alpha\beta - \gamma^2)\sigma_1 - \lambda(s_1 - 3\gamma) - \epsilon^2] = 0. \tag{4.5}$$

Since $\gamma\sigma_1 - \lambda = 0$ corresponds to trivial solutions it follows that for nontrivial solutions,

$$\sigma_1 = \frac{\lambda(s_1 - 3\gamma) + \epsilon^2}{\alpha\beta - \gamma^2} \tag{4.6}$$

With the value (4.6) of σ_1 the other elementary symmetric expressions (4.3) become, finally,

$$\begin{aligned} \sigma_2 &= \frac{[s_1\epsilon^2 + \lambda(s_1^2 - 3s_2)][\lambda(\gamma^2 + 2\alpha\beta - s_2) - \gamma\epsilon^2]}{(\alpha\beta - \gamma^2)^2(s_1^2 - 3s_2)} \\ \sigma_3 &= \frac{\epsilon^2[\lambda(\gamma^2 + 2\alpha\beta - s_2) - \gamma\epsilon^2]^2}{(\alpha\beta - \gamma^2)^2(s_1^2 - 3s_2)^2} \end{aligned} \tag{4.7}$$

Formulae (4.6) and (4.7) are what we were looking for. As before, recall that the expression of θ, however complicated, allows one to put the components of a nontrivial solution of the Swift-Soward system in proper order. When $\gamma = 1$, we recover the result of [3].

The Swift-Soward system could also have been treated directly as a degree $m = 3$ system, via Case B) of Section 2, but the computations would have been substantially more involved, as a result of ignoring part of the available symmetries.

Example 2. The Noonburg neural network system with three neurons. The solutions of this polynomial system describe the stationary points of a Lotka-Volterra system of ordinary differential equations measuring the activity levels at each cell for three interconnecting neurons [2]. It is the cyclic system of cubic polynomial equations associated to the polynomial $P(X_1, X_2, X_3) = X_1(\gamma X_1^2 + \alpha X_2^2 + \beta X_3^2) - \lambda X_1 + \epsilon$, where $\alpha, \beta, \gamma, \epsilon$, and λ are real parameters.

Although we could treat, as in Example 1, this system as quadratic, by replacing $P(X_1, X_2, X_3)$ with $X_2X_3P(X_1, X_2, X_3)$, then with $\delta(\gamma X_1^2 + \alpha X_2^2 + \beta X_3^2) + \epsilon X_2X_3 - \delta\lambda$, via the substitution $\Delta = X_1X_2X_3$, we choose to exemplify instead the machinery set up in Case B) of Section 2. The associated system (2.7) and equation (2.8) become in this case

$$\begin{cases} \gamma\sigma_1^4 - \lambda\sigma_1^2 + \epsilon\sigma_1 + (s_1+\gamma)\sigma_2^2 + 2(-2\gamma\sigma_1^2 + \lambda)\sigma_2 + 2(3\gamma - s_1)\sigma_1\sigma_3 = 0 \\ \epsilon\sigma_1 - (\alpha+\gamma)\sigma_2^2 + (\gamma\sigma_1^2 - \lambda)\sigma_2 + (s_1 - 3\gamma)\sigma_1\sigma_3 + (\alpha - \gamma)\sigma_1\theta = 0 \\ \epsilon\sigma_1 - (\beta+\gamma)\sigma_2^2 + (\beta\sigma_1^2 - \lambda)\sigma_2 + (s_1 - 3\beta)\sigma_1\sigma_3 + (\gamma - \beta)\sigma_1\theta = 0 \\ \theta^2 - (\sigma_1\sigma_2 - 3\sigma_3)\theta + (\sigma_1^3\sigma_3 + \sigma_2^3 + 9\sigma_3^2 - 6\sigma_1\sigma_2\sigma_3) = 0 \end{cases} \tag{4.8}$$

and

$$\gamma\sigma_1^3 - \lambda\sigma_1 + 3\epsilon + (\beta - 3\gamma)\sigma_1\sigma_2 + 3(\gamma - \beta)\sigma_3 + (\alpha - \beta)\theta = 0. \tag{4.9}$$

(Throughout this example s_1, s_2, and s_3 will denote the elementary symmetric expressions associated to α, β, and γ.)

The linear system in σ_3 and θ formed by the first two equations in (4.8) has in the generic case the solution (depending of σ_1 and σ_2),

$$\begin{aligned} \sigma_3 &= \frac{\gamma\sigma_1^4 - \lambda\sigma_1^2 + \epsilon\sigma_1 + (s_1+\gamma)\sigma_2^2 + 2(-2\gamma\sigma_1^2 + \lambda)\sigma_2}{2(s_1 - 3\gamma)\sigma_1} \\ \theta &= \frac{\gamma\sigma_1^4 - \lambda\sigma_1^2 + 3\epsilon\sigma_1 + (\beta - \alpha)\sigma_2^2 - 2\gamma\sigma_1^2\sigma_2}{2(\gamma - \alpha)\sigma_1}. \end{aligned} \tag{4.10}$$

Substituting (4.10) in Equation (4.9) and the fourth equation of (4.8) produces two equations in σ_1 and σ_2. A necessary and sufficient condition for these latter equations to hold is the vanishing of their resultant with respect to σ_2. The resultant in question factors to give

$$\begin{aligned}&\frac{2(s_1-3\gamma)(\alpha-\gamma)(s_1^2-5s_2+2\gamma^2+4\alpha\beta)}{\sigma_1}\times\\&(s_1\sigma_1^3-9\lambda\sigma_1+27\epsilon)^2(E_0\sigma_1^8+E_1\sigma_1^7+\cdots+E_7\sigma_1+E_8),\end{aligned}\tag{4.11}$$

where,

$$\begin{aligned}
E_0 &= \gamma(\alpha\beta-\gamma^2)^2 s_1(s_1^2-3s_2)\\
E_1 &= 0\\
E_2 &= \lambda(\alpha\beta-\gamma^2)(s_1^2-3s_2)[7s_1\gamma^2-(s_1^2+s_2)\gamma-s_1s_2]\\
E_3 &= \epsilon(\alpha\beta-\gamma^2)[8s_1^2\gamma^3+s_1(-13s_1^2+23s_2)\gamma^2+\\
&\quad(5s_1^4-s_2^2-12s_1^2s_2)\gamma+s_1s_2(s_1^2-3s_2)]\\
E_4 &= \lambda^2(s_1^2-3s_2)[3s_1\gamma^3+(7s_1^2-6s_2)\gamma^2-s_1(s_1^2+10s_2)\gamma+s_2(2s_1^2+s_2)]\\
E_5 &= -2\lambda\epsilon[12s_1^2\gamma^4-s_1(11s_1^2+15s_2)\gamma^3+\\
&\quad(s_1^4+12s_2^2+17s_1^2s_2)\gamma^2-\\
&\quad s_1(2s_1^4-s_1^2s_2+s_2^2)\gamma+s_2(3s_1^4-2s_2^2)]\\
E_6 &= (6\gamma^2-4s_1\gamma+s_1^2-s_2)(2s_1\gamma+s_1^2-s_2)^2\epsilon^2+\\
&\quad(s_1^2-3s_2)[3\gamma^3-5s_1\gamma^2-(s_1^2-10s_2)\gamma-s_1(s_1^2-2s_2)]\lambda^3\\
E_7 &= -\epsilon\lambda^2(\alpha-\beta)^2[(7s_1^2+3s_2)\gamma-s_1(s_1^2+5s_2)]\\
E_8 &= \lambda^4(\alpha-\beta)^2(s_1^2-3s_2).
\end{aligned}$$

So generically, either $s_1\sigma_1^3-9\lambda\sigma_1+27\epsilon=0$, or $E_0\sigma_1^8+E_1\sigma_1^7+\cdots+E_7\sigma_1+E_8=0$, as predicted. In the particular case $\alpha=\beta=1$, $\gamma=0$, $\epsilon=-1$, we recover the answer to an example treated in [1].

Example 3. A Swift-Soward convection-type system in four variables. This system is the adaptation of the system in Example 1 to four variables. Here $P(X_1,X_2,X_3,X_4)=X_1(\delta X_1^2+\alpha X_2^2+\beta X_3^2+\gamma X_4^2)-\epsilon X_2X_3X_4-\lambda X_1$, where α, β, γ, δ, ϵ and λ are complex parameters. Just as in Example 1 we can associate to it a quadratic system generated by the polynomial $P(X_1,X_2,X_3,X_4)=\delta X_1^2+\alpha X_1X_2+\beta X_1X_3+\gamma X_1X_4-\lambda X_1-\mu$, where μ is another complex parameter. For this latter polynomial the system in σ_1, σ_2, σ_3, σ_4, ω, and θ made up from the first four equations of (3.3), and

(3.5), becomes

$$\begin{cases} \delta\sigma_1^3 - \lambda\sigma_1^2 - \mu\sigma_1 + [(\alpha-3\delta)\sigma_1 + 2\lambda]\sigma_2 + (3\delta - 2\alpha - \beta)\sigma_3 + \\ (-\alpha+\beta)\sigma_1\omega + (-\alpha+\gamma)\theta = 0 \\ -\mu\sigma_1 + (\delta\sigma_1 - l)\sigma_2 + (\beta+\gamma-2\delta)\sigma_3 + (-\delta\sigma_1+\lambda)\omega + (\alpha-\delta)\theta = 0 \\ -\mu\sigma_1 + (\alpha+\gamma-\beta-\delta)\sigma_3 + ((\beta+\delta)\sigma_1 - 2\lambda)\omega = 0 \\ -\mu\sigma_1 + (\gamma\sigma_1 - \lambda)\sigma_2 + (\alpha+\beta-2\gamma)\sigma_3 + (-\gamma\sigma_1+\lambda)\omega + (\delta-\gamma)\theta = 0 \\ \theta^2 + (\sigma_1\omega - \sigma_1\sigma_2 + 2\sigma_3) + [-\omega^3 + (\sigma_1^2 - \sigma_2)\omega^2 + \\ (-\sigma_1^2\sigma_2 + \sigma_1\sigma_3 + \sigma_2^2)\omega + (\sigma_1^3\sigma_3 + \sigma_2^3 + 5\sigma_3^2 - 5\sigma_1\sigma_2\sigma_3)] = 0 \end{cases} \tag{4.12}$$

The linear system in σ_2, σ_3, ω, and θ generated by the first four equations of (4.12) has a solution complicated enough not to warrant insertion here, however, after substituting its components into the last equation of (4.12) we get, after factoring

$$\begin{aligned} &\frac{(2\delta^2 - 2\delta s_1 + s_1^2 - 2s_2)((\alpha+\beta+\gamma+\delta)\sigma_1^2 - 4\lambda\sigma_1 - 16\mu)^2}{(G_0\sigma_1 + G_1)^3} \times \\ &((\beta+\delta)\sigma_1 - 2\lambda)^2 (F_0\sigma_1^3 + F_1\sigma_1^2 + F_2\sigma_1 + F_3), \end{aligned} \tag{4.13}$$

where,

$$\begin{aligned}
F_0 &= \delta(-\delta^2 + s_2)[\delta^3 - (\beta^2 + 2s_2)\delta + (s_1^2 - 2s_2)\beta] \\
F_1 &= \lambda\{6\delta^5 - (2\beta + 3s_1)\delta^4 + [2\beta(-2\beta + s_1) + s_1^2 - 14s_2]\delta^3 + \\
&\quad [(s_1\beta + 3s_1^2 - 4s_2)\beta + 4s_1s_2]\delta^2 + \\
&\quad [(2s_2\beta - s_1^3)\beta + 6s_2^2 - s_1^2s_2]\delta - s_2(s_1^2 - 2s_2)\beta\} \\
F_2 &= 4\mu\delta^5 - [4(\beta + s_1)\mu + 11\lambda^2]\delta^4 + \\
&\quad [4(s_1\beta + s_1^2 - 4s_2)\mu + 2(3\beta + 5s_1)\lambda^2]\delta^3 + \\
&\quad [2s_1(-s_1^2 + 6s_2)\mu + \left((5\beta - 8s_1)\beta - 5s_1^2 + 19s_2\right)\lambda^2]\delta^2 + \\
&\quad [\left(2s_1(2s_2 - s_1^2)\beta + s_1^4 + 12s_2^2 - 8s_1^2s_2\right)\mu + (2(-s_1\beta + s_2)\beta + \\
&\quad s_1(s_1^2 - 8s_2))\lambda^2]\delta + (s_1^4 - 4s_1^2s_2 + 4s_2^2)\beta\mu + \\
&\quad [(-s_2\beta + s_1^3)\beta + s_2(s_1^2 - 4s_2)]\lambda^2 \\
F_3 &= -\lambda\{8\mu\delta^4 - 2[2(2\beta + 3s_1)\mu + 3\lambda^2]\delta^3 + \\
&\quad [12(s_1\beta + s_1^2 - 2s_2)\mu + (4\beta + 7s_1)\lambda^2]\delta^2 + \\
&\quad [\left(8(-s_1^2 + s_2)\beta + 2s_1(10s_2 - 3s_1^2)\right)\mu + \\
&\quad \left(2(\beta - 3s_1)\beta + 4(2s_2 - s_1^2)\right)\lambda^2]\delta + \\
&\quad [2s_1(s_1^2 - 2s_2)\beta + 2(s_1^4 - 6s_1^2s_2 + 8s_2^2)]\mu + \\
&\quad [s_1(-\beta + 2s_1)\beta + s_1(s_1^2 - 4s_2)]\lambda^2\}
\end{aligned}$$

$$\begin{aligned} G_0 &= -6\delta^4 + 6s_1\delta^3 + [2(3\beta - s_1)\beta + 10s_2 - 3s_1^2]\delta^2 + \\ &\quad [2(-2s_1\beta + 4s_2 - s_1^2)\beta + s_1(s_1^2 - 6s_2)]\delta + (s_1^2 - 2s_2)(\beta + s_1)\beta \\ G_1 &= 2\lambda(2\delta - s_1)[3\delta^2 - 2(\beta + s_1)\delta + (-\beta + 2s_1)\beta + s_1^2 - 4s_2] \end{aligned}$$

(In the formulae above s_1 and s_2 stand for the elementary symmetric expressions in α and γ only, namely, $s_1 = \alpha + \gamma$ and $s_2 = \alpha\gamma$.)

So generically, $(\alpha+\beta+\gamma+\delta)\sigma_1^2 - 4\lambda\sigma_1 - 16\mu = 0$ and $(\beta+\delta)\sigma_1 - 2\lambda = 0$ generate the trivial solutions of the system (1.2) associated to

$$P(X_1, X_2, X_3, X_4) = \alpha X_1^2 + \beta X_1X_2 + \gamma X_1X_3 + \delta X_1X_4 - \lambda X_1 - \mu,$$

as predicted, and $F_0\sigma_1^3 + F_1\sigma_1^2 + F_2\sigma_1 + F_3 = 0$ yields the three sets of nontrivial solutions.

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