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On the Neumann problem involving the Hardy - Sobolev potentials

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Abstract - We establish the existence of solutions for the Neumann problem involving two Hardy - Sobolev potentials with singularities at two distinct points.

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1. Introduction

In this paper we investigate the nonlinear Neumann problem

$$\begin{cases} -\Delta u & = \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)-2}u + \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)-2}u \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} & = 0 \text{ on } \partial\Omega,\ u > 0 \text{ on } \Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$. It is assumed that $0, \xi \in \partial\Omega$. $2^*(t_j)$ denote Hardy - Sobolev exponents given by $2^*(t_j) = \frac{2(N-t_j)}{N-2}$, $0 \le t_j \le 2$. In this paper we only consider the case $0 < t_j < 2$. If $t_j = 0$ for $j = 1, 2$, then $2^*(t_j) = 2^* = \frac{2N}{N-2}$ and this problem has an extensive literature. We refer to papers [1], [2], [6], [7], [10], [26]. The existence results in the case $t_1 = 0$ and $0 < t_2 < 2$ are given in [11]. If $t_j = 2$ for $j = 1, 2$, then $2^*(t_j) = 2$, $j = 1, 2$, and we have on the right hand side of equation (1.1) a sum of two Hardy potentials. In this situation we can look at (1.1) as an eigenvalue problem by replacing the right hand side of the equation by

$$\lambda\Big(\frac{P_1(x)}{|x|^2} + \frac{P_2(x)}{|x-\xi|^2}\Big)u$$

where $\lambda \in \mathbb{R}$ is an eigenvalue parameter (see [12]). For elliptic problems involving the Hardy potential we also refer to papers [5], [13], [14], [19], [20], [21], [22], [23], [25], where further bibliographical references can be found.

The coefficients P_j, $j = 1, 2$, are assumed to be continuous on $\bar{\Omega}$. Further assumptions on P_j will be formulated later. We look for solutions of problem (1.1) in a Sobolev space $H^1(\Omega)$ equipped with norm

$$\|u\|^2 = \int_\Omega \left(|\nabla u|^2 + u^2\right) dx.$$

By $H^1_\circ(\Omega)$ we denote a Sobolev space obtained as the closure of the space $C^\infty_\circ(\Omega)$ with respect to the norm

$$\|u\|^2_{H^1_\circ} = \int_\Omega |\nabla u|^2 \, dx.$$

Problems discussed in this paper are related to the optimal constant of the Hardy - Sobolev type. The best Hardy - Sobolev constant for the domain $\Omega \subset \mathbb{R}^N$ is defined by

$$S^s_H(\Omega) = \inf_{\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx = 1,\, u \in H^1_\circ(\Omega)} \int_\Omega |\nabla u|^2 \, dx, \tag{1.2}$$

where $2^*(s) = \frac{2(N-s)}{N-2}$, $0 \le s \le 2$. If $\Omega = \mathbb{R}^N$, we write S^s_H instead of $S^s_H(\Omega)$. If $s = 0$, then $S^0_H(\Omega) = S$ is the best Soblev constant which is independent of Ω. In the case $0 < s < 2$, $S^s_H(\Omega)$ depends on Ω (see [17], [18]). If $0 \le s < 2$, then S^s_H is attained by a family of functions

$$W^s_\epsilon(x) = \frac{C_N \epsilon^{\frac{N-2}{2-s}}}{\left(\epsilon^2 + |x|^{2-s}\right)^{\frac{N-2}{2-s}}}, \ \epsilon > 0, \tag{1.3}$$

where C_N is a normalizing positive constant depending on N and s. Obviously, W^s_ϵ satisfies the equation

$$-\Delta u = \frac{|u|^{2^*(s)-1}}{|x|^s} \ \text{ in } \mathbb{R}^N - \{0\}.$$

We also have

$$\int_{\mathbb{R}^N} |\nabla W^s_\epsilon|^2 \, dx = \int_{\mathbb{R}^N} \frac{\left(W^s_\epsilon\right)^{2^*(s)}}{|x|^s} \, dx = \left(S^s_H\right)^{\frac{N-s}{2-s}}.$$

From the definition of the Hardy - Sobolev constant $S^s_H(\Omega)$ it follows

$$S^s_H(\Omega) \left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s} \, dx\right)^{\frac{2}{2^*(s)}} \le \int_\Omega |\nabla u|^2 \, dx$$

for every $u \in H^1_\circ(\Omega)$. We need an extension of this inequality to the space $H^1(\Omega)$ (see [10]).

Lemma 1.1. *Let $0 \in \bar{\Omega}$. Then there exists a constant $K > 0$ such that*

$$\left(\int_\Omega \frac{|u|^{2^*(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*(s)}} \le K \int_\Omega \left(|\nabla u|^2 + u^2\right) dx \tag{1.4}$$

for every $u \in H^1(\Omega)$.

A solution $u \in H^1(\Omega)$ of (1.1) is understood in a distributional (or weak) sense, that is,

$$\int_\Omega \nabla u \nabla v\,dx = \int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)-2}uv\,dx + \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)-2}uv\,dx$$

for every $v \in H^1(\Omega)$. If $u \in H^1(\Omega)$ is a solution of (1.1) then

$$0 = \int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)-1}\,dx + \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)-1}\,dx$$

So if P_1 and P_2 are nonnegative and at least one of them not identically equal to 0, then problem (1.1) does not have a solution. Therefore, we assume

(P) $\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\,dx < \infty$, P_1 changes sign and $P_2(x) > 0$ on $\bar{\Omega}$.

We use the decomposition of the space $H^1(\Omega)$

$$H^1(\Omega) = V \oplus \mathbb{R}, \quad V = \{v \in H^1(\Omega) \mid \int_\Omega v(x)\,dx = 0\}.$$

This decomposition yields the following equivalent norm on $H^1(\Omega)$

$$\|u\|_V^2 = \||\nabla u|\|_2^2 + t^2,\ v \in V,\ \ t \in \mathbb{R}.$$

We note that inequality (1.4) in the space V takes the form: there exists a constant $K_1 > 0$ such that

$$\left(\int_\Omega \frac{|v|^{2^*(s)}}{|x|^s}\,dx\right)^{\frac{2}{2^*(x)}} \le K_1 \int_\Omega |\nabla v|^2\,dx$$

for every $v \in V$.

We frequently use in this paper the following qualitative property:

(S) there exists a constant $\eta > 0$ such that for every $t \in \mathbb{R}$ and $v \in V$ the inequality

$$\left(\int_\Omega |\nabla v|^2\,dx\right)^{\frac{1}{2}} \le \eta|t|$$

yields

$$\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|v+t|^{2^*(t_1)}\,dx \le \frac{|t|^{2^*(t_1)}}{2}\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\,dx.$$

This follows from the continuity of the embedding of $H^1(\Omega)$ into the space $L^{2^*(t_1)}\big(\Omega, \frac{1}{|x|^{t_1}}\big)$ (see also [3]). Solutions of problem (1.1) will be obtained as critical points of the variational functional

$$\begin{aligned} J(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{2^*(t_1)}\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\,dx \\ &- \frac{1}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x|^{t_2}}|u|^{2^*(t_2)}\,dx. \end{aligned}$$

To study problem (1.1) we distinguish three cases: (i) $2^*(t_1) < 2^*(t_2)$, (ii) $2^*(t_1) = 2^*(t_2)$ and (iii) $2^*(t_1) > 2^*(t_2)$. In the cases (i) and (ii) solutions are obtained via the mountain - pass principle. In the case (iii) we use a local minimization.

The paper is organized as follows. Sections 2 and 3 are devoted to the study of Palais - Smale sequences. In the final Section 4 we present the existence theorems for problem (1.1).

Throughout this paper, in a given Banach space we denote strong convergence by " $\rightarrow$ " and weak convergence by " $\rightharpoonup$ ". The norms in the Lebesgue spaces $L^p(\Omega)$, $1 \le p \le \infty$, are denoted by $\|\cdot\|_p$.

2. The mountain-pass geometry and (PS) sequences of J

We recall that a C^1 functional $\phi : X \to \mathbb{R}$ on a Banach space X satisfies the Palais - Smale condition at level c ($(PS)_c$ condition for short), if each sequence $\{x_n\} \subset X$ such that (*) $\phi(x_n) \to c$ and (**) $\phi'(x_n) \to 0$ in X^* is relatively compact in X. Finally, any sequence $\{x_n\}$ satisfying (*) and (**) is called a Palais - Smale sequence at level c (a $(PS)_c$ sequence for short).

We distinguish three cases: (i) $2^*(t_1) < 2^*(t_2)$, (ii) $2^*(t_1) = 2^*(t_2)$ and (iii) $2^*(t_1) > 2^*(t_2)$.

We begin with the case $2^*(t_1) < 2^*(t_2)$.

Proposition 2.1. *Suppose that* (**P**) *and* $2^*(t_1) < 2^*(t_2)$ *hold. Then every* $(PS)_c$ *sequence is bounded.*

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence. We have

$$\begin{aligned} J(u_n) &- \frac{1}{2^*(t_1)}\langle J'(u_n), u_n\rangle = \big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\int_\Omega |\nabla u_n|^2\,dx \\ &+ \big(\frac{1}{2^*(t_1)} - \frac{1}{2^*(t_2)}\big)\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u_n|^{2^*(t_2)}\,dx = c + o(1) + \epsilon_n\|u_n\|, \end{aligned}$$

where $\epsilon_n \to 0$. From this we deduce that there exists a constant $C > 0$ such that

$$\int_\Omega |\nabla u_n|^2\,dx,\ \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u_n|^{2^*(t_2)}\,dx \le C\bigl(1+\|u_n\|\bigr) \tag{2.1}$$

for every n. Let $d = \operatorname{diam}\Omega$ and $\bar m = \min_{x\in\bar\Omega} P_2(x)$. Then

$$\frac{\bar m}{d}\int_\Omega |u_n|^{2^*(t_2)}\,dx \le \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u_n|^{2^*(t_2)}\,dx \le C\bigl(1+\|u_n\|\bigr).$$

By the Hölder inequality we deduce from this

$$\int_\Omega u_n^2\,dx \le |\Omega|^{1-\frac{2}{2^*(t_2)}}\left(\int_\Omega |u_n|^{2^*(t_2)}\,dx\right)^{\frac{2}{2^*(t_2)}} \le \tilde C|\Omega|^{1-\frac{2}{2^*(t_2)}}\bigl(1+\|u_n\|^{\frac{2}{2^*(t_2)}}\bigr), \tag{2.2}$$

where $\tilde C >$ is a constant independent of n. Inequalities (2.1) and (2.2) yield the boundedness of $\{u_n\}$ in $H^1(\Omega)$. □

Proposition 2.2. *Suppose that* (**P**) *and* $2^*(t_1) < 2^*(t_2)$ *hold. Then there exist constants* $\kappa > 0$ *and* $\rho > 0$ *such that*

$$J(u) \ge \kappa \quad \textit{for} \quad \|u\| = \rho.$$

Proof. We use property (**S**). We distinguish two cases (i) $\|\nabla v\|_2 \le \eta|t|$ and (ii) $\|\nabla v\|_2 > \eta|t|$, where $\eta > 0$ is a constant from property (**S**) and $u = v+t$, $v \in V$, $t \in \mathbb{R}$. If (i) holds and $\|\nabla v\|_2^2 + t^2 = \rho^2$, then $t^2 \ge \frac{\rho^2}{1+\eta^2}$. By (**S**) we get

$$\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\,dx \le \frac{|t|^{2^*(t_1)}}{2}\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\,dx = -|t|^{2^*(t_1)}\alpha,$$

where $\alpha = -\frac{1}{2}\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\,dx > 0$. From this we derive the estimate of J from below

$$J(u) \ge \frac{\alpha\rho^{2^*(t_1)}}{2^*(t_1)\bigl(1+\eta^2\bigr)^{\frac{2^*(t_1)}{2}}} - \frac{1}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\,dx. \tag{2.3}$$

In the case (ii) we have

$$\|u\|_V \le \|\nabla v\|_2\bigl(1+\frac{1}{\eta^2}\bigr)^{\frac{1}{2}}. \tag{2.4}$$

It follows from Lemma 1.1 that

$$\left|\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\,dx\right| \le C_1\|u\|_V^{2^*(t_1)} \le C_1\|\nabla v\|_2^{2^*(t_1)}\bigl(1+\frac{1}{\eta^2}\bigr)^{\frac{2^*(t_1)}{2}}$$

for some constant $C_1 > 0$. Thus

$$J(u) \geq \frac{1}{2}\|\nabla v\|_2^2 - C_1\|\nabla v\|_2^{2^*(t_1)}\big(1+\frac{1}{\eta^2}\big)^{\frac{2^*(t_1)}{2}} - \frac{1}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\,dx.$$

Taking $\|\nabla v\|_2^2 \leq \rho^2$ small enough we derive from the above inequality that

$$J(u) \geq \frac{1}{4}\|\nabla v\|_2^2 - \frac{1}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\,dx.$$

If $\|u\|_V = \rho$, then combining (2.4) with the last inequality we get

$$J(u) \geq \frac{\rho^2\eta^2}{4\big(1+\eta^2\big)} - \frac{1}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\,dx. \tag{2.5}$$

Estimates (2.3) and (2.5) yield

$$J(u) \geq \min\left(\frac{\rho^2\eta^2}{4\big(1+\eta^2\big)}, \frac{\alpha\rho^{2^*(t_1)}}{2^*(t_1)\big(1+\eta^2\big)^{\frac{2^*(t_1)}{2}}}\right) - \frac{1}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\,dx.$$

Applying Lemma 1.1 to the integral on the right hand side gives

$$J(u) \geq \min\left(\frac{\rho^2\eta^2}{4\big(1+\eta^2\big)}, \frac{\alpha\rho^{2^*(t_1)}}{2^*(t_1)\big(1+\eta^2\big)^{\frac{2^*(t_1)}{2}}}\right) - C_2\rho^{2^*(t_2)}$$

for some constant $C_2 > 0$. Since $2 < 2^*(t_1) < 2^*(t_2)$, taking $\rho > 0$ sufficiently small we can find a constant $\kappa > 0$ such that

$$J(u) \geq \kappa \text{ for } \|u\|_V = \rho$$

which completes the proof. □

We now turn our attention to the case $2^*(t_1) = 2^*(t_2)$.

Proposition 2.3. *Suppose that* (**P**) *and* $2^*(t_1) = 2^*(t_2)$ *hold. Moreover, assume that*

$$\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\,dx + \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}\,dx \neq 0.$$

Then $(PS)_c$ *sequences of* J *are bounded in* $H^1(\Omega)$.

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence. We use the decomposition $u_n = v_n + t_n$, $v_n \in V$ and $t_n \in \mathbb{R}$. First we show that $\{t_n\}$ is bounded. Arguing by contradiction, assume $t_n \to \infty$ (the case $t_n \to -\infty$ can be treated in a similar way). We have

$$c + o(1) + \epsilon_n\|u_n\| = J(u_n) - \frac{1}{2^*(t_1)}\langle J'(u_n), u_n\rangle = \big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\int_\Omega |\nabla v_n|^2\,dx,$$

with $\epsilon_n \to 0$. This shows that

$$\|\nabla v_n\|_2^2 \leq C\big(1 + \|u_n\|_V\big) \tag{2.6}$$

for some constant $C > 0$ independent of n. Inequality (2.6) can be rewritten in the following form

$$\|\nabla\big(\frac{v_n}{t_n}\big)\|_2^2 \leq \frac{C}{t_n}\left(\frac{1}{t_n} + \left[\int_\Omega |\nabla\big(\frac{v_n}{t_n}\big)|^2\, dx + 1\right]^{\frac{1}{2}}\right).$$

Hence $\|\nabla\big(\frac{v_n}{t_n}\big)\|_2^2 \to 0$ and consequently $\frac{v_n}{t_n} \to 0$ in $L^{2^*(t_1)}(\Omega, \frac{1}{|x|^{t_1}})$ and $L^{2^*(t_1)}(\Omega, \frac{1}{|x-\xi|^{t_1}})$. On the other hand we have

$$\begin{aligned} c + o(1) + \epsilon_n\|u_n\|_V &= J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle \\ &= \big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\left(\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u_n|^{2^*(t_1)}\, dx\right. \\ &+ \left.\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}|u_n|^{2^*(t_1)}\, dx\right). \end{aligned}$$

Dividing this equality by $t_n^{2^*(t_1)}$ we get

$$\begin{aligned} \frac{1}{t_n^{2^*(t_1)}}\big(c &+ o(1) + \epsilon_n\|u_n\|_V\big) \\ &= \big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\left(\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|\frac{v_n}{t_n} + 1|^{2^*(t_1)}\, dx\right. \\ &+ \left.\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}|\frac{v_n}{t_n} + 1|^{2^*(t_1)}\, dx\right). \end{aligned}$$

Letting $n \to \infty$ we obtain

$$\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\, dx + \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}\, dx = 0$$

and we have arrived at a contradiction. Since $\{t_n\}$ is bounded, it follows from (2.6) that $\{\|\nabla v_n\|_2\}$ is also bounded and the result follows. □

In the case $2^*(t_1) = 2^*(t_2)$ we can obtain the mountain-pass geometry for a modified variational functional

$$\begin{aligned} J_\mu(u &= \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{2^*(t_1)}\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\, dx \\ &- \frac{\mu}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x|^{t_2}}|u|^{2^*(t_2)}\, dx, \end{aligned}$$

where $0 < \mu < \mu_\circ$ is a parameter and $\mu_\circ > 0$ is sufficiently small. The variational functional J_μ corresponds to the following Neumann problem

$$\left\{\begin{array}{ll} -\Delta u & = \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)-2}u + \mu\frac{P_2(x)}{|x-\xi|^{t_1}}|u|^{2^*(t_1)-2}u \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} & = 0 \ \text{ on } \partial\Omega,\ u > 0 \ \text{ on } \Omega. \end{array}\right. \tag{2.7}$$

Proposition 2.4. *Suppose that* $(\mathbf{P})$ *and* $2^*(t_1) = 2^*(t_2)$ *hold. Then there exist constants* $\mu_\circ > 0$, $\kappa > 0$ *and* $\rho > 0$ *such that*

$$J_\mu(u) \geq \kappa \quad \textit{for} \quad \|u\| = \rho$$

and $0 < \mu < \mu_\circ$

Proof. As in the proof of Proposition 2.2 we get

$$J_\mu(u) \geq \min\left(\frac{\rho^2\eta^2}{4\bigl(1+\eta^2\bigr)}, \frac{\alpha\rho^{2^*(t_1)}}{2^*(t_1)\bigl(1+\eta^2\bigr)^{\frac{2^*(t_1)}{2}}}\right) - \frac{\mu}{2^*(t_1)}\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}|u|^{2^*(t_1)}\,dx.$$

It then follows from Lemma 1.1 that

$$J_\mu(u) \geq \min\left(\frac{\rho^2\eta^2}{4\bigl(1+\eta^2\bigr)}, \frac{\alpha\rho^{2^*(t_1)}}{2^*(t_1)\bigl(1+\eta^2\bigr)^{\frac{2^*(t_1)}{2}}}\right) - \mu C_2\rho^{2^*(t_1)},$$

for some positive constant $C_2 > 0$. The result follows by taking $\mu_\circ$ sufficiently small. □

Problem (2.7) does not have a solution for μ large.

Proposition 2.5. *Suppose that assumptions of* Proposition 2.4 *hold. Then problem* (2.7) *does not admit a solution for*

$$\mu > \frac{-\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\,dx}{\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}\,dx}. \tag{2.8}$$

Proof. Suppose that u is a solution of problem (2.7). Let $\epsilon > 0$. Testing (2.7) with $\phi(x) = \bigl(u^2+\epsilon^2\bigr)^{-\frac{2^*(t_1)-1}{2}}$ we get

$$\begin{aligned} 0 &> -\bigl(2^*(t_1)-1\bigr)\int_\Omega |\nabla u|^2 u\bigl(u^2+\epsilon^2\bigr)^{-\frac{2^*(t_1)+1}{2}}\,dx \\ &= \int_\Omega \frac{P_1(x)}{|x|^{t_1}}\frac{|u|^{2^*(t_1)-1}}{\bigl(u^2+\epsilon^2\bigr)^{\frac{2^*(t_1)-1}{2}}}\,dx \\ &+ \mu\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}\frac{|u|^{2^*(t_1)-1}}{\bigl(u^2+\epsilon^2\bigr)^{\frac{2^*(t_1)-1}{2}}}\,dx. \end{aligned}$$

Hence

$$\mu \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}} \frac{u^{2^*(t_1)-1}}{\left(u^2+\epsilon^2\right)^{\frac{2^*(t_1)-1}{2}}}\, dx < -\int_\Omega \frac{P_1(x)}{|x|^{t_1}} \frac{u^{2^*(t_1)-1}}{\left(u^2+\epsilon^2\right)^{\frac{2^*(t_1)-1}{2}}}\, dx.$$

Letting $\epsilon \to 0$ we obtain

$$\mu \leq \frac{-\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\, dx}{\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}\, dx}$$

and the result follows. □

Remark 2.1. It is clear that problem (2.7) has no solution if

$$\frac{P_1(x)}{|x|^{t_1}} + \mu \frac{P_2(x)}{|x-\xi|^{t_1}} > 0 \quad \text{on} \quad \Omega. \tag{2.9}$$

Obviously inequality (2.9) yields (2.8).

Finally, we consider the case $2^*(t_1) > 2^*(t_2)$. As in the case $2^*(t_1) = 2^*(t_2)$ we consider the nonlinear Neumann problem involving a parameter $\mu > 0$

$$\begin{cases} -\Delta u &= \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)-2}u + \mu \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)-2}u \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial\Omega,\ u>0 \ \text{ on } \Omega, \end{cases} \tag{2.10}$$

where $0 < \mu < \mu_*$ with $\mu_* > 0$ small. Let

$$\begin{aligned} I_\mu(u) &= \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{2^*(t_1)}\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\, dx \\ &- \frac{\mu}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\, dx. \end{aligned}$$

Proposition 2.6. *Suppose* (**P**) *and* $2^*(t_1) > 2^*(t_2)$ *hold. Then there exist constants* $\mu_* > 0$, $\kappa > 0$ *and* $\rho > 0$ *such that*

$$I_\mu(u) \geq \kappa \quad \textit{for} \quad \|u\| = \rho \tag{2.11}$$

and $0 < \mu < \mu_*$. *Moreover,*

$$\inf_{\|u\|\leq\rho} I_\mu(u) < 0 \quad \textit{for} \quad 0 < \mu < \mu_*.$$

Proof. The proof of the first part is similar to that of Proposition 2.2. To show the second part observe that for a constant $t > 0$ we have

$$I_\mu(t) = -\frac{t^{2^*(t_1)}}{2^*(t_1)}\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\, dx - \mu\frac{t^{2^*(t_2)}}{2^*(t_2)}\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}\, dx.$$

Since $2^*(t_1) > 2^*(t_2)$, $I_\mu(t) < 0$ for $t > 0$ sufficiently small. □

3. Palais - Smale condition

We commence with the case $2^*(t_1) < 2^*(t_2)$.

Proposition 3.1. *Let $0, \xi \in \partial\Omega$. Suppose that $(\mathbf{P})$ and $2^*(t_1) < 2^*(t_2)$ hold. Moreover assume that $P_1(0) > 0$. Then $(PS)_c$ condition is satisfied for*

$$c < c^* := \min\Big(\frac{(2-t_1)}{4(N-t_1)}\frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{P_1(0)^{\frac{N-2}{2-t_1}}}, \frac{(2-t_2)}{4(N-t_2)}\frac{\big(S_H^{t_2}\big)^{\frac{N-t_2}{2-t_2}}}{P_1(\xi)^{\frac{N-2}{2-t_2}}}\Big) \tag{3.1}$$

Proof. Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence with c satisfying (3.1). By Proposition 2.1 $\{u_n\}$ is bounded in $H^1(\Omega)$. We may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*(t_1)}(\Omega, \frac{1}{|x|^{t_1}})$ and $L^{2^*(t_2)}(\Omega, \frac{1}{|x-\xi|^{t_2}})$. It follows from P.L. Lions' concentration - compactness principle (see [24]) that

$$|\nabla u_n|^2 \rightharpoonup \mu \geq |\nabla u|^2 + b_\circ\delta_0 + b_\xi\delta_\xi,$$

$$\frac{|u_n|^{2^*(t_1)}}{|x|^{t_1}} \rightharpoonup \frac{|u|^{2^*(t_1)}}{|x|^{t_1}} + a_\circ\delta_0,$$

and

$$\frac{|u_n|^{2^*(t_2)}}{|x-\xi|^{t_2}} \rightharpoonup \frac{|u|^{2^*(t_2)}}{|x-\xi|^{t_2}} + a_\xi\delta_\xi,$$

in the sense of measure, where $b_\circ, b_\xi, a_\circ, a_\xi$ are nonnegative constants and δ_0 and δ_ξ denote the Dirac measures assigned to 0 and ξ, respectively. The constants $b_\circ, b_\xi, a_\circ, a_\xi$ satisfy inequalities

$$\frac{a_\circ^{\frac{2}{2^*(t_1)}} S_H^{t_1}}{2^{\frac{2-t_1}{N-t_1}}} \leq b_\circ \quad \text{and} \quad \frac{a_\xi^{\frac{2}{2^*(t_2)}} S_H^{t_2}}{2^{\frac{2-t_2}{N-t_2}}} \leq b_\xi. \tag{3.2}$$

We have

$$\begin{aligned} c &= \lim_{n\to\infty}\big(J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle\big) \qquad (3.3)\\ &= \big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\,dx \\ &+ \big(\frac{1}{2} - \frac{1}{2^*(t_2)}\big)\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\,dx \\ &+ \big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)a_\circ P_1(0) + \big(\frac{1}{2} - \frac{1}{2^*(t_2)}\big)a_\xi P_2(\xi). \end{aligned}$$

We now observe that

$$0 \leq \int_\Omega |\nabla u|^2\,dx = \int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\,dx + \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\,dx.$$

Hence we derive from (3.3) that

$$\begin{aligned} c &\geq (\frac{1}{2}-\frac{1}{2^*(t_1)})\Big(\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\,dx \\ &+ \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}}|u|^{2^*(t_2)}\,dx\Big) \\ &+ (\frac{1}{2}-\frac{1}{2^*(t_1)})a_\circ P_1(0) + (\frac{1}{2}-\frac{1}{2^*(t_2)})a_\xi P_2(\xi) \\ &\geq (\frac{1}{2}-\frac{1}{2^*(t_1)})a_\circ P_1(0) + (\frac{1}{2}-\frac{1}{2^*(t_2)})a_\xi P_2(\xi). \end{aligned} \tag{3.4}$$

Let φ_δ, $\delta > 0$, be a family of C^1-functions concentrating at 0 as $\delta \to 0$. We derive from $\langle J'(u_n), u_n\varphi_\delta^2\rangle \to 0$ that

$$b_\circ \leq P_1(0)a_\circ \quad \text{and} \quad b_\xi \leq P_2(\xi)a_\xi. \tag{3.5}$$

To complete the proof it is sufficient to show that $a_\circ = a_\xi = 0$. Assume that $a_\circ > 0$, then (3.2) and (3.5) imply that

$$a_\circ \geq \frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{2P_1(0)^{\frac{N-t_1}{2-t_1}}}.$$

It then follows from (3.4) that

$$c \geq \frac{1}{2}(\frac{1}{2}-\frac{1}{2^*(t_1)})\frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{P_1(0)^{\frac{N-2}{2-t_1}}},$$

which is impossible. So $a_\circ = 0$. In a similar manner we show that one has $a_\xi = 0$. □

Remark 3.1. Inspection of the proof of Proposition 3.1 shows that if $P(0) \leq 0$, the $(PS)_c$ sequence cannot concentrate at 0. In this case (3.1) takes the form

$$c < c^* = \frac{(2-t_2)}{4(N-t_2)}\frac{\big(S_H^{t_2}\big)^{\frac{N-t_2}{2-t_2}}}{P_1(\xi)^{\frac{N-2}{2-t_2}}}.$$

We now consider the case $2^*(t_1) = 2^*(t_2)$.

Proposition 3.2. *Let $0, \xi \in \partial\Omega$. Let $(\mathbf{P})$ and $2^*(t_1) = 2^*(t_2)$ hold. Suppose that $P_1(0) > 0$, $0 < \mu$ and*

$$\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\,dx + \mu\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}\,dx \neq 0.$$

Then J_μ satisfies the $(PS)_c$ condition for

$$c < \tilde{c} := \min\Big(\frac{(2-t_1)}{4(N-t_1)} \frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{P_1(0)^{\frac{N-2}{2-t_1}}}, \frac{(2-t_2)}{4(N-t_2)} \frac{\big(S_H^{t_1}\big)^{\frac{N-t_2}{2-t_2}}}{\big(\mu P_2(\xi)\big)^{\frac{N-2}{2-t_1}}} \Big)$$

Proof. We argue as in the proof of Proposition 3.1. Let $\{u_n\} \subset H^1(\Omega)$ be a $(PS)_c$ sequence for J. By Proposition 2.3 $\{u_n\}$ is bounded in $H^1(\Omega)$. So we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$ and $L^{2^*(t_1)}(\Omega, \frac{1}{|x|^{2*(t_1)}}$. We have

$$\begin{aligned} c &= \lim_{n\to\infty} \big(J(u_n) - \frac{1}{2}\langle J'(u_n), u_n\rangle\big) = \big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\Big(\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\,dx \\ &+ \mu \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}|u|^{2^*(t_1)}\,dx\Big) \\ &+ \big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\big(a_\circ P_1(0) + \mu a_\xi P_2(\xi)\big), \end{aligned}$$

where $b_\circ$, b_ξ, $a_\circ$ and a_ξ satisfy

$$b_\circ \le P_1(0)a_\circ \quad\text{and}\quad b_\xi \le \mu P_2(\xi)a_\xi.$$

We now observe that

$$\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|u|^{2^*(t_1)}\,dx + \mu\int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}|u|^{2^*(t_1)}\,dx \ge 0.$$

If $a_\circ > 0$, then

$$c > \frac{1}{2}\big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{P_1(0)^{\frac{N-2}{2-t_1}}},$$

which is impossible. Similarly, if $a_\xi > 0$, then

$$c > \frac{1}{2}\big(\frac{1}{2} - \frac{1}{2^*(t_1)}\big)\frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{\big(\mu P_2(\xi)\big)^{\frac{N-2}{2-t_1}}},$$

which again gives a contradiction and the result follows. □

We now consider the case $2^*(t_1) > 2^*(t_2)$.

Proposition 3.3. *Let $0, \xi \in \partial\Omega$. Suppose that* (**P**) *and $2^*(t_1) > 2^*(t_2)$ hold. Moreover, assume that $P_1(0) > 0$ and $0 < \mu < \mu_*$. If $\{u_n\}$ is a bounded in $H^1(\Omega)$ a $(PS)_c$ sequence for the functional I_μ with*

$$c < \min\Big(\frac{(2-t_1)}{4(N-t_1)} \frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{P_1(0)^{\frac{N-2}{2-t_1}}}, \frac{(2-t_2)}{4(N-t_2)} \frac{\big(S_H^{t_2}\big)^{\frac{N-t_2}{2-t_2}}}{\big(\mu P_2(\xi)\big)^{\frac{N-2}{2-t_2}}} \Big), \tag{3.6}$$

then $\{u_n\}$ contains a subsequence converging weakly to nonzero solution of (2.10).

Proof. Since $\{u_n\}$ is a bounded sequence in $H^1(\Omega)$, we may assume that $u_n \rightharpoonup u$ in $H^1(\Omega)$, $L^{2^*(t_1)}\big(\Omega, \frac{1}{|x|^{t_1}}\big)$ and $L^{2^*(t_2)}\big(\Omega, \frac{1}{|x-\xi|^{t_2}}\big)$. Applying the P.L. Lions' concentration - compactness principle we get (3.4). If $u \equiv 0$ we derive a contradiction with (3.6). □

4. Existence of solutions

We commence with the case $2^*(t_1) < 2^*(t_2)$. Let $0, \xi \in \partial\Omega$. Assume that

$$c^* = \frac{(2-t_1)}{4(N-t_1)} \frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{P_1(0)^{\frac{N-2}{2-t_1}}} \quad \text{and} \quad P_1(0) > 0. \tag{4.1}$$

We choose $r_\circ > 0$ so that $P_1(x) > 0$ on $B(0, 2r_\circ) \subset \Omega$. Let ϕ be a C^1-function such that $\phi(x) = 1$ on $B(0, r_\circ)$, $\phi(x) = 0$ on $\mathbb{R}^N - B(0, 2r_\circ)$ and $0 \le \phi(x) \le 1$ on $\mathbb{R}^N$. To estimate the mountain-pass level of the functional J we use the function given by (1.3) with $s = t_1$. Let $w_{\epsilon,t_1}(x) = \phi(x) W_\epsilon^{t_1}(x)$ and define a function I on $H^1(\Omega)$ by

$$I(u) = \frac{\int_\Omega |\nabla u|^2\, dx}{\left(\int_\Omega \frac{|u|^{2^*(t_1)}}{|x|^{t_1}}\, dx\right)^{\frac{N-2}{N-t_1}}}.$$

Denoting by $H(0)$ a mean curvature of $\partial\Omega$ at 0, we have the following asymptotic estimate for $I(w_{\epsilon,t_1})$ (see [11], [17]):

$$I(w_{\epsilon,t_1}) = \begin{cases} \frac{S_H^{t_1}}{2^{\frac{2-t_1}{N-t_1}}} - H(0) a_N \epsilon^{\frac{2}{2-t_1}} + o\big(\epsilon^{\frac{2}{2-t_1}}\big) & , \ N \ge 4 \\ \frac{S_H^{t_1}}{2^{\frac{2-t_1}{N-t_1}}} - H(0) b_N \epsilon^{\frac{2}{2-t_1}} |\ln \epsilon| + o\big(\epsilon^{\frac{2}{2-t_1}}\big) & , \ N = 3, \end{cases} \tag{4.2}$$

where a_N and b_N are positive constants.

Theorem 4.1. *Let $0, \xi \in \partial\Omega$ and $H(0) > 0$. Suppose* (**P**), $2^*(t_1) < 2^*(t_2)$ *and* (4.1) *hold. If*

$$|P_1(x) - P_1(0)| = o\big(|x|^{\frac{2}{2-t_1}}\big)$$

for x close to 0, *then problem* (1.1) *admits a solution.*

Proof. By Proposition 2.2, the functional J has a mountain-pass structure. Since $2^*(t_1) < 2^*(t_2)$ there exists a function $v \in H^1(\Omega)$ such that $\|v\| > \rho$ and $J(v) < 0$. Let c be a mountain-pass level for J, that is,

$$c = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)),$$

where

$$\Gamma = \{\gamma \in C\big([0,1], H^1(\Omega)\big),\ \gamma(0) = 0,\ \gamma(1) = v\},$$

where $v = tw_{\epsilon,t_1}$ with $t > 0$ sufficiently large. It is clear that

$$\begin{aligned} c &\leq \max_{t\geq 0} J(tw_{\epsilon,t_1}) \leq \max_{t\geq 0}\bigg(\frac{t^2}{2}\int_\Omega |\nabla w_{\epsilon,t_1}|^2\,dx \\ &- \frac{t^{2^*(t_1)}}{2^*(t_1)}\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|w_{\epsilon,t_1}|^{2^*(t_1)}\,dx\bigg) \\ &= \frac{(2-t_1)}{2(N-t_1)}\frac{\Big(\int_\Omega |\nabla w_{\epsilon,t_1}|^2\,dx\Big)^{\frac{N-t_1}{2-t_1}}}{\Big(\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|w_{\epsilon,t_1}|^{2^*(t_1)}\,dx\Big)^{\frac{N-2}{2-t_1}}}. \end{aligned} \tag{4.3}$$

Obviously, the curve $\gamma(s) = stw_{\epsilon,t_1}$, $0 \leq s \leq 1$, with t sufficiently large, belongs to Γ. We now observe that

$$\int_\Omega \frac{P_1(x)}{|x|^{t_1}}|w_{\epsilon,t_1}|^{2^*(t_1)}\,dx = P_1(0)\int_\Omega \frac{|w_{\epsilon,t_1}|^{2^*(t_1)}}{|x|^{t_1}}\,dx + o\big(\epsilon^{\frac{2}{2-t_1}}\big). \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) we derive $c < c_*$. Thus by Proposition 3.1 the functional J satisfies the (PS) condition at the level c. The existence of a solution $u \neq 0$ of (1.1) follows from the mountain-pass principle. By Theorem 10 in [3] we may assume that $u \geq 0$ on Ω. The fact that $u > 0$ on Ω follows from Harnack inequality (see [16]). □

Similarly, we have

Theorem 4.2. *Let* $0, \xi \in \partial\Omega$, $H(\xi) > 0$. *Suppose that* (**P**), $2^*(t_1) < 2^*(t_2)$ *and*

$$c^* = \frac{(2-t_2)}{4(N-t_2)}\frac{\big(S_H^{t_2}\big)^{\frac{N-t_2}{2-t_2}}}{P_2(\xi)^{\frac{N-2}{2-t_2}}} \quad \textit{and} \quad P_1(0) > 0.$$

If

$$|P_2(\xi) - P_2(x)| = o\big(|x|^{\frac{2}{2-t_2}}\big)$$

for x *close to* ξ, *then problem* (1.1) *admits a solution.*

We now consider the case $2^*(t_1) = 2^*(t_2)$. We can always assume that $0 < \mu < \mu_\circ < \frac{P_1(0)}{P_2(\xi)}$ by taking $\mu_\circ$ smaller if necessary. Then

$$\tilde{c} = \frac{(2-t_1)}{4(N-t_1)}\frac{\big(S_H^{t_1}\big)^{\frac{N-t_1}{2-t_1}}}{P_1(0)^{\frac{N-2}{2-t_1}}}.$$

Propositions 2.3, 2.4, 3.2 and Remark 2.1 lead to the following existence theorem in the case $2^*(t_1) = 2^*(t_2)$.

Theorem 4.3. *Let* $0, \xi \in \partial\Omega$. *Let* $P_1(0) > 0$, $2^*(t_1) = 2^*(t_2)$, $H(0) > 0$, $0 < \mu < \mu_\circ$ *and*

$$\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\, dx + \mu \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_1}}\, dx < 0.$$

Moreover assume that $(\mathbf{P})$ *holds and that*

$$|P_1(x) - P_1(0)| = o\big(|x|^{\frac{2}{2-t_1}}\big)$$

for x *close to* 0, *then problem* (2.7) *admits a solution.*

Finally, in the case $2^*(t_1) > 2^*(t_2)$, by Proposition 2.6, the functional I_μ satisfies (2.11) and $\inf_{\|u\|_\rho} I_\mu(u) < 0$. Therefore we can apply the Ekeland variational principle and obtain the $(PS)_c$ sequence with $c = \inf_{\|u\|_\rho} I_\mu(u) < 0$ for $0 < \mu < \mu^*$. This sequence, according to Proposition 3.3, contains a subsequence weakly converging to nonzero solution of (2.10). This allows us to formulate the following existence result for problem (2.10):

Theorem 4.4. *Let* $0, \xi \in \partial\Omega$. *Suppose* $(\mathbf{P})$, $2^*(t_1) > 2^*(t_2)$ *and* $P_1(0) > 0$ *hold. Then problem* (2.10) *admits a solution.*

Theorems 4.3 and 4.4 continue to hold for $\mu = 0$, that is, for the following problem

$$\begin{cases} -\Delta u &= \frac{P(x)}{|x|^s}|u|^{2^*(s)-2}u \ \text{ in } \Omega, \\ \frac{\partial u}{\partial \nu} &= 0 \ \text{ on } \partial\Omega,\ u > 0 \ \text{ on } \Omega, \end{cases} \tag{4.5}$$

where $0 < s < 2$ and $P(x)$ is a continuous function on $\bar{\Omega}$. Moreover, we assume that

(R) The function $P(x)$ changes sign and $\int_\Omega \frac{P(x)}{|x|^s}\, dx < 0$.

The corresponding variational functional is given by

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx - \frac{1}{2^*(s)}\int_\Omega \frac{P(x)}{|x|^2}|u|^{2^*(s)}\, dx.$$

Repeating the arguments from Sections 2 and 3 we can show that I has a mountain - pass geometry. If $P(0) > 0$, then the $(PS)_c$ condition holds for

$$c < \frac{(2-s)}{4(N-s)} \frac{\big(S_H^s\big)^{\frac{N-s}{2-s}}}{P(0)}.$$

If $P(0) \le 0$, the $(PS)_c$ condition holds for every $c \in \mathbb{R}$. We can now state the following existence result for problem (4.5)

Theorem 4.5. *Let $0 \in \partial\Omega$, $0 < s < 2$, $P(0) > 0$ and $H(0) > 0$. Moreover, assume that* (**R**) *holds and*

$$|P(x) - P(0)| = o\big(|x|^{\frac{2}{2-s}}\big)$$

for x close to 0. Then problem (4.5) *admits a solution.*

The proof is similar to that of Theorem 4.1 and is omitted.

Remark 4.1. In the case $2^*(t_1) < 2^*(t_2)$, that is $t_1 > t_2$, a solution u of problem (1.1) satisfies the following estimate

$$\frac{\bar{m}}{d^{t_1}} \int_\Omega u^{\frac{2(t_1-t_2)}{N-2}}\, dx \leq \bar{m} \int_\Omega \frac{u^{\frac{2(t_1-t_2)}{N-2}}}{|x-\xi|^{t_2}}\, dx \leq$$

$$\leq \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}} u^{\frac{2(t_1-t_2)}{N-2}}\, dx \leq -\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\, dx,$$

where, $\bar{m} = \min_{x\in\bar{\Omega}} P_2(x)$. Indeed, taking as a test function $\phi(x) = \big(u^2 + \epsilon^2\big)^{-\frac{2^*(t_1)-1}{2}}$ (see the proof of Proposition 2.5) we get

$$\begin{aligned} 0 &> -(2^*(t_1)-1)\int_\Omega |\nabla u|^2 u\big(u^2+\epsilon^2\big)^{-\frac{2^*(t_1)+1}{2}}\, dx = \\ &= \int_\Omega \frac{P_1(x)}{|x|^{t_1}} \frac{u^{2^*(t_1)-1}}{\big(u^2+\epsilon^2\big)^{\frac{2^*(t_1)-1}{2}}}\, dx + \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}} \frac{u^{2^*(t_2)-1}}{\big(u^2+\epsilon^2\big)^{\frac{2^*(t_1)-1}{2}}}\, dx. \end{aligned}$$

Letting $\epsilon \to 0$ the estimate follows. In a similar, way one can show that a solution u of problem (2.10) (with $2^*(t_1) > 2^*(t_2)$) satisfies the following estimate

$$\begin{aligned} \frac{\bar{m}}{d^{t_1}} \int_\Omega u^{-\frac{2(t_1-t_2)}{N-2}}\, dx &\leq \bar{m}\mu \int_\Omega \frac{u^{-\frac{2(t_1-t_2)}{N-2}}}{|x-\xi|^{t_2}}\, dx \leq \int_\Omega \frac{P_2(x)}{|x-\xi|^{t_2}} u^{-\frac{2(t_1-t_2)}{N-2}}\, dx \\ &\leq -\int_\Omega \frac{P_1(x)}{|x|^{t_1}}\, dx. \end{aligned}$$

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