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Unicity properties and algebraic properties for the solutions of the functional equation $f \circ f + af + b1_{\mathbb{R}} = 0$ (II)

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Abstract - Throughout this paper we shall deal with the functional equation in the title, for real a, b and $b \neq 0$. This functional equation was completely solved in a previous paper. Namely we found all continuous solutions of the aforementioned functional equation. In this paper we shall establish bijections between the set of solutions and some special sets of functions.

Key words and phrases : iterative functional equation, continuous solution, homeomorphism.

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1. Introduction

We consider the functional equation

$$f \circ f(x) + af(x) + bx = 0, \quad x \in \mathbb{R} \tag{1.1}$$

where the signs of a and b are taken according to the conventions of [4].

In the first part of this paper (see [4]) we proved the existence and we presented the general form of the solutions of the aforementioned equation. Namely, the aim of the first part of this paper was to establish what conditions can guarantee the uniqueness of the continuous solution of this functional equation. More precisely, if two solutions coincide on an interval under some conditions, then they coincide everywhere. In the first part we established what conditions must fulfill this interval in each case. Let $\Delta = a^2 - 4b$ and let r_1, r_2 be the solutions of the equation $x^2 + ax + b = 0$. We proved the following theorem:

Theorem 1.1. (see [4, Theorem 2.1]) *Let us consider the functional equation* (1.1) *in case* $\Delta > 0$. *Then the following statements hold.*

(i) If $1 < r_1 < r_2$ *and two continuous solutions coincide on* $I = [a', b']$, $I \subset (0, \infty)$ *and* $\dfrac{b'}{a'} \geq r_2$, *then they coincide on* $(0, \infty)$.

A similar result holds if $I \subset (-\infty, 0)$.

If $I = [0, a']$ *the solutions coincide on* $(0, \infty)$. *A similar result holds if* $I = [a', 0]$.

(ii) If $r_2 < r_1 < -1$ *and two continuous solutions coincide on* $I = [a', b']$, $I \subset (0, \infty)$ *and* $\dfrac{b'}{a'} \geq r_2^2$ *(also if* $I \subset (-\infty, 0)$ *and* $I = [a', b']$ *and* $\dfrac{a'}{b'} \geq r_2^2$*), then they coincide on* $\mathbb{R}$.

If $0 \in I$ *the two solutions coincide on* $\mathbb{R}$.

(iii) If $r_1 < 1 < r_2$, *let us consider two solutions* f *and* g *such that* $f(0) \neq 0, g(0) \neq 0$ *and there exists* $a' > 0$ *such that* f*and* g *coincide on* $[0, a']$. *Then* f*and* g *coincide on* $\mathbb{R}$. *We have a similar result for the case* $[a', 0]$, $a' < 0$.

The aim of the paper is to prove the existence of several bijections between the set of solutions of equation (1.1) and some special sets of functions. We will consider three cases: $1 < r_1 < r_2$, $r_2 < r_1 < -1$ and $0 < r_1 < 1 < r_2$.

2. Algebraic properties of the solutions

A. Case $1 < r_1 < r_2$.

The set of continuous solutions of equation (1.1) (see [4, Theorem 1.2]) will be denoted in the sequel by S. From the Calibration Theorem (see [4]), we have for a solution f of equation (1.1) the inequality

$$r_1(x - y) \leq f(x) - f(y) \leq r_2(x - y), \qquad \forall x, y \in \mathbb{R}. \tag{2.1}$$

We notice that the set of continuous solutions of equation (1.1) on $(0, \infty)$, denoted by S_+, depends upon three parameters: $x_0 > 0$, $x_1 \in [x_0 r_1, x_0 r_2]$ and $f_0 : [x_0, x_1] \to [x_1, x_2]$. Here $(x_n)_{n \in \mathbb{Z}}$ is the sequence described in [4, Theorem 1.2], and f_0 is continuous and bijective which fulfills the inequality $r_1(x - y) \leq f_0(x) - f_0(y) \leq r_2(x - y)$ for $x > y$.

Under these assumptions, it follows from Theorem 1.1 that, starting with (x_0, x_1, f_0) under the previous conditions, they uniquely determine the solution on $(0, \infty)$, because two solutions which coincide on $[x_0, x_1]$ coincide on $(0, \infty)$, according to Theorem 1.1.

Notation 2.1. In the sequel we will use the following notation.

(i) $V(x_1) = \{f_0 : [x_0, x_1] \to [x_1, x_2] \,|\, x_1 \in [r_1 x_0, r_2 x_0],\ x_2 = a x_1 - b x_0,$ f_0 continuous, bijective and fulfills (2.1) for $x > y,\ x, y \in [x_0, x_1]\}$.

(ii) $V = \bigcup\limits_{x_1 \in [x_0 r_1, x_0, r_2]} V(x_1)$.

(iii) $S_+ = \{f : (0,\infty) \to (0,\infty) \,|\, f$ is a continuous solution of equation (1.1) on $(0,\infty)\}$.

(iv) $S_- = \{f : (-\infty,0) \to (-\infty,0) \,|\, f$ is a continuous solution of equation (1.1) on $(0,\infty)\}$.

(v) $B = \Big\{ h : [0,1] \to [0,1] \mid h(0) = 0,\ h$ is continuous, increasing,

$$h(1) \in \left[\frac{r_1}{r_2}, 1\right];\quad \frac{h(x)-h(y)}{x-y} \in \left[\frac{r_1}{r_2}, 1\right] \quad \text{for } x, y \in [0,1], x > y \Big\}.$$

Lemma 2.1. *There exists a bijection $H : V \to S_+$.*

Proof. Let $f_0 \in V$. There exists $x_1 \in [x_0 r_1, x_0 r_2]$ such that $f_0 \in V(x_1)$. We define $H(f_0) = f$, where f is the only function that fulfills the conditions $f(x_0) = x_1$, $f \in S_+$ and $f(x) = f_0(x)$ for all $x \in [x_0, x_1]$ (see Theorem 1.1). Then it follows that f fulfills (2.1), for all $x > y \geq 0$. Hence H is well defined.

a) H is injective. Let $f_1, f_2 \in S_+$ be such that $H\left(f_0^1\right) = f_1$; $H\left(f_0^2\right) = f_2$ and $f_1 = f_2$. Let $x_0 > 0$ and $x_1 \in [x_0 r_1, x_0 r_2]$ be such that $f_1(x_0) = f_2(x_0) = x_1$, $f_0^1 = f_1|_{[x_0,x_1]}$; $f_0^2 = f_2|_{[x_2,x_1]}$. Because $f_1|_{[x_0,x_1]} = f_2|_{[x_0,x_1]}$, it follows that $f_0^1 = f_0^2$, so H is injective.

b) H is surjective. Let $f \in S_+$. We look for $f_0 \in V$ which fulfills the condition $H(f_0) = f$. Consider $x_0 > 0$; $x_1 = f(x_0)$; $f_0 = f|_{[x_0,x_1]}$; $x_2 = f(x_1)$; $f_0 : [x_0, x_1] \to [x_1, x_2]$. Let us prove that $f_0 \in V(x_1)$. Because $f \in S$ we have $r_1 x_0 \leq f(x_0) \leq r_2 x_0$ (see [4, Lemma 1.1]). So $x_1 \in [r_1 x_0, r_2 x_0]$. Since $f(f(x_0)) = a \cdot f(x_0) - b x_0$, we have $x_2 = a x_1 - b x_0$. Because f fulfills (2.1), it follows that f_0 fulfills (2.1) (this results from the fact that $f_0 = f|_{[x_0,x_1]}$). Hence H is surjective and it follows that H is bijective. □

Theorem 2.1. *There exists a bijection $F : B \to S_+$.*

Proof. Firstly we shall prove that there exists $T : B \to V$, T bijective. Let $x_0 > 0$ and $x_1 \in [r_1 x_0, r_2 x_0]$, $x_2 = a x_1 - b x_0$.

Let $h \in B$. If $x \in [x_0, x_1]$, there exists $t \in [0,1]$ such that $t = \dfrac{x - x_0}{x_1 - x_2}$ and conversely. We define now φ with the aid of h:

$$\varphi : [x_0, x_1] \to [x_1, x_2], \quad \varphi(x) = \frac{x_2 - x_1}{h(1)} h\left(\frac{x - x_0}{x_1 - x_0}\right) - x_1.$$

We can take $x_1 = x_0 \dfrac{h(1) r_2 - b}{h(1) r_2 - a + 1}$ (namely, we shall prove that for this x_1 one has $x_1 \in [r_1 x_0, r_2 x_0]$). According to the definition of the set B, $h(1) \in \left[\dfrac{r_1}{r_2}, 1\right]$.

Let us consider $g : \left[\frac{r_1}{r_2}, 1\right] \to \mathbb{R}$, given by $g(x) = x_0 \frac{(xr_2 - b)}{xr_2 - a + 1}$. Then

$$g'(x) = x_0 \frac{(r_2(xr_2 - a + 1) - (xr_2 - b) r_2)}{(xr_2 - a + 1)^2} = x_0 \frac{-ar_2 + r_2 + br_2}{(xr_2 - a + 1)^2} =$$

$$= x_0 r_2 \frac{b - a + 1}{(xr_2 - a + 1)^2} = \frac{x_0 r_2 (r_1 - 1)(r_2 - 1)}{(xr_2 - a + 1)^2}.$$

It follows that g is increasing, so $g(x) \in \left[g\left(\frac{r_1}{r_2}\right), g(1)\right]$. We have

$$g\left(\frac{r_1}{r_2}\right) = \frac{x_0(r_1 - b)}{r_1 - r_1 - r_2 + 1} = \frac{x_0 r_1 (1 - r_2)}{1 - r_2} = x_0 r_1.$$

Similarly we can prove that $g(1) = x_0 r_2$. Hence $x_1 \in [x_0 r_1, x_0 r_2]$.

We define $T : B \to V$ by $T(h) = \varphi$.

a) We shall prove that T is well defined; more precisely, starting with $h \in B$ and defining φ as above, let us prove that $\varphi \in V(x_1)$. Obviously $\varphi(x_0) = x_1$ and $\varphi(x_1) = x_2$; because h is increasing it follows that φ is increasing. Hence φ is surjective and increasing, therefore φ is bijective.

Let us prove that $\frac{\varphi(y) - \varphi(x)}{y - x} \in [r_1, r_2]$, for $y > x$. By using the relations

$$\begin{cases} x = x_0 + t_1(x_1 - x_0) \\ y = x_0 + t_2(x_1 - x_0), \end{cases}$$

we deduce that $y - x = (t_2 - t_1)(x_1 - x_0)$. Therefore $y > x$ if and only if $t_2 > t_1$. According to the definition of x_1 it follows that

$$x_1 h(1) r_2 + x_1(-a + 1) = x_0 h(1) r_2 - x_0 b.$$

One easily deduces the relation

$$h(1) = \frac{1}{r_2}\left(\frac{x_2 - x_1}{x_1 - x_0}\right). \tag{2.2}$$

But

$$h(t_1) = \frac{\varphi(x) \cdot h(1)}{x_2 - x_1} - x_1 \cdot \frac{h(1)}{x_2 - x_1} \text{ and}$$

$$h(t_2) = \frac{\varphi(y) \cdot h(1)}{x_2 - x_1} - x_1 \cdot \frac{h(1)}{x_2 - x_1}.$$

It follows that

$$\frac{h(t_2) - h(t_1)}{t_2 - t_1} = \frac{h(1)}{(x_2 - x_1)} \cdot \left(\frac{\varphi(y) - \varphi(x)}{y - x}\right) \cdot \left(\frac{y - x}{t_2 - t_1}\right) =$$

$$= \frac{\varphi(y)-\varphi(x)}{y-x} \cdot \frac{h(1)}{x_2-x_1} \cdot (x_1-x_0) =$$

$$= \left(\frac{\varphi(y)-\varphi(x)}{y-x}\right) \cdot \frac{1}{r_2} \cdot \frac{x_2-x_1}{x_1-x_0} \cdot \frac{x_1-x_0}{x_2-x_1} = \frac{1}{r_2}\left(\frac{\varphi(y)-\varphi(x)}{y-x}\right). \tag{2.3}$$

Hence

$$\frac{\varphi(y)-\varphi(x)}{y-x} = r_2 \cdot \left(\frac{h(t_2)-h(t_1)}{t_2-t_1}\right).$$

Since $\frac{h(t_2)-h(t_1)}{t_2-t_1} \in \left[\frac{r_1}{r_2}, 1\right]$, one has $\frac{\varphi(y)-\varphi(x)}{y-x} \in [r_1, r_2]$ and therefore $\varphi \in V(x_1)$. Hence T is well defined. We shall prove that $C = [r_1, r_2]$, where

$$C = \bigcup_{x_1 \in [x_0 r_1, x_0 r_2]} \bigcup_{\varphi \in V(x_1)} \left\{ \frac{\varphi(y)-\varphi(x)}{y-x} \middle| y > x,\ x, y \in [x_0, x_1] \right\}.$$

Obviously $C \subset [r_1, r_2]$. On the other hand, one has $C_1 \subset C$, where

$$C_1 = \bigcup_{x_1 \in [x_0 r_1,\ x_0 r_2]} \bigcup_{\varphi\ \in\ W\ (x_1)} D(x_1, \varphi) \quad \text{with}$$

$$D(x_1, \varphi) = \{\frac{\varphi(y)-\varphi(x)}{y-x} \mid y > x, x, y \in [x_0, x_1]\}$$

and $W\ (x_1) = V\ (x_1) \bigcap \{\ \varphi : [0,1] \to \mathbb{R} \mid \varphi(x) = px + q\}$.

We shall prove that $C_1 = [r_1, r_2]$. Let $\alpha \in [x_0 r_1, x_0 r_2]$ be arbitrarily taken. We look for φ linear having the form $\varphi(x) = px + q$ such that $p = \alpha$.

Hence $\varphi(x_0) = x_1$, $\varphi(x_1) = x_2$ if and only if $px_0 + q = x_1$ and $px_1 + q = x_2$. Consequently $p\,(x_1 - x_0) = x_2 - x_1$, that is

$$p = \frac{x_2-x_1}{x_1-x_0} = \frac{ax_1 - bx_0 - x_1}{x_1-x_0} = \frac{(a-1)x_1 - bx_0}{x_1-x_0},$$

with $x_1 \in [x_0 r_1, x_0 r_2]$.

We write $g_1(x) = \frac{(a-1)x - bx_0}{x - x_0}$, $g_1 : [x_0 r_1, x_0 r_2] \to \mathbb{R}$. Then

$$g_1'(x) = \frac{(a-1)\,(x-x_0) - ((a-1)x - bx_0)}{(x-x_0)^2} = \frac{x_0\,(b-a+1)}{(x-x_0)^2} \geq 0. \tag{2.4}$$

It follows that g_1 is increasing ($g_1 \uparrow$). We have

$$g_1(x_0 r_1) = \frac{x_0 r_1^2 - x_0 r_1}{x_0 r_1 - x_0} = r_1$$

and similarly $g_1\,(x_0 r_2) = r_2$. Because g_1 is continuous and increasing, it follows that $g_1([x_0 r_1, x_0 r_2]) = [r_1, r_2]$. Hence, for all $\alpha \in [x_0 r_1, x_0 r_2]$ there exist p and $\varphi(x) = px + q$, such that $p = \alpha$.

Consequently $C_1 = [r_1, r_2]$ and $C = [r_1, r_2]$.

b) We shall prove that T is injective. Let $h_1, h_2 \in B$, $h_1 \neq h_2$. Let us prove that $T(h_1) \neq T(h_2)$, $T(h_1) = \varphi_1$ and $T(h_2) = \varphi_2$, where $\varphi_1, \varphi_2 \in V$. There exist x_1 and x_1' such that $\varphi_1 \in V(x_1)$ and $\varphi_2 \in V(x_1')$.

1) If $x_1 \neq x_1'$ we have $\varphi_1(x_0) = x_1$ and $\varphi_2(x_0) = x_1'$ with $x_1 \neq x_1'$, hence we have $\varphi_1 \neq \varphi_2$.

2) If $\varphi_1, \varphi_2 \in V(x_1)$ one has $x_1 = x_1'$ and so $x_2 = x_2'$. On the other hand, $h_1(1) = \frac{1}{r_2}\left(\frac{x_2 - x_1}{x_1 - x_0}\right) = \frac{1}{r_2}\left(\frac{x_2' - x_1'}{x_1' - x_0}\right) = h_2(1)$. Because $h_1 \neq h_2$, there exists $\alpha \in (0,1)$ such that $h_1(\alpha) \neq h_2(\alpha)$. We choose $x_\alpha = x_0 + \alpha(x_1 - x_0)$, $x_\alpha \in (x_0, x_1)$. It follows that

$$\varphi_1(x_\alpha) = \frac{x_2 - x_1}{h_1(1)} \cdot h_1\left(\frac{x_\alpha - x_0}{x_1 - x_0}\right) + x_1 = \frac{x_2 - x_1}{h_1(1)} \cdot h_1(\alpha) + x_1.$$

Similarly

$$\varphi_2(x_\alpha) = \left(\frac{x_2' - x_1'}{h_2(1)}\right) \cdot h_2(\alpha) + x_1'.$$

Since $x_1 = x_1'$, $x_2 = x_2'$ and $h_1(\alpha) \neq h_2(\alpha)$, it follows that $\varphi_1(x_\alpha) \neq \varphi_2(x_\alpha)$. Hence $\varphi_1 \neq \varphi_2$, which implies that T is injective.

c) We shall prove that T is surjective. Let $\varphi \in V$; there exists x_1 such that $\varphi \in V(x_1)$. Let us prove that there exists $h \in B$ such that $T(h) = \varphi$. Because $\varphi \in V(x_1)$, it follows that $\frac{x - x_0}{x_1 - x_0} \in [0,1]$, for all $x \in [x_0, x_1]$. We denote $t = \frac{x - x_0}{x_1 - x_0}$; $t \in [0,1]$. Let us define

$$h(t) = \left(\frac{\varphi(x_0 + t(x_1 - x_0)) - x_1}{x_2 - x_1}\right)\left(\frac{x_2 - x_1}{x_1 - x_0}\right) \cdot \frac{1}{r_2} \text{ for any } t \in [0,1].$$

Let us prove that $h \in B$. Obviously h is continuous, h is increasing, $h(0) = 0$ and $h(1) = \frac{1}{r_2}\left(\frac{x_2 - x_1}{x_1 - x_0}\right)$. Since $x_2 = ax_1 - bx_0$ one has $h(1) = \frac{1}{r_2}\left(\frac{(a-1)x_1 - bx_0}{x_1 - x_0}\right)$. Due to (2.2), if $x_1 \in [r_1 x_0, r_2 x_0]$, one has $\frac{1}{r_2}\left(\frac{x_2 - x_1}{x_1 - x_0}\right) \in \left[\frac{r_1}{r_2}, 1\right]$, for all $x_1 \in [r_1 x_0, r_2 x_0]$. Hence $h(1) \in \left[\frac{r_1}{r_2}, 1\right]$. Thus it follows that $h(1)$ is well defined. Let $0 \leq t_1 < t_2 \leq 1$ and write

$$\begin{cases} x = x_0 + t_1(x_1 - x_0) \\ y = x_0 + t_2(x_1 - x_0). \end{cases}$$

Due to (2.3) we have:

$$\frac{\varphi(y) - \varphi(x)}{y - x} = r_2\left(\frac{h(t_2) - h(t_1)}{t_2 - t_1}\right).$$

Consequently $\dfrac{\varphi(y)-\varphi(x)}{y-x}\in[r_1,r_2]$, which implies $\dfrac{h(t_2)-h(t_1)}{t_2-t_1}\in\left[\dfrac{r_1}{r_2},1\right]$ and thus it follows that $h\in B$. Let us prove now that $T(h)=\varphi$. We have

$$T(h)(t)=\left(\frac{x_2-x_1}{h(1)}\right)\left(\frac{\varphi(x)-x_1}{x_2-x_1}\right)\left(\frac{x_2-x_1}{x_1-x_0}\right)\cdot\frac{1}{r_2}+x_1=$$

$$=\left(\frac{\varphi(x)-x_1}{h(1)}\right)\left(\frac{x_2-x_1}{x_1-x_0}\right)\cdot\frac{1}{r_2}+x_1.$$

Since $h(1)=\dfrac{x_2-x_1}{(x_1-x_0)\,r_2}$, it follows that $T(h)(t)=\varphi(x)$. We deduce that $T(h)=\varphi$, which means that T is surjective. In conclusion, T is bijective.

We define $F=H\circ T$; $H:V\to S_+$, $T:B\to V$ and so $H\circ T:B\to S_+$. Since H and T are bijective, it follows that F is also bijective. □

Remark 2.1. In the same way it can be proved that there exists a bijection $F_2:B\to S_-$.

Corollary 2.1. *There exists a bijection $F:B\times B\to S$.*

Proof. According to Theorem 2.1, there exists $F_1:B\to S_+$ bijective and there exists $F_2:B\to S_-$ bijective. We now define F as follows:

$$F(h)(x)=\begin{cases}F_1(h_1)(x), & x\geq 0\\ F_2(h_2)(x), & x<0,\end{cases}$$

where $h\in B\times B$, $h=(h_1,h_2)$.

a) We shall prove that F is injective. Since $h\neq h'$, it follows that $h_1\neq h_1'$ or $h_2\neq h_2'$. We deduce that $F_1(h_1)\neq F_1(h_1')$ or $F_2(h_2)\neq F_2(h_2')$, because F_1, F_2 are injective. Hence $F(h)\neq F(h')$.

b) We shall prove that F is surjective.

Let $f\in S$. Then $f(x)=\begin{cases}f_1(x), & x\geq 0\\ f_2(x), & x<0.\end{cases}$

According to Theorem 2.1, there exists $F_1:B\to S_+$ which satisfies the relation $F_1(h_1)=f_1$ and $F_2:B\to S_-$ which satisfies the relation $F_2(h_2)=f_2$ (recall that F_1, F_2 are surjective). Then

$$F(h)(x)=\begin{cases}F_1(h_1)(x), & x\geq 0\\ F_2(h_2)(x), & x<0,\end{cases}$$

which shows that $F(h)=f$. □

B. Case $r_2<r_1<-1$.

The set of solutions of equation (1.1) is given by [4, Theorem 1.3]. From the Calibration Theorem, we have for a solution f of equation (1.1) the inequality

$$-r_1(x-y)\leq f(x)-f(y)\leq -r_2(x-y),\qquad \forall x,y\in\mathbb{R}. \tag{2.5}$$

Let us observe that the set of solutions on $\mathbb{R}$ depends upon three parameters: $x_0 > 0$, $x_1 \in [x_0 r_2, x_0 r_1]$, $f_0 : [x_0, x_2] \to [x_3, x_1]$, where $(x_n)_{n\in\mathbb{Z}}$ is the one of [4, Theorem 1.3] and f_0 is continuous, bijective and satisfies (2.5) for $x, y \in [x_0, x_2]$. From these statements it follows from Theorem 1.1 that starting with (x_0, x_1, f_0) under previous conditions, these parameters uniquely determine the solution on $\mathbb{R}$, because two solutions which coincide on $[x_0, x_2]$ coincide on $\mathbb{R}$, due to Theorem 1.1.

Notation 2.2. In the sequel we will use the following notation.

(i) $V(x_1)$ is the set of functions $f_0 : [x_0, x_2] \to [x_3, x_1]$ with $x_1 \in [r_2 x_0, r_1 x_0]$, where $x_{n+2} = -a x_{n+1} - b x_n$ for $n = 0, 1$ and f_0 is continuous, bijective and satisfies (2.5).

(ii) $V = \bigcup\limits_{x_1 \in [x_0 r_2, x_0 r_1]} V(x_1)$.

(iii) $S = \{f : \mathbb{R} \to \mathbb{R} \mid f \text{ is a solution of equation (1.1)}\}$.

(iv) B has the same meaning like in the case **A**.

Lemma 2.2. *There exists a bijection $H : V \to S$.*

Proof. Let $f_0 \in V$. There exists $x_1 \in [x_0 r_2, x_0 r_1]$, such that $f_0 \in V(x_1)$. We define $H(f_0) = f$, where f is the unique function which fulfills the conditions $f(x_0) = x_1$, $f \in S$, $f(x) = f_0(x)$ for all $x \in [x_0, x_2]$. Then it will follow that f satisfies (2.5), for all $x > y$; $x, y \in \mathbb{R}$, so H is well defined.

a) We prove that H is injective.

Indeed, let $f_0^1, f_0^2 \in S$ be such that $H(f_0^1) = H(f_0^2)$ on $\mathbb{R}$. Hence $H\left(f_0^1\right) = f_1$; $H\left(f_0^2\right) = f_2$. Let $x_0 > 0$, $x_1 \in [x_0 r_2, x_0 r_1]$, $x_2 = -a x_1 - b x_0$. Then $f_1(x_0) = f_2(x_0) = x_1$; $f_0^1 = f_1|_{[x_0, x_2]}$, $f_0^2 = f_2|_{[x_0, x_2]}$. Since $f_1|_{[x_0, x_2]} = f_2|_{[x_0, x_2]}$, we have $f_0^1 = f_0^2$ and it follows that H is injective.

b) We prove that H is surjective.

Indeed, let $f \in S$. We look for $f_0 \in V$ with $H(f_0) = f$. Let $x_0 > 0$. Write $x_1 = f(x_0)$; $x_2 = f(x_1)$; $x_3 = f(x_2)$. Define $f_0 : [x_0, x_2] \to [x_3, x_1]$, given via $f_0(x) = f(x)$. Let us prove that it holds $f_0 \in V(x_1)$. Since $f \in S$, it follows that $r_2 x_0 \le f(x_0) \le r_1 x_0$ (see [4, Lemma 1.2]), so $x_1 \in [r_2 x_0, r_1 x_0]$. Because $f(f(x_0)) = -a \cdot f(x_0) - b x_0$, we have $x_2 = -a x_1 - b x_0$ and because $f(f(x_1)) = -a \cdot f(x_1) - b x_1$, we have $x_3 = -a x_2 - b x_1$. Since f satisfies (2.5) and $f_0 = f|_{[x_0, x_2]}$, it follows that f_0 satisfies (2.5). Consequently H is surjective, therefore it is bijective. □

Theorem 2.2. *There exists a bijection $F : B \to S$.*

Proof. Firstly we shall prove that there exists a bijection $T : B \to V$, T. Let $x_0 > 0$ and $x_1 \in [r_2 x_0, r_1 x_0]$. Consequently x_2, x_3 are given by the relations $x_2 + a x_1 + b x_0 = 0$; $x_3 + a x_2 + b x_1 = 0$.

Let $h \in B$. We now try to define $\varphi : [x_0, x_2] \to [x_3, x_1]$. We notice that $x \in [x_0, x_2]$ if and only if $\frac{x - x_0}{x_2 - x_0} \in [0,1]$. Therefore there exists $t \in [0,1]$ such that $t = \frac{x - x_0}{x_2 - x_0}$. Hence $t \in [0,1]$ if and only if $x \in [x_0, x_2]$. We can define now φ as follows:

$$\varphi : [x_0, x_2] \to [x_3, x_1], \varphi(x) = \frac{x_3 - x_1}{h(1)} h\left(\frac{x - x_0}{x_2 - x_0}\right) + x_1$$

$$\varphi(x_0) = x_1; \quad \varphi(x_2) = x_3.$$

Since $x_3 - x_1 < 0$ and h is increasing, it follows that $\varphi \downarrow$.

We can take $x_1 = \frac{x_0 (r_2 h(1)(1+b) + ab)}{1 - a^2 + b - a r_2 h(1)}$ (namely, we shall prove that for this x_1 one has $x_1 \in [r_2 x_0, r_1 x_0]$).

According to the definition of the set B, it holds $h(1) \in \left[\frac{r_1}{r_2}, 1\right]$.

We now consider $g : \left[\frac{r_1}{r_2}, 1\right] \to \mathbb{R}$, given by $g(x) = \frac{x_0 (r_2(1+b)x + ab)}{-a r_2 x + 1 - a^2 + b}$. Taking into account that $\frac{1 - a^2 + b}{a r_2} > 1$, it will follow that g is well defined. It is well known that the function $x \mapsto \frac{mx + n}{px + q}$ is strictly decreasing if and only if $\begin{vmatrix} m & n \\ p & q \end{vmatrix} < 0$. We have

$$\begin{vmatrix} (1+b)r_2 & ab \\ -a r_2 & 1 - a^2 + b \end{vmatrix} = \begin{vmatrix} r_2 + r_1 r_2^2 & -r_1^2 r_2 - r_2^2 r_1 \\ r_2^2 + r_1 r_2 & 1 - r_1^2 - r_2^2 - r_1 r_2 \end{vmatrix} =$$

$$= \begin{vmatrix} r_2 + r_1 r_2^2 & r_2 - r_1^2 r_2 \\ r_2^2 + r_1 r_2 & 1 - r_1^2 \end{vmatrix} = r_2 \left(1 - r_1^2\right) \cdot \begin{vmatrix} 1 + r_1 r_2 & r_2 \\ r_1 + r_2 & 1 \end{vmatrix} =$$

$$= r_2 \left(1 - r_1^2\right) \left(1 - r_2^2\right) < 0.$$

Therefore it follows that g is decreasing. Then, for $x \in \left[\frac{r_1}{r_2}, 1\right]$, it will follow that $g(x) \in \left[g\left(\frac{r_1}{r_2}\right), g(1)\right]$. But $g\left(\frac{r_1}{r_2}\right) = x_0 r_1$ and $g(1) = x_0 r_2$, so $x_1 \in [x_0 r_2, x_0 r_1]$.

Let us define $T(h) = \varphi$; $T : B \to V$.

a) We shall prove that T is well defined. More precisely, starting with $h \in B$ and defining φ as previously, let us show that $\varphi \in V(x_1)$. Obviously $\varphi(x_0) = x_1$ and $\varphi(x_2) = x_3$. Since h is increasing it follows that φ is decreasing. Therefore φ is surjective and decreasing, i.e. is bijective.

Let us prove that $\dfrac{\varphi(y)-\varphi(x)}{x-y} \in [r_1, r_2]$, for $x > y$. By the equalities

$$\begin{cases} x = x_0 + t_1 (x_2 - x_0) \\ y = x_0 + t_2 (x_2 - x_0), \end{cases}$$

we get $x - y = (t_1 - t_2)(x_2 - x_0)$. Hence $x > y$ if and only if $t_1 > t_2$. We have

$$\frac{\varphi(y)-\varphi(x)}{x-y} = \frac{x_3 - x_1}{h(1)} \cdot \left(\frac{h(t_2)-h(t_1)}{t_1 - t_2}\right) \cdot \left(\frac{t_1 - t_2}{x-y}\right) =$$

$$= \frac{x_1 - x_3}{h(1)} \cdot \left(\frac{h(t_1)-h(t_2)}{t_1 - t_2}\right) \cdot \left(\frac{1}{x_2 - x_0}\right).$$

According to the definition of x_1, it follows that

$$x_0 r_2 h(1)(1+b) + abx_0 = x_1 \left(1 - a^2 + b\right) - a x_1 r_2 h(1).$$

We deduce

$$h(1) = \frac{x_1 \left(1 - a^2 + b\right) - abx_0}{r_2 \left(x_0(1+b) + ax_1\right)} = \frac{1}{r_2} \left(\frac{x_1 - x_3}{x_0 - x_2}\right). \tag{2.6}$$

Consequently

$$\frac{\varphi(y)-\varphi(x)}{x-y} = -\left(\frac{h(t_1)-h(t_2)}{t_1 - t_2}\right) \cdot r_2. \tag{2.7}$$

Since $\dfrac{h(t_1)-h(t_2)}{t_1 - t_2} \in \left[\dfrac{r_1}{r_2}, 1\right]$, it follows that $\dfrac{\varphi(y)-\varphi(x)}{x-y} \in [-r_1, -r_2]$. Hence $\varphi \in V(x_1)$ and then it follows that T is well defined.

We shall prove that if

$$C = \bigcup_{x_1 \in [x_0 r_2, x_0 r_1]} \; \bigcup_{\varphi \in V(x_1)} \left\{ \frac{\varphi(y)-\varphi(x)}{y-x} \;\middle|\; y > x;\; x, y \in [x_0, x_2] \right\},$$

then $C = [r_2, r_1]$. It obviously holds $C \subset [r_2, r_1]$. On the other hand, one has $C_1 \subset C$, where

$$C_1 = \bigcup_{x_1 \in [x_0 r_2,\, x_0 r_1]} \; \bigcup_{\varphi \,\in\, W\,(x_1)} D(x_1, \varphi) \text{ with}$$

$$D(x_1, \varphi) = \{\frac{\varphi(y)-\varphi(x)}{y-x} \mid y > x,\; x, y \in [x_0, x_2]\} \text{ and}$$

$$W\,(x_1) = V\,(x_1) \bigcap \{\,\varphi : [0,1] \to \mathbb{R} \mid \varphi(x) = px + q\}.$$

We shall prove that $C_1 = [r_2, r_1]$. Let $\alpha \in [r_2 x_0, r_1 x_0]$. We look for φ linear ($\varphi(x) = px + q$) such that $p = \alpha$. So $\varphi(x_0) = x_1, \varphi(x_2) = x_3$, that is

$px_0+q=x_1, px_2+q=x_3$. These equalities imply that $p\left(x_2-x_0\right)=x_3-x_1$. The latter relation is equivalent to

$$p=\frac{x_3-x_1}{x_2-x_0}=\frac{x_1-x_3}{x_0-x_2}=\frac{x_1-x_1(a^2-b)-abx_0}{x_0+ax_1+bx_0}=\frac{x_1\left(1-a^2+b\right)-abx_0}{ax_1+(b+1)x_0},$$

$x_1\in\left[r_2x_0,r_1x_0\right]$.

We consider $g_1:[x_0r_2,x_0r_1]\to\mathbb{R}$, given by $g_1(x)=\dfrac{\left(1-a^2+b\right)x-abx_0}{ax+(b+1)x_0}$. Obviously $-\left(\dfrac{b+1}{a}\right)=\dfrac{r_1r_2+1}{r_1+r_2}>r_1$. Hence g_1 is well defined. We now establish the monotony of g_1. One has

$$\begin{vmatrix} 1-a^2+b & -ab \\ a & b+1 \end{vmatrix}=(b+1)^2-a^2=$$

$$=(a+b+1)(b+1-a)=(r_1^2-1)(r_2^2-1)>0,$$

and it follows g_1 increasing. It is clear that $g_1\left(x_0r_1\right)=\dfrac{x_0\left(r_1^3-r_1\right)}{x_0\left(r_1^2-1\right)}=r_1$ and $g_1\left(x_0r_2\right)=\dfrac{x_0\left(r_2^3-r_2\right)}{x_0\left(r_2^2-1\right)}=r_2$. Since g_1 is continuous and increasing, it follows that $g_1([x_0r_2,x_0r_1])=\left[r_2,r_1\right]$. Consequently, for all $\alpha\in\left[x_0r_2,x_0r_1\right]$ there exists φ linear ($\varphi(x)=px+q$) such that $p=\alpha$. Hence $C_1=\left[r_2,r_1\right]$ and it follows that $C=[r_2,r_1]$.

b) We shall prove that T is injective.

Let $h_1,h_2\in B$ with $h_1\neq h_2$. Let us prove that $T(h_1)\neq T(h_2)$. Next, we will denote by φ_1 the function $T(h_1)$ and by φ_2 the function $T(h_2)$, and $\varphi_1,\varphi_2\in V$. There exist x_1, x_1' such that $\varphi_1\in V(x_1)$ and $\varphi_2\in V(x_1')$.

1) If $x_1\neq x_1'$, then $\varphi_1(x_0)=x_1$ and $\varphi_2(x_0)=x_1'$, with $x_1\neq x_1'$, so $\varphi_1\neq\varphi_2$.

2) If $x_1=x_1'$ hence $\varphi_1,\varphi_2\in V(x_1)$, we have $x_2=x_2'$ with obvious notations. But $h_1(1)=\dfrac{1}{r_2}\left(\dfrac{x_1-x_3}{x_0-x_2}\right)=\dfrac{1}{r_2}\left(\dfrac{x_1'-x_3'}{x_0-x_2'}\right)=h_2(1)$. Since $h_1\neq h_2$, there exists $\alpha\in(0,1)$ such that $h_1(\alpha)\neq h_2(\alpha)$. We choose $x_\alpha=x_0+(x_2-x_0)$; $x_\alpha\in(x_0,x_2)$.

We shall prove that $\varphi_1(\alpha)\neq\varphi_2(\alpha)$. One has

$$\varphi_1\left(x_\alpha\right)=\frac{x_3-x_1}{h_1(1)}\cdot h_1\left(\frac{x_\alpha-x_0}{x_2-x_0}\right)+x_1=\frac{x_3-x_1}{h_1(1)}\cdot h_1(\alpha)+x_1.$$

In the same way

$$\varphi_2\left(x_\alpha\right)=\frac{x_3'-x_1'}{h_2(1)}\cdot h_2(\alpha)+x_1'.$$

Since $x_1 = x'_1$, $x_3 = x'_3$ and $h_1(\alpha) \neq h_2(\alpha)$, it follows that $\varphi_1(x_\alpha) \neq \varphi_2(x_\alpha)$. Consequently $\varphi_1 \neq \varphi_2$ and it follows that T is injective.

c) We shall show that T is surjective. Let $\varphi \in V$; there exists x_1 such that $\varphi \in V(x_1)$. Let us prove that there exists $h \in B$ with $T(h) = \varphi$. Since $\varphi \in V(x_1)$, it follows that $\dfrac{x - x_0}{x_2 - x_0} \in [0,1]$ for $x \in [x_0, x_1]$. We can write $t = \dfrac{x - x_0}{x_2 - x_0} \in [0,1]$.

We define $h(t) = \dfrac{\varphi(x) - x_1}{x_3 - x_1} \cdot \dfrac{x_1 - x_3}{r_2(x_0 - x_2)}$. Hence $h(t) = \dfrac{\varphi(x) - x_1}{(x_2 - x_0)\, r_2}$.

1) Let us prove that $h \in B$. Obviously h is continuous and increasing. One has $h(0) = 0$ and $h(1) = \dfrac{1}{r_2}\left(\dfrac{x_1 - x_3}{x_0 - x_2}\right)$ (see (2.6)). According to (2.7), if $x_1 \in [x_0 r_1, x_0 r_2]$, it follows that $h(1) \in \left[\dfrac{r_1}{r_2}, 1\right]$. Consequently h is well defined in 1.

Let $0 \leq t_1 < t_2 \leq 1$, and write

$$x = x_0 + t_1(x_2 - x_0), \quad y = x_0 + t_2(x_2 - x_0).$$

One has the equality $\dfrac{\varphi(y) - \varphi(x)}{y - x} = r_2\left(\dfrac{h(t_2) - h(t_1)}{t_2 - t_1}\right)$ (see (2.7)). Hence $\dfrac{\varphi(y) - \varphi(x)}{y - x} \in [r_2, r_1]$ if and only if $\dfrac{h(t_2) - h(t_1)}{t_2 - t_1} \in \left[\dfrac{r_1}{r_2}, 1\right]$ and $h \in B$.

2) Let us prove that $T(h) = \varphi$. One has

$$T(h)(x) = \left(\frac{x_3 - x_1}{h(1)}\right) \cdot \left(\frac{\varphi(x) - x_1}{(x_2 - x_0)\, r_2}\right) + x_1 =$$

$$= \frac{x_3 - x_1}{\left(\dfrac{x_1 - x_3}{x_0 - x_2}\right) \cdot \dfrac{1}{r_2}} \cdot \frac{\varphi(x) - x_1}{(x_2 - x_0) \cdot r_2} + x_1 = \varphi(x).$$

Consequently $T(h) = \varphi$ and T is bijective.

We define $F = H \circ T$, $H : V \to S$ and $T : B \to V$, so $H \circ T : B \to S$. Since H and T are bijective, it follows that F is also bijective. □

C. Case $0 < r_1 < 1 < r_2$.

The set of solutions is given by [4, Theorem 1.4]. Let us observe that the set of solutions on $\mathbb{R}$, for which $f(x) > x$, depends upon two parameters: $x_1 > 0$ and the function $f_0 : [0, x_1] \to [x_1, x_2]$, where $(x_n)_{n \in \mathbb{Z}}$ is that one of [4, Theorem 1.4] and f_0 is continuous bijective and verifies the relation (2.1). Under these conditions it follows from Theorem 1.1 that, starting with (x_1, f_0) under previous conditions, these parameters determine uniquely the solution on $\mathbb{R}$, which additionally fulfills the condition $f(0) = x_1$ (because two solutions f and g with $f(0) = g(0) = x_1$ which coincide on $[0, x_1]$, coincide on $\mathbb{R}$ according to Theorem 1.1).

Notation 2.3. In the sequel we will use the following notation.

(i) $V(x_1) = \{f_0 : [0, x_1] \to [x_1, ax_1]\ f_0$ is continuous, bijective, increasing, f_0 satisfies (2.1)$\}$.

(ii) $V = \bigcup\limits_{x_1 \in (0,\infty)} V(x_1)$.

(iii) $S_+ = \{f : \mathbb{R} \to \mathbb{R} \mid f$ is (strictly) increasing and is solution of equation (1.1), $f(x) > x\}$.

(iv) $B = \{h : [0,1] \to [0,1], h(0) = 0,\ h$ is continuous, increasing,

$$h(1) = \frac{r_1 + r_2 - 1}{r_2};\ \frac{h(x) - h(y)}{x - y} \in \left[\frac{r_1}{r_2}, 1\right];\ x > y;\ x, y \in [0,1]\Big\}.$$

Lemma 2.3. *There exists a bijection $H : V \to S_+$.*

Proof. Let $f_0 \in V$. There exists $x_1 > 0$ such that $f_0 \in V(x_1)$. We define $H(f_0) = f$, where f is the unique function which fulfills the conditions $f(x_0) = x_1$, $f \in S_+$ and $f(x) = f_0(x)$ for all $x \in [0, x_1]$. Then it follows that f satisfies (2.1), for all $x > y \geq 0$. Hence H is well defined.

a) We shall prove that H is injective. Let $f_1, f_2 \in S_+$ with $H\left(f_0^1\right) = f_1$; $H\left(f_0^2\right) = f_2$ and $f_1 = f_2$. Let $x_1 > 0$, such that $f_1(0) = f_2(0) = x_1$, $f_0^1 = f_1|_{[0,x_1]}$; $f_0^2 = f_2|_{[0,x_1]}$. Since $f_1|_{[0,x_1]} = f_2|_{[0,x_1]}$ it follows that $f_0^1 = f_0^2$, so H is injective.

b) We shall prove that H is surjective. Let $f \in S_+$. We look for $f_0 \in V$ with $H(f_0) = f$. Take $x_1 = f(0)$; $f_0 = f|_{[0,x_1]}$; $x_2 = f \circ f(0)$; then one has $f_0 : [0, x_1] \to [x_1, x_2]$. We shall prove that $f_0 \in V(x_1)$. Since $f \in S_+$, it follows that $x_1 > 0$. The equation $f(f(0)) = a \cdot f(0) - b \cdot 0$ implies $x_2 = ax_1$. Since f verifies (2.1), f_0 verifies (2.1), because $f_0 = f|_{[0,x_1]}$. Hence H is surjective and then it follows that H is bijective. □

Theorem 2.3. *There exists a bijection $F : B \to S_+$.*

Proof. Firstly we shall prove that there exists $T : B \to V$, T bijective. Consider $x_1 > 0$ and put $x_2 = ax_1$.

Let $h \in B$. If $x \in [0, x_1]$, then there exists $t \in [0,1]$ such that $t = \dfrac{x}{x_1}$. Consequently $t \in [0,1]$ if and only if $x \in [0, x_1]$. We now define φ by

$$\varphi : [0, x_1] \to [x_1, ax_1], \quad \varphi(x) = x_1 r_2 h\left(\frac{x}{x_1}\right) + x_1.$$

For $x = x_1$ it holds

$$\varphi(x_1) = x_1 r_2 h(1) + x_1 = x_1 r_2 \cdot \frac{(a-1)}{r_2} + x_1 = ax_1 = x_2.$$

Hence $\varphi(x) \in [x_1, ax_1]$, because φ is increasing.

We define $T : B \to V$ by $T(h) = \varphi$.

a) We shall prove that T is well defined. More precisely, starting with $h \in B$ and defining φ as above, let us prove that $\varphi \in V(x_1)$.

Obviously $\varphi(0) = x_1$ and $\varphi(x_1) = ax_1$. Since φ is increasing, continuous, $\varphi(0) = x_1$ and $\varphi(x_1) = ax_1$, it follows that φ is bijective.

Let us prove that $\dfrac{\varphi(y) - \varphi(x)}{y - x} \in [r_1, r_2]$. For $x_1 > y > x > 0$ put $x = t_1 x_1$, $y = t_2 x_1$, where t_1, t_2 are in $[0, 1]$. One has $y - x = (t_2 - t_1) x_1$ and $y > x$ if and only if $t_2 > t_1$. Then we have:

$$\begin{cases} \varphi(y) = x_1 r_2 h(t_2) + x_1 \\ \varphi(x) = x_1 r_2 h(t_1) + x_1 \end{cases}$$

and it follows $\varphi(y) - \varphi(x) = x_1 r_2 (h(t_2) - h(t_1))$. We deduce that

$$\frac{\varphi(y) - \varphi(x)}{y - x} = x_1 r_2 \cdot \frac{h(t_2) - h(t_1)}{t_2 - t_1} \cdot \left(\frac{t_2 - t_1}{y - x} \right) =$$

$$= \frac{h(t_2) - h(t_1)}{t_2 - t_1} \cdot x_1 r_2 \cdot \frac{1}{x_1}.$$

In conclusion, it holds

$$\frac{\varphi(y) - \varphi(x)}{y - x} = r_2 \cdot \left(\frac{h(t_2) - h(t_1)}{t_2 - t_1} \right). \tag{2.8}$$

Since $\dfrac{h(t_2) - h(t_1)}{t_2 - t_1} \in \left[\dfrac{r_1}{r_2}, 1 \right]$, we have $\dfrac{\varphi(y) - \varphi(x)}{y - x} \in [r_1, r_2]$. Hence $\varphi \in V(x_1)$. Then it follows that T is well defined.

We shall prove that if

$$C = \bigcup_{x_1 \in [0, \infty]} \bigcup_{\varphi \in V(x_1)} \left\{ \frac{\varphi(y) - \varphi(x)}{y - x} \,\middle|\, y > x;\ x, y \in [0, x_1] \right\},$$

then $C = [r_1, r_2]$. We have $C \subset [r_1, r_2]$ and it remains to prove the inclusion $[r_1, r_2] \subset C$. We look for functions $\varphi : [0, x_1] \to [x_1, ax_1]$ $(a = r_1 + r_2)$ of the form

$$\varphi(x) = \begin{cases} rx + c & x \in [0, x_1') \\ r'x + c' & x \in [x_1', x_1], \end{cases}$$

where $r \neq r'$ and we study what values can take the ratio $\dfrac{\varphi(x) - \varphi(y)}{x - y}$, where $0 \leq y < x \leq x'$, in order to have $\varphi \in V(x_1)$. Consequently the following necessary conditions are compulsory: $\varphi(0) = x_1$, $\varphi(x_1) = ax_1$, $rx_1' + c = r'x_1' + c'$ and $r_1 \leq r$, $r' \leq r_2$. Consequently

$$\begin{cases} c = x_1 \\ r'x_1 + c' = ax_1 \\ rx_1' + c = r'x_1' + c,' \end{cases} \quad \text{which implies} \quad \begin{cases} c = x_1 \\ c' = (a - r')x_1 \\ rx_1' + x_1 = r'x_1' + (a - r')x_1. \end{cases}$$

It follows that
$$x_1' = \frac{x_1\,(1-a+r')}{r'-r}.$$
Hence one has
$$0 < x_1' < x_1 \text{ if and only if } \frac{1-a+r'}{r'-r} > 0$$
and
$$\frac{x_1\,(1-a+r')}{r'-r} < x_1 \text{ if and only if } \frac{1-a+r'}{r'-r} < 0.$$

If $r' > r$, then $\begin{cases} 1-a+r<0 \\ 1-a+r'>0 \end{cases}$, so $\begin{cases} r<a-1 \\ r'>a-1. \end{cases}$ Then $r \in [r_1, a-1)$ and $r' \in (a-1, r_2]$. Hence if $r < r'$, then r can take any value from the interval $[r_1, a-1)$.

If $r' < r$, then $\begin{cases} 1-a+r>0 \\ 1-a+r'<0 \end{cases}$, so $\begin{cases} r>a-1 \\ r'<a-1. \end{cases}$ Then $r \in (a-1, r_2]$ and $r' \in [r_1, a-1)$. Therefore if $r > r'$, then r can take any value from the interval $(a-1, r_2]$.

Our next objective is to prove that $\varphi \in V(x_1)$. We observe that it is enough to prove that
$$0 \le \alpha < \beta \le x_1 \text{ implies that } r_1(\beta-\alpha) \le \varphi(\beta) - \varphi(\alpha) \le r_2(\beta-\alpha).$$
If $\alpha, \beta \in [0, x_1']$ or $\alpha, \beta \in [x_1', x_1]$ it is obvious that
$$r_1(\beta-\alpha) \le \varphi(\beta) - \varphi(\alpha) \le r_2(\beta-\alpha),$$
according to the previous considerations.

Let us suppose that $\alpha < x_1' < \beta$. We shall prove that the inequality $\varphi(\beta) - \varphi(\alpha) \le r_2(\beta-\alpha)$ holds. Indeed, one has
$$\varphi(\alpha) = r\alpha + c = r\alpha + x_1, \varphi(\beta) = r'\beta + c' = r'\beta + (a-r')x_1,$$
deducing that $\varphi(\beta) - \varphi(\alpha) \le r_2(\beta-\alpha)$ if and only if
$$r'\beta + ax_1 - r'x_1 - r\alpha - x_1 \le r_2\beta - r_2\alpha.$$
The latter inequality is equivalent to
$$\beta > \alpha\frac{r_2-r}{r_2-r'} + \frac{x_1(a-r'-1)}{r_2-r'}.$$

Since $\beta > x_1'$, it is sufficient to prove that
$$x_1' > \alpha\left(\frac{r_2-r}{r_2-r'}\right) + x_1\left(\frac{a-r'-1}{r_2-r'}\right).$$

By replacing x_1', we get the inequality

$$x_1\left(\frac{1-a+r'}{r'-r}\right) > \alpha\left(\frac{r_2-r}{r_2-r'}\right) + x_1\left(\frac{a-r'-1}{r_2-r'}\right).$$

By direct computation one shows that this is equivalent to

$$x_1\left(1-a+r'\right)\frac{r_2-r}{(r'-r)(r_2-r')} > \alpha\frac{r_2-r}{r_2-r'}.$$

The latter inequality is equivalent to

$$\alpha < x_1\left(\frac{1-a+r'}{r'-r}\right),$$

that is $\alpha < x_1'$. We already know that this is true, so $\varphi(\beta)-\varphi(\alpha) \le r_2(\beta-\alpha)$.

We prove now that $\varphi(\beta)-\varphi(\alpha) \ge r_1(\beta-\alpha)$, that is

$$r'\beta + ax_1 - r'x_1 - r\alpha - x_1 \ge r_1\beta - r_1\alpha.$$

This relation is equivalent to

$$\beta > \alpha\left(\frac{r-r_1}{r'-r_1}\right) + x_1\left(\frac{r'-a+1}{r'-r_1}\right).$$

Since $\beta > x_1$, it is enough to prove that

$$x_1' > \alpha\left(\frac{r-r_1}{r'-r_1}\right) + x_1\left(\frac{r'-a+1}{r'-r_1}\right).$$

By replacing x_1', we get the equivalent inequality

$$x_1\left(\frac{1-a+r'}{r'-r}\right) > \alpha\left(\frac{r-r_1}{r'-r_1}\right) + x_1\left(\frac{r'-a+1}{r'-r_1}\right).$$

It is easy to check that this relation is equivalent to

$$x_1\left(1-a+r'\right)\frac{r-r_1}{(r'-r)(r'-r_1)} > \alpha\left(\frac{r-r_1}{r'-r_1}\right),$$

that is $\alpha < x_1'$. From our hypothesis we know that this inequality is true. Hence

$$\varphi(\beta)-\varphi(\alpha) \ge r_1(\beta-\alpha).$$

Thus we have $\varphi \in V\left(x_1\right)$.

We have proved also that: $[r_1, r_2] - \{a-1\} \subset C$.

Obviously, choosing $\varphi(x) = (a-1)x + x_1$, we have

$$\varphi(0) = x_1,\ \ \varphi(x_1) = ax_1 \ \text{ and } \ \frac{\varphi(y)-\varphi(x)}{y-x} = a-1.$$

Since $a-1 \in (r_1, r_2)$, it follows that $\varphi \in V(x_1)$. Hence $[r_1, r_2] \subset C$. Then it follows that $C = [r_1, r_2]$.

b) Let us prove that T is injective. Let h_1, $h_2 \in B$ with $h_1 \neq h_2$. We shall prove that $T(h_1) \neq T(h_2)$. Put $T(h_1) = \varphi_1$ and $T(h_2) = \varphi_2$, where $\varphi_1, \varphi_2 \in V$. There exist x_1 and x_1' such that $\varphi_1 \in V(x_1)$ and $\varphi_2 \in V(x_1')$. If $x_1 \neq x_1'$ we have $\varphi_1(0) = x_1$ and $\varphi_2(0) = x_1'$, so $\varphi_1 \neq \varphi_2$. If $x_1 = x_1'$, hence $\varphi_1, \varphi_2 \in V(x_1)$ we have $x_2 = x_2' = ax_1$. Since $h_1 \neq h_2$, there exists $\alpha \in (0,1)$ such that $h_1(\alpha) \neq h_2(\alpha)$. We choose $x_\alpha = x_1 \cdot \alpha$, hence $x_\alpha \in (0, x_1)$. We have

$$\varphi_1(x_\alpha) = x_1 r_2 h_1(\alpha) + x_1,$$
$$\varphi_2(x_\alpha) = x_1' r_2 h_2(\alpha) + x_1'.$$

Since $x_1 = x_1'$ and $h_1(\alpha) \neq h_2(\alpha)$, we deduce that $\varphi_1(x_\alpha) \neq \varphi_2(x_\alpha)$. Consequently $\varphi_1 \neq \varphi_2$ and we deduce that T is injective.

c) Let us prove that T is surjective. Let $\varphi \in V$; there exists x_1 such that $\varphi \in V(x_1)$. Let us prove that there exists $h \in B$ such that $T(h) = \varphi$. For any $x \in [0, x_1]$, write $t = \dfrac{x}{x_1} \in [0,1]$. We define $h : [0,1] \to [0,1]$ via

$$h(t) = \frac{\varphi(tx_1) - x_1}{x_1 r_2}.$$

1) Let us prove that $h \in B$. Obviously h is continuous and increasing. One has $h(0) = 0$, $h(1) = \dfrac{\varphi(x_1) - x_1}{x_1 r_2} = \dfrac{a-1}{r_2}$. Hence h is well defined at $t = 1$. Let $0 \leq t_1 < t_2 \leq 1$, and write $x = t_1 x_1$, $y = t_2 x_1$. According to relation (2.8) we have:

$$\frac{\varphi(y) - \varphi(x)}{y - x} = r_2 \left(\frac{h(t_2) - h(t_1)}{t_2 - t_1} \right).$$

Hence $\dfrac{\varphi(y) - \varphi(x)}{y - x} \in [r_1, r_2]$ if and only if $\dfrac{h(t_2) - h(t_1)}{t_2 - t_1} \in \left[\dfrac{r_1}{r_2}, 1 \right]$ and this implies that $h \in B$.

2) Let us prove that $T(h) = \varphi$. One has

$$T(h)(x) = x_1 r_2 \left(\frac{\varphi(x) - x_1}{r_2 x_1} \right) + x_1 = \varphi(x).$$

Consequently it follows that $T(h) = \varphi$. Thus it follows that T is surjective and finally that T is bijective.

We define $F = H \circ T$, $H : V \to S_+$ and $T : B \to V$, so $H \circ T : B \to S_+$. Since H and T are bijective, it follows that F is also bijective. □

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