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On the existence of solutions for nonconvex impulsive hyperbolic differential inclusions

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Abstract - We consider a Darboux problem associated to a nonconvex impulsive hyperbolic differential inclusion and we prove a Filippov type existence result. This result allows to obtain a relaxation theorem for the problem considered.

Key words and phrases : set-valued map, impulsive hyperbolic differential inclusion, relaxation.

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1. Introduction

In this paper we study hyperbolic impulsive differential inclusions of the form

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x \partial y}(x,y) \in F(x,y,u(x,y)) \quad a.e.\,(x,y) \in J_1 \times J_2, x \neq x_k, k=1,\dots,m\\
&\Delta u(x_k,y) = I_k(u(x_k,y)), \quad k=1,\dots,m,\\
&u(x,0) = \Phi(x), \quad x \in J_1,\\
&u(0,y) = \Psi(y), \quad y \in J_2,
\end{aligned} \tag{1.1}$$

where $J_1 = [0,T_1]$, $J_2 = [0,T_2]$, $F(.,.,.) : J_1 \times J_2 \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ is a set-valued map with non-empty values, $0 = x_0 < x_1 < \ldots < x_m < x_{m+1} = T_1$, $I_k \in C(\mathbb{R}^n,\mathbb{R}^n)$, $k = 1,\dots,m$ and $\Delta u|_{x=x_k} = u(x_k^+,y) - u(x_k^-,y)$, where $u(x_k^+,y) = \lim_{h\to 0+, v\to y} u(x_k+h,v)$ is the right limit and $u(x_k^-,y) = \lim_{h\to 0+, v\to y} u(x_k-h,v)$ is the left limit of $u(x,y)$ at (x_k,y).

Existence of solutions of problem (1.1) has been studied by many authors using fixed point techniques (see [1,2,4,5], etc.). For a detailed discussion on this topic with an exhaustive bibliography we refer to [2].

The aim of this note is to show that Filippov's ideas (see [3]) can be suitably adapted in order to obtain the existence of solutions of problem (1.1). We recall that for a first order differential inclusion defined by a lipschitzian set-valued map with nonconvex values Filippov's theorem (see

[3]) consists in proving the existence of a solution starting from a given ‘almost’ solution. Moreover, the result provides an estimate between the starting ‘quasi’ solution and the solution of the differential inclusion.

As an application of our main result we obtain a relaxation theorem for the problem considered. Namely, we prove that the solution set of the problem (1.1) is dense in the set of the relaxed solutions; i.e. the set of solutions of the differential inclusion whose right hand side is the convex hull of the original set-valued map.

Our results are extensions of previous results of Tuan ([6,7]) obtained for hyperbolic differential inclusions without impulses. In fact, in the proof of our theorems we essentially use several technical results due to Tuan ([6,7]).

The paper is organized as follows: in Section 2 we briefly recall some preliminary results that we will use in the sequel and in Section 3 we prove the main results of the paper.

2. Preliminaries

In this section we sum up some basic facts that we are going to use later.

Let (X, d) be a metric space. The Pompeiu-Hausdorff distance of the closed subsets $A, B \subset X$ is defined by $d_H(A, B) = \max\{d^*(A, B), d^*(B, A)\}$, $d^*(A, B) = \sup\{d(a, B);\, a \in A\}$, where $d(x, B) = \inf\{d(x, y); y \in B\}$. With $\overline{co}(A)$ we denote the closed convex hull of a set $A \subset X$.

Consider $J_1 = [0, T_1]$, $J_2 = [0, T_2]$ and $\Pi = [0, T_1] \times [0, T_2]$.

Let $C(\Pi, \mathbb{R}^n)$ be the Banach space of all continuous functions from Π to $\mathbb{R}^n$ with the norm $\|u\|_\infty = \sup\{\|u(x, y)\|;\, (x, y) \in \Pi\}$ where $\|\cdot\|$ is the Euclidian norm on $\mathbb{R}^n$, and let $L^1(\Pi, \mathbb{R}^n)$ be the Banach space of functions $u(\cdot, \cdot) : \Pi \to \mathbb{R}^n$ which are integrable, normed by

$$\|u\|_{L^1} = \int_0^{T_1}\int_0^{T_2} \|u(x, y)\| dxdy.$$

We denote by $AC^1(\Pi, \mathbb{R}^n)$ the space of absolutely continous functions $u(\cdot, \cdot)$ defined on Π with an integrable derivative $u_{xy}(\cdot, \cdot)$. We recall that a function $u(\cdot, \cdot)$ is said to be absolutely continous on Π if there exist $f(\cdot, \cdot) \in L^1(\Pi, \mathbb{R}^n)$, $g(\cdot) \in L^1(J_1, \mathbb{R}^n)$ and $h(\cdot) \in L^1(J_2, \mathbb{R}^n)$ such that

$$u(x, y) = \int_0^x\int_0^y f(\overline{x}, \overline{y})d\overline{y}d\overline{x} + \int_0^x g(\overline{x})d\overline{x} + \int_0^y h(\overline{y})d\overline{y} + u(0, 0) \quad \forall (x, y) \in \Pi.$$

It is well known that the space of absolutely continous functions is a Banach space endowed with the norm

$$\|u\|_{AC} = \int_0^{T_1}\int_0^{T_2} \|u_{xy}(x, y)\| dxdy + \int_0^{T_1} \|u_x(x, 0)\| dx +$$

$$+\int_0^{T_2} \|u_y(0,y)\|dy + \|u(0,0)\|.$$

A set-valued map $F(\cdot,\cdot):\Pi\times\mathbb{R}^n\to\mathcal{P}(\mathbb{R}^n)$ is called measurable if for any $w\in\mathbb{R}^n$ the function $(x,y)\mapsto d(w,F(x,y,u))=\inf\{\|w-v\|;\ v\in F(x,y,u)\}$ is measurable.

In order to define the solution of (1.1) we consider the space

$$\begin{aligned}\Omega=\{&u(\cdot,\cdot):\Pi\to\mathbb{R}^n\mid u_k(\cdot,\cdot)\in C(\Gamma_k,\mathbb{R}^n),\quad k=0,1,\ldots,m,\\ &\exists u(x_k^-,\cdot),\ u(x_k^+,\cdot),\quad k=1,\ \ldots,m\ \text{ with }\ u(x_k^-,\cdot)=u(x_k,\cdot)\},\end{aligned}$$

where $u_k(\cdot,\cdot)$ is the restriction of $u(\cdot,\cdot)$ to Γ_k, $\Gamma_k=(x_k,x_{k+1})\times[0,T_2]$, $k=0,1,\ldots,m$. Ω is a Banach space with the norm

$$\|u\|_\Omega=\max\{\|u_k\|_\infty;\quad k=0,1,\ldots,m\}.$$

Definition 2.1. (see [2]) *A function $u(\cdot,\cdot)\in\Omega\cap AC(\Gamma_k,\mathbb{R}^n)$, $k=1,\ldots,m$ is said to be a solution of* (1.1) *if there exists $v(\cdot,\cdot)\in L^1(\Pi,\mathbb{R}^n)$ such that $v(x,y)\in F(x,y,u(x,y))$ a.e* (Π) *and*

$$u(x,y)=z(x,y)+\int_0^x\int_0^y v(\overline{x},\overline{y})d\overline{y}d\overline{x}+\sum_{0<x_k<x} I_k(u(x_k,y)),$$

where $z(x,y)=\Phi(x)+\Psi(y)-\Phi(0)$.

We recall now some results that we are going to use in the next section.

Lemma 2.1. (see [6]) *Let $H(\cdot,\cdot):\Pi\to\mathcal{P}(\mathbb{R}^n)$ be a compact valued measurable multifunction and $v(\cdot,\cdot):\Pi\to\mathbb{R}^n$ a measurable function.*

Then there exists a measurable selection $h(\cdot,\cdot)$ of $H(\cdot,\cdot)$ such that

$$\|v(x,y)-h(x,y)\|=d(v(x,y),H(x,y)),\quad a.e.\ (\Pi).$$

Lemma 2.2. (see [7]) *Let $F(\cdot,\cdot):\Pi\to\mathcal{P}(\mathbb{R}^n)$ be a compact valued measurable multifunction such that there exists a constant $M\ge 0$ which verifies the condition*

$$d(0,F(x,y,0))\le M<+\infty.$$

Then for every $\overline{\varepsilon}>0$, and every measurable function $\overline{v}(\cdot,\cdot):\Pi\to\mathbb{R}^n$ which satisfies $\overline{v}(x,y)\in\overline{co}F(x,y)$ a.e. (Π), *there exists a measurable function $v(\cdot,\cdot):\Pi\to\mathbb{R}^n$ such that $v(x,y)\in F(x,y)$ a.e.* (Π) *and*

$$\sup_{(x,y)\in\Pi}\left|\left|\int_0^x\int_0^y(\overline{v}(\overline{x},\overline{y})-v(\overline{x},\overline{y}))d\overline{y}d\overline{x}\right|\right|\le\overline{\varepsilon}.$$

3. The main results

Consider next $\Phi(\cdot) \in C(J_1, \mathbb{R}^n)$, $\Psi(\cdot) \in C(J_2, \mathbb{R}^n)$. In order to prove our main result one needs the following assumption.

Hypothesis 3.1. Let $F(.,.,.) : \Pi \times \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ be a set-valued map with non-empty, compact values that verifies the following conditions.
(i) For all $u \in \mathbb{R}^n$, $F(\cdot, \cdot, u)$ is measurable.
(ii) There exists $L(\cdot, \cdot) \in L^1(\Pi, \mathbb{R}_+)$ such that for almost all $(x, y) \in \Pi$, $F(x, y, \cdot)$ is $L(x, y)$ - Lipschitz in the sense that

$$d_H(F(x, y, u_1), F(x, y, u_2)) \leq L(x, y)\|u_1 - u_2\|, \quad \forall u_1, u_2 \in \mathbb{R}^n.$$

(iii) There exist constants $c_k \geq 0$ such that

$$\|I_k(x) - I_k(y)\| \leq c_k\|x - y\|, \quad \forall x, y \in \mathbb{R}^n.$$

In what follows $g(\cdot, \cdot) \in L^1(\Pi, \mathbb{R}^n)$ is given such that there exists $\lambda(\cdot, \cdot) \in L^1(\Pi, \mathbb{R}_+)$ which satisfies

$$d(g(x, y), F(x, y, w(x, y))) \leq \lambda(x, y),$$

where $w(\cdot, \cdot)$ is a solution of the hyperbolic impulsive differential equation

$$\begin{aligned} &\frac{\partial^2 w}{\partial x \partial y}(x, y) = g(x, y), \quad a.e.\,(x, y) \in J_1 \times J_2, x \neq x_k, k = 1, \ldots, m \\ &\Delta w(x_k, y) = I_k(w(x_k, y)), \quad k = 1, \ldots, m, \\ &w(x, 0) = \Phi(x), \quad x \in J_1, \\ &w(0, y) = \Psi(y), \quad y \in J_2. \end{aligned} \tag{3.1}$$

Theorem 3.1. *Assume that Hypothesis* 3.1 *is satisfied and consider* $g(\cdot, \cdot)$, $\lambda(\cdot, \cdot)$, $w(\cdot, \cdot)$ *as above.*

If $\|L\|_1 + \sum_{k=1}^m c_k < 1$, *then the differential inclusion* (1.1) *has at least one solution* $u(\cdot, \cdot)$ *which satisfies*

$$\|u - w\|_\Omega \leq \frac{\|\lambda\|_1}{1 - \sum_{k=1}^m c_k - \|L\|_1}. \tag{3.2}$$

Proof. We set $v_0(x, y) = g(x, y)$, $u_0(x, y) = w(x, y)$. It follows from Lemma 2.1 and Hypothesis 3.1 that there exists a measurable function $v_1(\cdot, \cdot)$ such that $v_1(x, y) \in F(x, y, u_0(x, y))$ a.e. (Π) and for almost all $(x, y) \in \Pi$

$$\|g(x, y) - v_1(x, y)\| = d(g(x, y), F(x, y, u_0(x, y))) \leq \lambda(x, y).$$

Consider $u_1(\cdot,\cdot)$ the solution of the problem

$$\begin{aligned}
&\frac{\partial^2 u_1}{\partial x \partial y}(x,\, y) = v_1(x,y) \quad a.e.\,(x,y) \in J_1 \times J_2, x \neq x_k, k = 1,\dots,m,\\
&\Delta u_1(x_k,y) = I_k(u_1(x_k,y)), \quad k = 1,\dots,m,\\
&u_1(x,0) = \Phi(x), \quad x \in J_1,\\
&u_1(0,y) = \Psi(y), \quad y \in J_2.
\end{aligned} \tag{3.3}$$

Therefore,

$$u_1(x,y) = z(x,y) + \int_0^x\int_0^y v_1(\overline{x},\overline{y})d\overline{y}d\overline{x} + \sum_{0<x_k<x} I_k(u_1(x_k,y)).$$

For all $(x,y) \in \Pi$ we have

$$\begin{aligned}
\|u_1(x,y) - u_0(x,y)\| &\leq \| \int_0^x\int_0^y v_1(\overline{x},\overline{y})d\overline{y}d\overline{x} + \sum_{0<x_k<x} I_k(u_1(x_k,y)) -\\
&- \int_0^x\int_0^y v_0(\overline{x},\overline{y})d\overline{y}d\overline{x} - \sum_{0<x_k<x} I_k(u_0(x_k,y))\| \leq\\
\leq \int_0^x\int_0^y \|v_1(\overline{x},\overline{y}) - v_0(\overline{x},\overline{y})\|d\overline{y}d\overline{x} &+ \sum_{k=1}^m \|I_k(u_1(x_k,y)) - I_k(u_0(x_k,y))\| \leq\\
&\leq \int_0^x\int_0^y \lambda(\overline{x},\overline{y})d\overline{x}d\overline{y} + \sum_{k=1}^m c_k\|u_1(x_k,y) - u_0(x_k,y)\| \leq\\
&\leq \int_0^x\int_0^y \lambda(\overline{x},\overline{y})d\overline{x}d\overline{y} + \sum_{k=1}^m c_k\|u_1 - u_0\|_\Omega \leq\\
&\leq \int_0^{T_1}\int_0^{T_2} \lambda(\overline{x},\overline{y})d\overline{x}d\overline{y} + \sum_{k=1}^m c_k\|u_1 - u_0\|_\Omega.
\end{aligned}$$

Thus,

$$\|u_1 - u_0\|_\Omega \leq \int_0^{T_1}\int_0^{T_2} \lambda(\overline{x},\overline{y}) + \sum_{k=1}^m c_k\|u_1 - u_0\|_\Omega.$$

Therefore,

$$\|u_1 - u_0\|_\Omega \leq \frac{\|\lambda\|_1}{1 - \sum_{k=1}^m c_k}. \tag{3.4}$$

From Lemma 2.1 and Hypothesis 3.1 we deduce the existence of a measurable function $v_2(\cdot,\cdot)$ such that $v_2(x,y) \in F(x,y,u_0(x,y))$ a.e. (Π) and for almost all $(x,y) \in \Pi$

$$\begin{aligned}
&\|v_2(x,y) - v_1(x,y)\| \leq d(v_1(x,y), F(x,y,u_1(x,y))) \leq\\
&\leq d_H(F(x,y,u_0(x,y)), F(x,y,u_1(x,y))) \leq L(x,y)\|u_1(x,y) - u_0(x,y)\|.
\end{aligned}$$

Consider $u_2(\cdot,\cdot)$ the solution of the problem

$$\begin{aligned}
&\frac{\partial^2 u_2}{\partial x \partial y}(x,y) = v_2(x,y) \quad a.e.\,(x,y) \in J_1 \times J_2, x \neq x_k, k = 1,\dots,m,\\
&\Delta u_2(x_k,y) = I_k(u_2(x_k,y)), \quad k = 1,\dots,m,\\
&u_2(x,0) = \Phi(x), \quad x \in J_1,\\
&u_2(0,y) = \Psi(y), \quad y \in J_2.
\end{aligned} \tag{3.5}$$

It follows

$$u_2(x,y) = z(x,y) + \int_0^x \int_0^y v_2(\overline{x},\overline{y}) d\overline{y} d\overline{x} + \sum_{0<x_k<x} I_k(u_2(x_k,y)).$$

For all $(x,y) \in \Pi$ we have

$$\begin{aligned}
\|u_2(x,y) - u_1(x,y)\| \leq \| \int_0^x \int_0^y v_2(\overline{x},\overline{y}) d\overline{y} d\overline{x} + \sum_{0<x_k<x} I_k(u_2(x_k,y)) -\\
- \int_0^x \int_0^y v_1(\overline{x},\overline{y}) d\overline{y} d\overline{x} - \sum_{0<x_k<x} I_k(u_1(x_k,y))\| \leq\\
\leq \int_0^x \int_0^y \|v_2(\overline{x},\overline{y}) - v_1(\overline{x},\overline{y})\| d\overline{y} d\overline{x} + \sum_{k=1}^m \|I_k(u_2(x_k,y)) - I_k(u_1(x_k,y))\| \leq\\
\leq \int_0^x \int_0^y L(\overline{x},\overline{y}) \|u_1(\overline{x},\overline{y}) - u_0(\overline{x},\overline{y})\| d\overline{y} d\overline{x} + \sum_{k=1}^m c_k \|u_2(x_k,y) - u_1(x_k,y)\|.
\end{aligned}$$

From (3.4) we have

$$\|u_2 - u_1\|_\Omega \leq \int_0^{T_1} \int_0^{T_2} L(\overline{x},\overline{y}) \frac{\|\lambda\|_1}{1 - \sum_{k=1}^m c_k} d\overline{y} d\overline{x} + \sum_{k=1}^m c_k \|u_2 - u_1\|_\Omega.$$

Hence,

$$\|u_2 - u_1\|_\Omega \leq \frac{\|\lambda\|_1 \cdot \|L\|_1}{(1 - \sum_{k=1}^m c_k)^2}. \tag{3.6}$$

Assume that for some $p \geq 1$ we have constructed $(u_i)_{i=1}^p$ with u_p satisfying

$$\|u_p - u_{p-1}\|_\Omega \leq \frac{\|L\|_1^{p-1} \cdot \|\lambda\|_1}{(1 - \sum_{k=1}^m c_k)^p}.$$

Using Lemma 2.1 and Hypothesis 3.1 we obtain that there exists a measurable function $v_{p+1}(x,y) \in F(x,y,u_p(x,y))$ a.e. (Π) such that for almost all $(x,y) \in \Pi$ holds

$$\begin{aligned}
\|v_{p+1}(x,y) - v_p(x,y)\| \leq d(v_{p+1}(x,y), F(x,y,u_{p-1}(x,y))) \leq\\
\leq d_H(F(x,y,u_p(x,y)), F(x,y,u_{p-1}(x,y))) \leq L(x,y)\|u_p(x,y) - u_{p-1}(x,y)\|.
\end{aligned}$$

We consider $u_{p+1}(\cdot,\cdot)$ the solution of the problem

$$\begin{aligned}&\frac{\partial^2 u_{p+1}}{\partial x \partial y}(x,y) = v_{p+1}(x,y) \quad a.e.\,(x,y)\in J_1\times J_2, x\neq x_k, k=1,\dots,m,\\ &\Delta u_{p+1}(x_k,y) = I_k(u_{p+1}(x_k,y)), \quad k=1,\dots,m,\\ &u_{p+1}(x,0)=\Phi(x), \quad x\in J_1,\\ &u_{p+1}(0,y)=\Psi(y), \quad y\in J_2.\end{aligned} \tag{3.7}$$

Therefore

$$u_{p+1}(x,y) = z(x,y) + \int_0^x\int_0^y v_{p+1}(\overline{x},\overline{y})d\overline{y}d\overline{x} + \sum_{0<x_k<x} I_k(u_{p+1}(x_k,y)). \tag{3.8}$$

We have

$$\begin{aligned}\|u_{p+1}(x,y)-u_p(x,y)\| \le \| \int_0^x\int_0^y v_{p+1}(\overline{x},\overline{y})d\overline{y}d\overline{x}+\\ + \sum_{0<x_k<x} I_k(u_{p+1}(x_k,y)) - \int_0^x\int_0^y v_p(\overline{x},\overline{y})d\overline{y}d\overline{x} - \sum_{0<x_k<x} I_k(u_p(x_k,y))\| \le\\ \int_0^x\int_0^y \|v_{p+1}(\overline{x},\overline{y}) - v_p(\overline{x},\overline{y})\|d\overline{y}d\overline{x} + \sum_{k=1}^m \|I_k(u_{p+1}(x_k,y)) - I_k(u_p(x_k,y))\| \le\\ \int_0^x\int_0^y L(\overline{x},\overline{y})\|u_p(\overline{x},\overline{y}) - u_{p-1}(\overline{x},\overline{y})\|d\overline{y}d\overline{x} + \sum_{k=1}^m c_k\|u_{p+1}(x_k,y)-u_p(x_k,y)\|.\end{aligned}$$

So,

$$\|u_{p+1}-u_p\|_\Omega \le \frac{\|L\|_1^{p-1}\cdot\|\lambda\|_1}{(1-\sum_{k=1}^m c_k)^p}\int_0^{T_1}\int_0^{T_2} L(\overline{x},\overline{y})d\overline{y}d\overline{x} + (\sum_{k=1}^m c_k)\|u_{p+1}-u_p\|_\Omega.$$

We obtain

$$\|u_{p+1}-u_p\|_\Omega \le \frac{\|L\|_1^{p}\cdot\|\lambda\|_1}{(1-\sum_{k=1}^m c_k)^{p+1}}, \quad \forall p\ge 1. \tag{3.9}$$

Therefore $(u_p(\cdot,\cdot))_{p\ge 0}$ is a Cauchy sequence in the Banach space Ω, so it converges to $u(\cdot,\cdot)\in\Omega$. Since, for almost all $(x,y)\in\Pi$ one has

$$\|v_{p+1}(x,y)-v_p(x,y)\| \le L(x,y)\|u_p(x,y)-u_{p-1}(x,y)\| \le L(x,y)\|u_p-u_{p-1}\|_\Omega,$$

it follows that $(v_p(\cdot,\cdot))_p$ is a Cauchy sequence in the Banach space $L^1(\Pi,\mathbb{R}^n)$ and thus it converges to $v(\cdot,\cdot)\in L^1(\Pi,\mathbb{R}^n)$. Passing to the limit in (3.8) and using Lebesgue dominated convergence theorem we get

$$u(x,y) = z(x,y) + \int_0^x\int_0^y v(\overline{x},\overline{y})d\overline{y}d\overline{x} + \sum_{0<x_k<x} I_k(u(x_k,y)).$$

Moreover, since the values of $F(.,.,.)$ are compact and since one has $v_{p+1}(x,y) \in F(x,y,u_p(x,y))$ a.e. (Π) for any $p \geq 0$, passing to the limit with $p \to \infty$ we obtain $v(x,y) \in F(x,y,u(x,y))$ a.e. (Π) and $u(.,.)$ is a solution for the problem (1.1).

It remains to prove the estimation (3.2). It holds

$$\begin{aligned}
\|u_p - u_0\|_\Omega &\leq \|u_p - u_{p-1}\|_\Omega + \ldots + \|u_2 - u_1\|_\Omega + \|u_1 - u_0\|_\Omega \\
&\leq \frac{\|L\|_1^{p-1} \cdot \|\lambda\|_1}{(1-\sum_{k=1}^m c_k)^p} + \cdots + \frac{\|L\|_1 \cdot \|\lambda\|_1}{(1-\sum_{k=1}^m c_k)^2} + \frac{\|\lambda\|_1}{1-\sum_{k=1}^m c_k} \\
&= \frac{\|\lambda\|_1}{1-\sum_{k=1}^m c_k} \cdot \frac{1-\left(\frac{\|L\|_1}{1-\sum_{k=1}^m c_k}\right)^p}{1-\frac{\|L\|_1}{1-\sum_{k=1}^m c_k}} \leq \frac{\|\lambda\|_1}{1-\sum_{k=1}^m c_k - \|L\|_1}
\end{aligned}$$

and the proof is complete. □

Remark 3.1. If in Theorem 3.1 one takes $g = 0$, $w = 0$ and $\lambda = L$, then Theorem 3.1 yields Theorem 10.5 in [2].

Moreover, in this case our result provides an a priori estimation for the solution of problem (1.1) of the form

$$\|u\|_\Omega \leq \frac{\|L\|_1}{1-\sum_{k=1}^m c_k - \|L\|_1}.$$

As we already pointed out, Theorem 3.1 allows to obtain a relaxation theorem for problem (1.1). In what follows, we are concerned also with the convexified (relaxed) impulsive hyperbolic differential inclusion

$$\begin{aligned}
&\frac{\partial^2 u}{\partial x \partial y}(x,y) \in \overline{co}F(x,y,u(x,y)) \quad a.e.\ (x,y) \in J_1 \times J_2, x \neq x_k, k = 1,\ldots,m, \\
&\Delta u(x_k,y) = I_k(u(x_k,y)) \quad k = 1,\ldots,m, \\
&u(x,0) = \Phi(x), \quad x \in J_1, \\
&u(0,y) = \Psi(y), \quad y \in J_2.
\end{aligned} \tag{3.10}$$

Hypothesis 3.2. We assume that Hypothesis 3.1 is satisfied and there exists $M \geq 0$ such that $d(0, F(x,y,0)) \leq M < +\infty$ a.e. (Π).

Theorem 3.2. *We assume that Hypothesis* 3.2 *is satisfied and let* $\overline{u}(\cdot,\cdot)$ *be a solution of the convexified problem* (3.10).

If $\|L\|_1 + \sum_{k=1}^m c_k < 1$*, then, for all* $\varepsilon > 0$*, there exists a solution* $u(\cdot,\cdot)$ *of the problem* (1.1) *such that*

$$\|u - \overline{u}\|_\Omega \leq \varepsilon.$$

Proof. Let $\overline{u}(\cdot,\cdot)$ be a solution of the convexified problem. Then there exists $\overline{v}(\cdot,\cdot) \in L^1(\Pi, \mathbb{R}^n)$, $\overline{v}(x,y) \in \overline{co}F(x,y,\overline{u}(x,y))$ a.e. (Π) such that

$$\overline{u}(x,y) = z(x,y) + \int_0^x\int_0^y \overline{v}(\overline{x},\overline{y})d\overline{y}d\overline{x} + \sum_{0<x_k<x} I_k(\overline{u}(x_k,y)).$$

From Lemma 2.2, for all $\delta > 0$, there exists $\tilde{v}(x,y) \in F(x,y,\overline{u}(x,y))$ a.e. (Π) such that

$$\sup_{(x,y)\in\Pi} \left\| \int_0^x\int_0^y (\tilde{v}(\overline{x},\overline{y}) - \overline{v}(\overline{x},\overline{y}))d\overline{y}d\overline{x} \right\| \leq \delta.$$

Consider $\tilde{u}(\cdot,\cdot)$ the solution of the problem

$$\begin{aligned}
&\frac{\partial^2 \tilde{u}}{\partial x \partial y}(x,\, y) = \tilde{v}(x,y) \quad a.e.\,(x,y) \in J_1 \times J_2, x \neq x_k, k = 1,\dots,m,\\
&\Delta\tilde{u}(x_k,y) = I_k(\tilde{u}(x_k,y)), \quad k = 1,\dots,m,\\
&\tilde{u}(x,0) = \Phi(x), \quad x \in J_1,\\
&\tilde{u}(0,y) = \Psi(y), \quad y \in J_2.
\end{aligned} \tag{3.11}$$

Therefore,

$$\tilde{u}(x,y) = z(x,y) + \int_0^x\int_0^y \tilde{v}(\overline{x},\overline{y})d\overline{y}d\overline{x} + \sum_{0<x_k<x} I_k(\tilde{u}(x_k,y)).$$

We have

$$\begin{aligned}
\|\tilde{u}(x,y) - \overline{u}(x,y)\| = \| &\int_0^x\int_0^y \tilde{v}(\overline{x},\overline{y})d\overline{y}d\overline{x} + \sum_{0<x_k<x} I_k(\tilde{u}(x_k,y)) - \\
&\int_0^x\int_0^y \overline{v}(\overline{x},\overline{y})d\overline{y}d\overline{x} - \sum_{0<x_k<x} I_k(\overline{u}(x_k,y))\| \leq \\
\leq || \int_0^x\int_0^y \tilde{v}(\overline{x},\overline{y}) - \overline{v}(\overline{x},\overline{y})d\overline{y}d\overline{x}|| &+ \sum_{k=1}^m \|I_k(\tilde{u}(x_k,y)) - I_k(\overline{u}(x_k,y))\| \leq \\
\leq \delta + \sum_{k=1}^m c_k \|\tilde{u}(x_k,y) - \overline{u}(x_k,y)\| &\leq \delta + \sum_{k=1}^m c_k \|\tilde{u} - \overline{u}\|_\Omega.
\end{aligned}$$

Hence

$$\|\tilde{u} - \overline{u}\|_\Omega \leq \delta + \sum_{k=1}^m c_k \|\tilde{u} - \overline{u}\|_\Omega$$

and then

$$\|\tilde{u} - \overline{u}\|_\Omega \leq \frac{\delta}{1 - \sum_{k=1}^m c_k}. \tag{3.12}$$

We now apply Theorem 3.1 for the 'quasi' solution $\tilde{u}(\cdot,\cdot)$. One has

$$\lambda(x,y) = d(\tilde{v}(x,y), F(x,y,\tilde{u}(x,y))) \le d_H(F(x,y,\overline{u}(x,y)), F(x,y,\tilde{u}(x,y)))$$
$$\le L(x,y)\|\tilde{u}(x,y) - \overline{u}(x,y)\| \le L(x,y)\cdot \frac{\delta}{1-\sum_{k=1}^m c_k},$$

which shows that $\lambda(\cdot,\cdot) \in L^1(\Pi, \mathbb{R}^n)$.

From Theorem 3.1 there exists a solution $u(\cdot,\cdot)$ of (1.1) such that

$$\|u-\tilde{u}\|_\Omega \le \frac{\|L\|_1}{1-\sum_{k=1}^m c_k - \|L\|_1}\cdot \frac{\delta}{1-\sum_{k=1}^m c_k}. \tag{3.13}$$

From (3.12) and (3.13) it follows that

$$\|u-\overline{u}\|_\Omega \le \|u-\tilde{u}\|_\Omega + \|\tilde{u}-\overline{u}\|_\Omega \le$$
$$\le \frac{\|L\|_1}{1-\sum_{k=1}^m c_k - \|L\|_1}\cdot \frac{\delta}{1-\sum_{k=1}^m c_k} + \frac{\delta}{1-\sum_{k=1}^m c_k} = \frac{\delta}{1-\sum_{k=1}^m c_k - \|L\|_1}.$$

Since $\delta > 0$ is arbitrary, it is enough to take

$$\delta = (1-\sum_{k=1}^m c_k - \|L\|_1)\cdot \varepsilon$$

in order to obtain the conclusion of the theorem. □

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