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The generalized Rayleigh wave in an isotropic solid subject to initial fields

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Abstract - In this paper we solve the problem of generalized Rayleigh wave ($\overset{\circ}{P}_2$) propagation in an isotropic semi-infinite plane subject to initial electro-mechanical fields. We suppose an initial transverse electric field and a particular choice of the incremental stress field according to the boundary surface $x_2 = 0$, in the realistic assumption concerning the material constants and the initial mechanical stress field. In this way we succeed to generalize the well-known Rayleigh equation (1885), and to establish the condition for the existence and uniqueness of the solution of this equation into the interval $(0, 1)$. At the same time, the numerical solutions obtained for silica (SiO_2), are analyzed.

Key words and phrases : generalized Rayleigh wave, isotropic semi-infinite plane, initial electro-mechanical fields.

Mathematics Subject Classification (2000) **:** 74J15, 74F15, 74B15.

1. Introduction

The mathematical problems related to the behaviour of piezoelectric bodies subject to incremental deformations and electric fields superposed on an initial large static deformation and electric fields is a topic of considerable and increasing interest. The problem of waves propagation in piezoelectric crystals subject to initial electro-mechanical fields have attracted considerable attention, due to their complexity and multiple applications (see papers [4–15]).

The monograph [2] establishes the basic equations of the theory of piezoelectric bodies subject to infinitesimal deformations and fields superposed on initial mechanical and electric fields.

An useful development of the equations of electromagnetism in material continua may be found in [19].

The problem of waves propagation in elastic crystals and in piezoelectric crystals, is presented in [3]. The plane waves propagation in a piezoelectric semi-infinite plane when the sagittal plane is normal to a direct, respective inverse axis of order two, or the propagation of the Rayleigh/Bleustein-Gulyaev wave type, presented in this monograph, are considered as being classical.

An alternative derivation of the equations obtained in [2] is described in [1]. In addition, it is presented a qualitative analysis of the problem of harmonic waves propagation in piezoelectric crystals subject to initial fields, in the particular case of homogeneous initial state and non-polarizable environment.

Using the alternative derivation obtained in [1], the mathematical problem describing the propagation of plane guided waves in a non-magnetizable electroelastic semi-infinite plane (called *sagittal plane*) subject to initial fields, is presented in [16, 17]. In this problem the volumetric charge density and mechanical body forces, static or incremental, are neglected, the initial state is homogeneous, the environment is non-polarizable, the body conducts neither heat nor electricity, and the quasi-electrostatic approximation is adopted.

In this context, for a piezoelectric solid from the monoclinic system, and for a particular choice both of the sagittal plane x_1x_2 and of the initial electric field, the mathematical problem decomposes in [17]. It is obtained mechanical and piezoelectric waves generalizing the classical guided waves which are related to the case of the absent initial fields (see, to compare [3]). These are $\overset{\circ}{P}_2$ and $\overline{\overset{\circ}{TH}}$ waves, which stand for the sagittal plane normal to a direct dyad axis and for the initial electric field normal to sagittal plane, respective $\overline{\overset{\circ}{P}}_2$ and $\overset{\circ}{TH}$ waves, which stand for the sagittal plane normal to an inverse dyad axis and for the initial electric field parallel to sagittal plane (see [17]). These waves are called the generalized Rayleigh, or Bleustein-Gulyaev waves.

In this paper we solve the problem of generalized Rayleigh wave ($\overset{\circ}{P}_2$) propagation in the isotropic semi-infinite plane subject to electro-mechanical initial fields (see also [18]). The initial electric field is normal to sagittal plane x_1x_2, and the boundary surface $x_2 = 0$ is free from incremental stresses. In this way, in realistic assumptions concerning the material constants and the initial mechanical stress field, we succeed to generalize the well-known Rayleigh equation (1885), and to obtain the polarization directions. We shall see that these derivations essentially depend on the shear stress field component $\overset{\circ}{S}_{12}$. Moreover, we establish the condition for the existence and uniqueness into the interval $(0, 1)$ of the solution of the generalized Rayleigh equation.

At the same time, we give the numerical solutions obtained for an isotropic material as SiO_2.

2. Basic assumptions

We suppose that we are in the following hypothesis:

the isotropic solid is elastic, with linear behaviour; the isotropic solid conducts neither heat, nor electricity; the environment of the body is not polarizable; the isotropic solid is non-magnetizable.

We shall use the quasi-electrostatic approximation of the equation of balance; the body is homogeneous, and the initial state as well; the initial static mechanical body forces and the volumetric static charge density are neglected; the incremental mechanical body forces and the incremental volumetric charge density are neglected.

The isotropic solid is semi-infinite, occupying the region $x_2 > 0$.

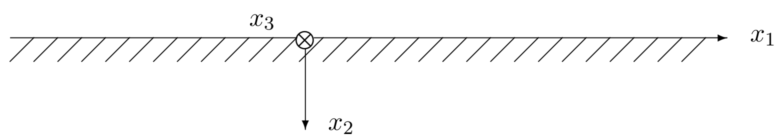

Fig. 1. The isotropic semi-infinite plane

The waves propagate along x_1 axis; the plane x_1x_2 containing the surface normal and the propagation direction is the *sagittal plane* (see Figure 1); the guide of waves has properties invariant with time t and with x_1 variable; we neglect the effects of diffraction in x_3 direction.

We suppose *normal modes* which have the form

$$u_j(\vec{x}, t) = a_{\circ j}(x_2, x_3)\, \mathrm{e}^{\mathrm{i}(\omega t - kx_1)}\ ,\ j = \overline{1,4}\ ,\quad V = \frac{\omega}{k}\ .$$

The initial static electric field is normal to sagittal plane, $(\overset{\circ}{E}_1 = \overset{\circ}{E}_2 = 0)$; the $\overset{\circ}{E}_3$ component of the initial static electric field and the initial static mechanical stress field $\overset{\circ}{S}$ are considered as parameters of the problem.

3. The $\overset{\circ}{P}_2$ generalized problem

Following the results obtained in [16, 17] for monoclinic crystals, the mathematical problem related to the propagation of the plane guided wave $\overset{\circ}{P}_2$ in deformable isotropic semi-infinite plane subject to initial electromechanical fields (see [18]), has the following form:

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{11} - \overset{\circ}{\rho} V^2 & \overset{\circ}{\Gamma}_{12} \\ \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{22} - \overset{\circ}{\rho} V^2 \end{pmatrix} \begin{pmatrix} a_{\circ 1} \\ a_{\circ 2} \end{pmatrix} (x_2) = \overrightarrow{0}\ , \tag{3.1}$$

where

$$\overset{\circ}{\Gamma}_{11} = (\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11}) + 2\mathrm{i}\overset{\circ}{S}_{12}\frac{\partial}{\partial X_2} - (\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})\frac{\partial^2}{\partial X_2^2} \ ,$$
$$\overset{\circ}{\Gamma}_{12} = \mathrm{i}(\overset{\circ}{C}_{12} + \overset{\circ}{C}_{66})\frac{\partial}{\partial X_2} \ , \tag{3.2}$$
$$\overset{\circ}{\Gamma}_{22} = (\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}) + 2\mathrm{i}\overset{\circ}{S}_{12}\frac{\partial}{\partial X_2} - (\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})\frac{\partial^2}{\partial X_2^2} \ ,$$

are *complex differential operators in non-dimensional variable X_2, depending on the initial stress field $\overset{\circ}{S}$, only.*

The mechanical boundary conditions on the plane $x_2 = 0$ are:

$$-\Sigma_{21} = \left\{k\left[-\mathrm{i}\overset{\circ}{S}_{12} + (\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})\frac{\partial}{\partial X_2}\right]a_{\circ 1}(x_2) - \right.$$
$$\left. -k\mathrm{i}\overset{\circ}{C}_{66}\, a_{\circ 2}(x_2)\right\}\mathrm{e}^{\mathrm{i}(\omega t - kx_1)} \ ,$$
$$-\Sigma_{22} = \left\{k\left[-\mathrm{i}\overset{\circ}{S}_{12} + (\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22})\frac{\partial}{\partial X_2}\right]a_{\circ 2}(x_2) - \right. \tag{3.3}$$
$$\left. -k\mathrm{i}\overset{\circ}{C}_{12}\, a_{\circ 1}(x_2)\right\}\mathrm{e}^{\mathrm{i}(\omega t - kx_1)} \ ,$$

depending on the initial stress field $\overset{\circ}{S}$ only, for given stresses $-\Sigma_{21}$, $-\Sigma_{22}$.

Here the elastic constants $\overset{\circ}{C}_{11}$, $\overset{\circ}{C}_{12}$, $\overset{\circ}{C}_{66}$ are linked by relations

$$\overset{\circ}{C}_{11} = \lambda + \mu,\ \overset{\circ}{C}_{12} = \lambda,\ \overset{\circ}{C}_{66} = \mu = (\overset{\circ}{C}_{11} - \overset{\circ}{C}_{12})/2 \ ,$$

where λ and μ are Lamé's coefficients.

The solution of this problem is $\left(a_{\circ 1}, a_{\circ 2}, 0, 0\right) \overset{not}{=} \overrightarrow{a}_{\overset{\circ}{P}_2}$.

Concluding, this problem defines a *mechanical guided wave, polarized in the sagittal plane x_1x_2, depending on the initial stress field $\overset{\circ}{S}$, only.*

If we search the solution of the problem (3.1) - (3.3) in the form

$$\overrightarrow{a_\circ}(x_2) = \overrightarrow{a_\circ}\mathrm{e}^{-\mathcal{X}kx_2} = \overrightarrow{a_\circ}\mathrm{e}^{-\mathcal{X}X_2} \ ,$$

with $\mathcal{X}$ a complex number, so that $Re(\mathcal{X}k) > 0$, and we follow the derivation rules

$$\frac{\partial}{\partial X_2}(\cdot) = -\mathcal{X}(\cdot) \ ,\ \frac{\partial^2}{\partial X_2^2}(\cdot) = \mathcal{X}^2(\cdot) = -(\mathrm{i}\mathcal{X})^2(\cdot) \ ,$$

then our mathematical problem becomes

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{11} - \overset{\circ}{\rho} V^2 & \overset{\circ}{\Gamma}_{12} \\ \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{22} - \overset{\circ}{\rho} V^2 \end{pmatrix} \begin{pmatrix} a_{\circ 1} \\ a_{\circ 2} \end{pmatrix} = \overrightarrow{0} \,, \tag{3.4}$$

called *the fundamental system*. Here

$$\begin{aligned} \overset{\circ}{\Gamma}_{11} &= (\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11}) - 2\overset{\circ}{S}_{12}(\mathrm{i}\mathcal{X}) + (\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})(\mathrm{i}\mathcal{X})^2 \,, \\ \overset{\circ}{\Gamma}_{12} &= -(\overset{\circ}{C}_{12} + \overset{\circ}{C}_{66})(\mathrm{i}\mathcal{X}) \,, \\ \overset{\circ}{\Gamma}_{22} &= (\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}) - 2\overset{\circ}{S}_{12}(\mathrm{i}\mathcal{X}) + (\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})(\mathrm{i}\mathcal{X})^2 \,. \end{aligned} \tag{3.5}$$

The boundary conditions on $x_2 = 0$ are:

$$\begin{aligned} -\Sigma_{21} &= -k\mathrm{i}\left\{\left[\overset{\circ}{S}_{12} - (\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})(\mathrm{i}\mathcal{X})\right] a_{\circ 1} + \overset{\circ}{C}_{66} a_{\circ 2}\right\} \mathrm{e}^{\mathrm{i}(\omega t - kx_1)} \,, \\ -\Sigma_{22} &= -k\mathrm{i}\left\{\overset{\circ}{C}_{12} a_{\circ 1} + \left[\overset{\circ}{S}_{12} - (\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})(\mathrm{i}\mathcal{X})\right] a_{\circ 2}\right\} \mathrm{e}^{\mathrm{i}(\omega t - kx_1)} \,. \end{aligned} \tag{3.6}$$

The displacement vector has the form:

$$\overrightarrow{u} = \overrightarrow{a_\circ} \mathrm{e}^{\mathrm{i}(\omega t - kx_1) - \mathcal{X} k x_2}, \quad \text{where} \quad Re(\mathcal{X}k) > 0 \,. \tag{3.7}$$

4. The analysis of $\overset{\circ}{P}_2$ problem

For the fundamental system (3.4) the compatibility condition is

$$(\overset{\circ}{\Gamma}_{11} - \overset{\circ}{\rho} V^2)(\overset{\circ}{\Gamma}_{22} - \overset{\circ}{\rho} V^2) - \overset{\circ}{\Gamma}{}^2_{12} = 0 \,,$$

or, in the equivalent form

$$\begin{aligned} &\left\{\overset{\circ}{\rho} V^2 - \left[(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})(\mathrm{i}\mathcal{X})^2 - 2\overset{\circ}{S}_{12}(\mathrm{i}\mathcal{X}) + (\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11})\right]\right\} \cdot \\ &\cdot \left\{\overset{\circ}{\rho} V^2 - \left[(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})(\mathrm{i}\mathcal{X})^2 - 2\overset{\circ}{S}_{12}(\mathrm{i}\mathcal{X}) + (\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})\right]\right\} = 0 \,. \end{aligned} \tag{4.1}$$

The equation (4.1) is a fourth degree relation in $(\mathrm{i}\mathcal{X})$ depending on the unknown parameter V. We suppose that $\mathcal{X} = Re\,\mathcal{X} + \mathrm{i}\,Im\,\mathcal{X}$. Then the condition (4.1) reduces now to the following relations

$$\begin{aligned}
&(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22})\Big[(Im\,\mathcal{X})^2-(Re\,\mathcal{X})^2\Big]+\\
&\quad+2\overset{\circ}{S}_{12}(Im\,\mathcal{X})+(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{11}-\overset{\circ}{\rho}\,V^2)=0\ ,\\
&\Big[(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22})(Im\,\mathcal{X})+\overset{\circ}{S}_{12}\Big](Re\,\mathcal{X})=0\ ,
\end{aligned} \tag{4.2}$$

(according to the first factor from (4.1)), or

$$\begin{aligned}
&(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})\Big[(Im\mathcal{X})^2-(Re\mathcal{X})^2\Big]+\\
&\quad+2\overset{\circ}{S}_{12}(Im\mathcal{X})+(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11}-\overset{\circ}{\rho}\,V^2)=0\ ,\\
&\Big[(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})(Im\,\mathcal{X})+\overset{\circ}{S}_{12}\Big](Re\,\mathcal{X})=0\ ,
\end{aligned} \tag{4.3}$$

(according to the second factor from (4.1)).

Taking into consideration that in most of the practical situations

$$\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22}>0\ ,\ \overset{\circ}{C}_{66}+\overset{\circ}{S}_{22}>0\ ,\ \text{and}\ \ Re(\mathcal{X}k)>0\,,$$

then the system (4.2) has a unique solution denoted by $\mathcal{X}_{(1)}$, where

$$\begin{aligned}
Re\mathcal{X}_{(1)}&=\frac{\sqrt{(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22})(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{11}-\overset{\circ}{\rho}\,V^2)-\overset{\circ}{S}{}_{12}^{2}}}{(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22})}\ ,\\
Im\mathcal{X}_{(1)}&=\frac{-\overset{\circ}{S}_{12}}{(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22})}\ ,
\end{aligned} \tag{4.4}$$

if

$$V^2<\frac{(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{11})(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22})-\overset{\circ}{S}{}_{12}^{2}}{\overset{\circ}{\rho}(\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22})}\ . \tag{4.5}$$

For the system (4.3), the unique solution denoted by $\mathcal{X}_{(2)}$, is

$$\begin{aligned}
Re\mathcal{X}_{(2)}&=\frac{\sqrt{(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11}-\overset{\circ}{\rho}\,V^2)-\overset{\circ}{S}{}_{12}^{2}}}{(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})}\ ,\\
Im\mathcal{X}_{(2)}&=\frac{-\overset{\circ}{S}_{12}}{(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})}\ ,
\end{aligned} \tag{4.6}$$

if

$$V^2<\frac{(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11})(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})-\overset{\circ}{S}{}_{12}^{2}}{\overset{\circ}{\rho}(\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})}\ . \tag{4.7}$$

We observe that if the restriction (4.7) is satisfied then the restriction (4.5) is satisfied, too. We also observe that if $\overset{\circ}{S}_{12} = 0$ then $\overset{\circ}{\mathcal{X}}_1$ and $\overset{\circ}{\mathcal{X}}_2$ become real numbers.

In what follows, we shall suppose that the restriction (4.7) is fulfilled.

Hence, in most of the practical situations, the equation (4.1) has two solutions which verify the condition $Re(\mathcal{X}k) > 0$, that is $\mathcal{X}_{(1)}$ and $\mathcal{X}_{(2)}$ given by (4.4) and (4.6), respectively.

Solving the fundamental system (3.4) we obtain the polarization directions

$$\overrightarrow{a}_{\circ(1)} = \big(1, -\mathrm{i}\mathcal{X}_{(1)}, 0, 0\big) \ \text{ and } \ \overrightarrow{a}_{\circ(2)} = \big(\mathrm{i}\mathcal{X}_{(2)}, 1, 0, 0\big) \,, \tag{4.8}$$

corresponding to the displacement $\overrightarrow{u}_{(1)}$, $\overrightarrow{u}_{(2)}$, in the form (3.7).

Therefore, the general solution of the system (3.4) is

$$\overrightarrow{u} = A_1 \overrightarrow{u}_{(1)} + A_2 \overrightarrow{u}_{(2)},$$

where

$$\begin{aligned} &\overrightarrow{u} = \overrightarrow{a}(x_2)\mathrm{e}^{\mathrm{i}(\omega t - k x_1)} \,, \text{ with } \overrightarrow{a}(x_2) = (a_1, a_2, a_3, a_4)(x_2) \ \text{ and} \\ &a_1(x_2) = A_1 \mathrm{e}^{-\mathcal{X}_{(1)} k x_2} + A_2 \mathrm{i}\mathcal{X}_{(2)} \mathrm{e}^{-\mathcal{X}_{(2)} k x_2} \,, \qquad a_3(x_2) = 0 \,, \\ &a_2(x_2) = -A_1 \mathrm{i}\mathcal{X}_{(1)} \mathrm{e}^{-\mathcal{X}_{(1)} k x_2} + A_2 \mathrm{e}^{-\mathcal{X}_{(2)} k x_2} \,, \qquad a_4(x_2) = 0 \,. \end{aligned} \tag{4.9}$$

Here A_1 and A_2 are constant quantities, $\overrightarrow{a}(x_2)$ being named *the polarization direction*.

Taking into account (4.9), in the case of homogeneous mechanical boundary conditions

$$-\Sigma_{21} = 0 \quad \text{and} \quad -\Sigma_{22} = 0 \,, \text{ for } x_2 = 0 \,,$$

the boundary conditions (3.6) take the form

$$\begin{aligned} &\Big[(2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})\mathcal{X}_{(1)} + \mathrm{i}\overset{\circ}{S}_{12}\Big] A_1 + \\ &\qquad + \Big[\mathrm{i}(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})\mathcal{X}_{(2)}^2 - \overset{\circ}{S}_{12}\mathcal{X}_{(2)} + \mathrm{i}\overset{\circ}{C}_{66}\Big] A_2 = 0, \\ &\Big[\mathrm{i}(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})\mathcal{X}_{(1)}^2 - \overset{\circ}{S}_{12}\mathcal{X}_{(1)} - \mathrm{i}\overset{\circ}{C}_{12}\Big] A_1 - \\ &\qquad - \Big[(2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})\mathcal{X}_{(2)} + \mathrm{i}\overset{\circ}{S}_{12}\Big] A_2 = 0. \end{aligned} \tag{4.10}$$

This is a system of equations, the unknown variables being A_1 and A_2.

The compatibility condition for this system is

$$
\begin{aligned}
&-\left[\overset{\circ}{S}{}_{12}^{2}+(2\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})^{2}\right]\mathcal{X}_{(1)}\mathcal{X}_{(2)}-\\
&-\mathrm{i}\overset{\circ}{S}_{12}\left(4\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11}+\overset{\circ}{S}_{22}-\overset{\circ}{\rho}V^{2}\right)\left(\mathcal{X}_{(1)}+\mathcal{X}_{(2)}\right)+\\
&+\left(2\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11}-\overset{\circ}{\rho}V^{2}\right)^{2}+\overset{\circ}{S}{}_{12}^{2}=0\,.
\end{aligned}
\tag{4.11}
$$

Using the usual decomposition into real and imaginary part for $\mathcal{X}_{(1)}$ and $\mathcal{X}_{(2)}$ respectively, the condition (4.11) is now equivalent to the system

$$
\begin{aligned}
&-\left[\overset{\circ}{S}{}_{12}^{2}+(2\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})^{2}\right]\left(Re\mathcal{X}_{(1)}Re\mathcal{X}_{(2)}-Im\mathcal{X}_{(1)}Im\mathcal{X}_{(2)}\right)+\\
&+\overset{\circ}{S}_{12}\left(4\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11}+\overset{\circ}{S}_{22}-\overset{\circ}{\rho}\,V^{2}\right)\left(Im\mathcal{X}_{(1)}+Im\mathcal{X}_{(2)}\right)+\\
&+\left(2\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11}-\overset{\circ}{\rho}\,V^{2}\right)^{2}+\overset{\circ}{S}{}_{12}^{2}=0\,,\\
&-\left[\overset{\circ}{S}{}_{12}^{2}+(2\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22})^{2}\right]\left(Re\mathcal{X}_{(1)}Im\mathcal{X}_{(2)}+Im\mathcal{X}_{(1)}Re\mathcal{X}_{(2)}\right)-\\
&-\overset{\circ}{S}_{12}\left(4\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11}+\overset{\circ}{S}_{22}-\overset{\circ}{\rho}\,V^{2}\right)\left(Re\mathcal{X}_{(1)}+Re\mathcal{X}_{(2)}\right)=0\,.
\end{aligned}
\tag{4.12}
$$

Substituting $Im\mathcal{X}_{(1)}$ from (4.4) and $Im\mathcal{X}_{(2)}$ from (4.6), the system (4.12) becomes equivalent with the system

$$
\begin{aligned}
&l\left[Re\mathcal{X}_{(1)}Re\mathcal{X}_{(2)}\right]=mV^{4}+nV^{2}+p\,,\\
&\overset{\circ}{S}_{12}\Big[(\widetilde{q}-\overset{\circ}{\rho}\,V^{2})(Re\mathcal{X}_{(1)}+Re\mathcal{X}_{(2)})-\\
&\qquad -b\left(\frac{1}{r}Re\mathcal{X}_{(1)}+\frac{1}{w}Re\mathcal{X}_{(2)}\right)\Big]=0\,,
\end{aligned}
\tag{4.13}
$$

where

$$
\begin{aligned}
&\overset{\circ}{C}_{11}+\overset{\circ}{S}_{22}\overset{not}{=}w\,,\ \overset{\circ}{C}_{66}+\overset{\circ}{S}_{22}\overset{not}{=}r\,,\ 2\overset{\circ}{C}_{66}+\overset{\circ}{S}_{22}\overset{not}{=}\widetilde{d}\,,\\
&2\overset{\circ}{C}_{66}+\overset{\circ}{S}_{11}\overset{not}{=}\widetilde{e}\,,\ \overset{\circ}{S}{}_{12}^{2}+\widetilde{d}^{2}\overset{not}{=}b\,,\ \widetilde{d}+\widetilde{e}\overset{not}{=}\widetilde{q}\,,\\
&bwr\overset{not}{=}l\,,\ \overset{\circ}{\rho}{}^{2}wr\overset{not}{=}m\,,\ \overset{\circ}{\rho}\left[\overset{\circ}{S}{}_{12}^{2}(w+r)-2\widetilde{e}wr\right]\overset{not}{=}n\,,\\
&\overset{\circ}{S}{}_{12}^{2}\left[b-\widetilde{q}(w+r)+wr\right]+\widetilde{e}^{2}wr\overset{not}{=}p\,.
\end{aligned}
\tag{4.14}
$$

It is important now to observe that if $\overset{\circ}{S}_{12} = 0$, then (4.13) is equivalent to (4.13.1). It means that the second equation of the system (4.13) is fulfilled, **the system being reduced to the equation (4.13.1), for the real numbers $\mathcal{X}_{(1)}$ and $\mathcal{X}_{(2)}$**. Consequently, to solve the system (4.13) **depends essentially on $\overset{\circ}{S}_{12}$ component**.

5. The velocity equation. Special case $\overset{\circ}{S}_{12} = 0$

From (4.4) and (4.6) it yields that $Re\mathcal{X}_{(1)} = \mathcal{X}_{(1)}$ and $Re\mathcal{X}_{(2)} = \mathcal{X}_{(2)}$, and the equation (4.13.2) is fulfilled. Therefore, the system (4.13) is reduced to the velocity equation (4.13.1), for the real numbers $\mathcal{X}_{(1)}$ and $\mathcal{X}_{(2)}$. Then, the velocity equation becomes

$$R^4 - 8R^3 + \left[\frac{24\,\overset{\circ}{V}_L^{\,2}\,\overset{\circ}{V}_T^{\,2} - 16\overset{\circ}{K}\left(\overset{\circ}{V}_T^{\,2} - \dfrac{\overset{\circ}{S}_{11}}{2\overset{\circ}{\rho}}\right)^2}{\overset{\circ}{V}_L^{\,2}\,\overset{\circ}{V}_T^{\,2}}\right]R^2 - \left[\frac{32\,\overset{\circ}{V}_L^{\,2}\,\overset{\circ}{V}_T^{\,2} - 16\overset{\circ}{K}(\overset{\circ}{V}_L^{\,2} + \overset{\circ}{V}_T^{\,2})\left(\overset{\circ}{V}_T^{\,2} - \dfrac{\overset{\circ}{S}_{11}}{2\overset{\circ}{\rho}}\right)}{\overset{\circ}{V}_L^{\,2}\,\overset{\circ}{V}_T^{\,2}}\right]R + 16(1 - \overset{\circ}{K}) = 0\,, \tag{5.1}$$

where

$$\sqrt{\frac{\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11}}{\overset{\circ}{\rho}}} \overset{not}{=} \overset{\circ}{V}_L \quad , \quad \sqrt{\frac{\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}}{\overset{\circ}{\rho}}} \overset{not}{=} \overset{\circ}{V}_T\,,$$ are the longitudinal,

respectively the transverse speed in the inactive initial field case, and

$$\frac{(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})(2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})^4}{(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})(2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})^4} \overset{not}{=} \overset{\circ}{K}\,,$$

$$\frac{V^2}{\left(\overset{\circ}{V}_T^{\,2} - \dfrac{\overset{\circ}{S}_{11}}{2\overset{\circ}{\rho}}\right)} \overset{not}{=} R\,. \tag{5.2}$$

It represents a **generalization** of the equation

$$R^3 - 8(R-1)\left(R - 1 - \frac{C_{12}}{C_{11}}\right) = 0\,, \tag{5.3}$$

derived by lord Rayleigh (1885), where $R = V^2/V_T^2$, $V_T = \sqrt{C_{66}/\rho}$, and $R \in (0,1)$ if $0 < V < V_T$; the equation (5.3) has an unique solution $\widetilde{R}$ into the interval $(0,1)$, solution that leads to an unique velocity $V = V_T\sqrt{\widetilde{R}}$.

Indeed, we notice that if $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22}$, then $\overset{\circ}{K} = 1$ and **the last term from (5.1) vanishes**. So, this equation becomes one of third degree in R, that is

$$R^3 - 8(R-1)\left[(R-1) - \frac{2\overset{\circ}{C}_{66}\overset{\circ}{C}_{12} + \overset{\circ}{S}_{11}(2\overset{\circ}{C}_{11} - 2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})}{2(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})}\right] - \left[\frac{4\overset{\circ}{S}_{11}(2\overset{\circ}{C}_{11} - 2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})}{(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})}\right] = 0\,, \tag{5.4}$$

or

$$R^3 - 8R^2 + AR - B = 0\,, \tag{5.5}$$

where

$$A = 4\,\frac{5\overset{\circ}{S}{}_{11}^{2} + 2\overset{\circ}{S}_{11}(3\overset{\circ}{C}_{11} + \overset{\circ}{C}_{66}) + 2\overset{\circ}{C}_{66}(3\overset{\circ}{C}_{11} - 2\overset{\circ}{C}_{66})}{(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})}\,,$$
$$B = 8\,\frac{2\overset{\circ}{S}{}_{11}^{2} + \overset{\circ}{S}_{11}(3\overset{\circ}{C}_{11} - \overset{\circ}{C}_{66}) + 2\overset{\circ}{C}_{66}(\overset{\circ}{C}_{11} - \overset{\circ}{C}_{66})}{(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})}\,, \tag{5.6}$$

and

$$R = \frac{V^2}{\overset{\circ}{V}{}_T^{2} - \dfrac{\overset{\circ}{S}_{11}}{2\overset{\circ}{\rho}}}\,,\ \text{belongs to}\ \left(0, 1 + \frac{\overset{\circ}{S}_{11}}{2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}}\right) \overset{not}{=} I. \tag{5.7}$$

Please note that $1 + \dfrac{\overset{\circ}{S}_{11}}{2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}} \in (0,2)$.

In particular, if $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = 0$ *then the equation* (5.4) *becomes the Rayleigh's equation* (5.3).

Therefore, **the equation (5.1), as well as the equation (5.4) or (5.5), represent generalizations of the equation (5.3) derived by lord Rayleigh (1885)**. The equation (5.1) is obtained from (4.13) considering $\overset{\circ}{S}_{12} = 0$. The equation (5.4) or (5.5) is obtained from (5.1) considering, in addition, that $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22}$.

6. Polarization directions. Special case $\overset{\circ}{S}_{12} = 0$

Returning to the system (4.10) of boundary conditions, we obtain

$$\mathcal{X}_{(1)} = \sqrt{\frac{\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11} - \overset{\circ}{\rho}\, V^2}{\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22}}}\ ,\ \mathcal{X}_{(2)} = \sqrt{\frac{\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11} - \overset{\circ}{\rho}\, V^2}{\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22}}}\ , \tag{6.1}$$

and

$$A_1 = -\frac{\mathrm{i}[(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})\mathcal{X}_{(2)}^2 + \overset{\circ}{C}_{66}]}{(2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})\mathcal{X}_{(1)}}\, A_2 \ \Leftrightarrow\ A_2 = \mathrm{i}A_1\sqrt{\frac{\mathcal{X}_{(1)}}{\mathcal{X}_{(2)}}}\ . \tag{6.2}$$

From (4.8), that is

$$\overrightarrow{a_{\circ}}_{(1)} = \big(1, -\mathrm{i}\mathcal{X}_{(1)}, 0, 0\big) \quad \text{and} \quad \overrightarrow{a_{\circ}}_{(2)} = \big(\mathrm{i}\mathcal{X}_{(2)}, 1, 0, 0\big)\ ,$$

and from (6.1), we obtain $\overrightarrow{a_{\circ}}_{(1)}$ and $\overrightarrow{a_{\circ}}_{(2)}$, and following (6.2) and (4.9), we find the components of the polarization direction

$$\begin{gathered} a_{\circ 1}(x_2) = A_1\left[\mathrm{e}^{-\mathcal{X}_{(1)}kx_2} - \sqrt{\mathcal{X}_{(1)}\mathcal{X}_{(2)}}\,\mathrm{e}^{-\mathcal{X}_{(2)}kx_2}\right], \\ a_{\circ 3}(x_2) = 0 \quad , \quad a_{\circ 4}(x_2) = 0\,, \\ a_{\circ 2}(x_2) = \mathrm{i}A_1\sqrt{\frac{\mathcal{X}_{(1)}}{\mathcal{X}_{(2)}}}\left[\mathrm{e}^{-\mathcal{X}_{(2)}kx_2} - \sqrt{\mathcal{X}_{(1)}\mathcal{X}_{(2)}}\,\mathrm{e}^{-\mathcal{X}_{(1)}kx_2}\right]. \end{gathered} \tag{6.3}$$

The corresponding displacement vector is

$$\overrightarrow{u} = \overrightarrow{a_{\circ}}(x_2)\mathrm{e}^{\mathrm{i}(\omega t - kx_1)}\,. \tag{6.4}$$

These relations are *identical* in form with those obtained by lord Rayleigh (1885). The only difference between them is the different expressions for $\mathcal{X}_{(1)}$ and $\mathcal{X}_{(2)}$, (see (4.4) and (4.6)). Note that, in the absence of initial fields, these expressions provide the same known quantities obtained by lord Rayleigh, as we see in what follows.

Hence, in the special case $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22}$ we obtain

$$\mathcal{X}_{(1)} = \sqrt{1 - \frac{V^2}{\overset{\circ}{V}_L^{\,2}}}\ ,\ \text{where}\ \ \overset{\circ}{V}_L^{\,2} = \frac{\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11}}{\overset{\circ}{\rho}}\ ,$$

$$\mathcal{X}_{(2)} = \sqrt{1 - \frac{V^2}{\overset{\circ}{V}_T^{\,2}}}\ ,\ \text{where}\ \ \overset{\circ}{V}_T^{\,2} = \frac{\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}}{\overset{\circ}{\rho}}\ .$$

If $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = \overset{\circ}{S}_{12} = 0$, (i.e. in the absence of initial fields), then these last relations take place for the classical speed values $\overset{\circ}{V}_L^{\,2} = \overset{\circ}{C}_{11}/\overset{\circ}{\rho}$ (the longitudinal elastic wave speed), and $\overset{\circ}{V}_T^{\,2} = \overset{\circ}{C}_{66}/\overset{\circ}{\rho}$ (the transverse elastic wave speed).

7. The solutions of the generalized equation (5.5). Special case $\overset{\circ}{S}_{12} = 0$

We focus our attention more on the equation (5.5), that is the equation obtained from the generalized equation (5.1) considering in addition that $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22}$.

We define the associated function

$$f : [0, 2] \to \mathbb{R} \quad , \quad f(R) = R^3 - 8R^2 + AR - B \, , \tag{7.1}$$

where A and B are the constants defined by (5.6).

In order to study the existence and the uniqueness of the solution of this equation, we define:

$$\begin{aligned}
\overset{\circ}{S}_1 &= \frac{-(5\overset{\circ}{C}_{11} + \overset{\circ}{C}_{12}) + \sqrt{9\overset{\circ}{C}_{11}^{\,2} + 10\overset{\circ}{C}_{11}\overset{\circ}{C}_{12} + 17\overset{\circ}{C}_{12}^{\,2}}}{8} < 0 \, , \\
\overset{\circ}{S}_2 &= \frac{-(5\overset{\circ}{C}_{11} + 9\overset{\circ}{C}_{12}) + \sqrt{49\overset{\circ}{C}_{11}^{\,2} + 66\overset{\circ}{C}_{11}\overset{\circ}{C}_{12} + 81\overset{\circ}{C}_{12}^{\,2}}}{12} > 0 \, , \\
\overset{\circ}{S}_3 &= \frac{(5\overset{\circ}{C}_{12} - 3\overset{\circ}{C}_{11}) - \sqrt{9\overset{\circ}{C}_{11}^{\,2} - 18\overset{\circ}{C}_{11}\overset{\circ}{C}_{12} + 13\overset{\circ}{C}_{12}^{\,2}}}{2} < 0 \, , \\
\overset{\circ}{S}_4 &= \frac{(5\overset{\circ}{C}_{12} - 3\overset{\circ}{C}_{11}) + \sqrt{9\overset{\circ}{C}_{11}^{\,2} - 18\overset{\circ}{C}_{11}\overset{\circ}{C}_{12} + 13\overset{\circ}{C}_{12}^{\,2}}}{2} > 0 \, ,
\end{aligned} \tag{7.2}$$

$$\min\{\overset{\circ}{S}_1,\ \overset{\circ}{S}_3\} = a \ < 0 \, , \quad \min\{\overset{\circ}{S}_2,\ \overset{\circ}{S}_4\} = b \ > 0 \, , \quad J = (a,\ b) \, .$$

- If $\overset{\circ}{S}_{11} \in (0, b)$, then $1 + \dfrac{\overset{\circ}{S}_{11}}{2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}} > 1$ and the equation (5.5) **has an unique solution $\widetilde{R}$ into the interval** $(0, 1)$. The corresponding velocity is

$$\overset{\circ}{V} = \sqrt{\left(\overset{\circ}{V}_T^{\,2} - \frac{\overset{\circ}{S}_{11}}{2\overset{\circ}{\rho}}\right)\widetilde{R}} \quad , \quad 0 < \overset{\circ}{V} < \overset{\circ}{V}_T \, . \tag{7.3}$$

- If $\overset{\circ}{S}_{11} \in (a, 0)$, then $0 < 1 + \dfrac{\overset{\circ}{S}_{11}}{2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}} \overset{not}{=} c < 1$ and the equation (5.5)

has an unique solution $\widetilde{R} \in (0, c)$ **if, in addition, the condition**

$$f\,(c) > 0\ , \tag{7.4}$$

is satisfied; the corresponding velocity $\overset{\circ}{V}$ results from (7.3) in the same way.

8. Numerical example. Special case $\overset{\circ}{S}_{12} = 0$

We shall consider now that the isotropic material is SiO_2 (silica).

In the absence of the initial fields, the classical Rayleigh equation (5.3) has the solution $V = V_T\sqrt{\widetilde{R}}$, where $V_T = 3763.3105$ m/s, $\widetilde{R} = 0.8204850$, so $V = 3408.8302$ m/s. In the case of active initial fields, considering $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = (p\%)\overset{\circ}{C}_{11}$ and $\overset{\circ}{S}_{12} = 0$, where $p \in \{-5, -4, \ldots, 4, 5\}$, we shall establish the admissible solutions of the generalized Rayleigh's equation (5.5), for every situation concerning the parameter p. We shall see that, in each case, the above mentioned equation has a single admissible solution $\overset{\circ}{V}$. We shall estimate the relative variation $(\overset{\circ}{V} - V)/V \overset{not}{=} \Delta_r$, where V is the propagation velocity according to the absent initial fields situation (obtained here in the case $p = 0$). This numerical example confirms all the results related to the existence and the uniqueness of the solution of the equation (5.5).

In this way, considering

-the material constants:

$$\overset{\circ}{C}_{11} = 7.85 \cdot 10^{10}\ N/m^2\ ,\ \overset{\circ}{C}_{12} = 1.61 \cdot 10^{10}\ N/m^2\ ,$$
$$\overset{\circ}{C}_{66} = 3.12 \cdot 10^{10}\ N/m^2\ ,\ \overset{\circ}{\rho} = 2203\ kg/m^3\ ,$$

-the parametric initial stress field:

$$\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = (p\%)\overset{\circ}{C}_{11}\ ,\ p \in \mathbb{Z}\ ,\ |p| \le 5\ ,$$

-the longitudinal (L), (transverse (T)) generalized velocity:

$$\overset{\circ}{V}_L = \sqrt{\frac{\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11}}{\overset{\circ}{\rho}}}\ ,\ \overset{\circ}{V}_T = \sqrt{\frac{\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}}{\overset{\circ}{\rho}}}\ ,$$

-the definition domain for the generalized equation: $I = \left(0, 1 + \dfrac{\overset{\circ}{S}_{11}}{2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}}\right)$,

-the generalized velocity solution: $\overset{\circ}{V} = \sqrt{\left(\overset{\circ}{V}_T^{\,2} - \dfrac{\overset{\circ}{S}_{11}}{2\,\overset{\circ}{\rho}}\right)\widetilde{R}}\ ,$

-the admissibility condition for generalized velocity solution: $\overset{\circ}{V} \in (0, \overset{\circ}{V}_T)$,

-the relative variation: $\Delta_r = \dfrac{\overset{\circ}{V} - V}{V}$,

-the positive domain for the $\overset{\circ}{S}_{11}$ *stress component (taction case)*

$\big(0 \, , \, b \big) = (\, 0 \, , \, 0.833866 \cdot 10^{10} \,) \; N/m^2$,

domain for which the existence and uniqueness is established ,

-the negative domain for the $\overset{\circ}{S}_{11}$ *stress component (compression case)*

$\big(a \, , \, 0 \big) = (\, -1.7416486 \cdot 10^{10} \, ; \, 0 \,) \; N/m^2$,

domain for which the existence and uniqueness is established if the condition $f(c) > 0$ *, where f is given by* (7.1) *and* $1 + \overset{\circ}{S}_{11} / (2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}) \overset{not}{=} c$ *, is satisfied ,*

we obtain the numerical data of Table 1.

Table 1. Numerical data obtained in the special case $\overset{\circ}{S}_{12} = 0$

$\overset{\circ}{S}_{11} = (p\%)\overset{\circ}{C}_{11}$	Longitudinal generalized velocity	Transverse generalized velocity	The domain of definition for the generalized equation	The solution of generalized equation	The corresponding velocity solution	The relative variation	The supplementary condition $f(\cdot) > 0$
p	$\overset{\circ}{V}_L$	$\overset{\circ}{V}_T$	$I = (0;\cdot)$	$\widetilde{R}$	$\overset{\circ}{V}$	Δ_r	$f(\cdot)$
-5	5818.2099	3518.6424	(0;0.93287730)	0.70798034	3065.3034	≃ -10%	1.3901
-4	5848.7519	3568.9181	(0;0.94701316)	0.73239215	3138.5599	≃ -7.9%	1.2982
-3	5879.1352	3618.4954	(0;0.96077942)	0.75578399	3209.3348	≃ -5.8%	1.2140
-2	5909.3623	3667.4025	(0;0.97419037)	0.77822337	3277.8479	≃ -3.8%	1.1366
-1	5939.4356	3715.6659	(0;0.9872596)	0.79977171	3344.2905	≃ -1.8%	1.0655
0	**5969.3574**	**3763.3105**	**(0;1)**	**0.820485**	**3408.8302**	**= 0.0%**	**—**
1	5999.1299	3810.3593	(0;1.0124238)	0.84041443	3471.6145	≃ 1.8%	—
2	6028.754	3856.8342	(0;1.0245428)	0.85960691	3532.7742	≃ 3.6%	—
3	6058.2361	3902.7558	(0;1.0363678)	0.8781055	3592.4254	≃ 5.3%	—
4	6087.5739	3948.1432	(0;1.0479097)	0.89594985	3650.6721	≃ 7.0%	—
5	6116.7711	3993.0148	(0;1.0591783)	0.91317646	3707.6074	≃ 8.7%	—

Taking into account the results obtained in Table 1, we observe that, in realistic assumptions concerning the parametric initial stress field, that is $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = (p\%)\overset{\circ}{C}_{11}$, (where $p \in \{-5, -4, \ldots, 4, 5\}$), and $\overset{\circ}{S}_{12} = 0$, the solution $\overset{\circ}{V}$ of the equation (5.5) is close to the solution of the classical Rayleigh's equation, in the sense that $|\Delta_r| \leq 10\%$, and in the case of $p = 0$, both of these equations coincide.

We plot for each integer value of p, so that $p \in \{-5, -4, \ldots, 4, 5\}$, the graph of function f given by (7.1). For a better view we give a zoom on the $[0, 1]$ interval.

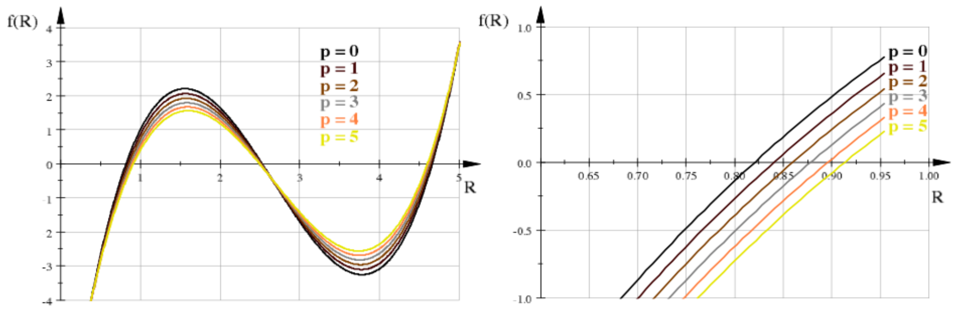

Fig. 2. The graph of the function f for $p \in \{0 \ldots, 4, 5\}$ (traction case)

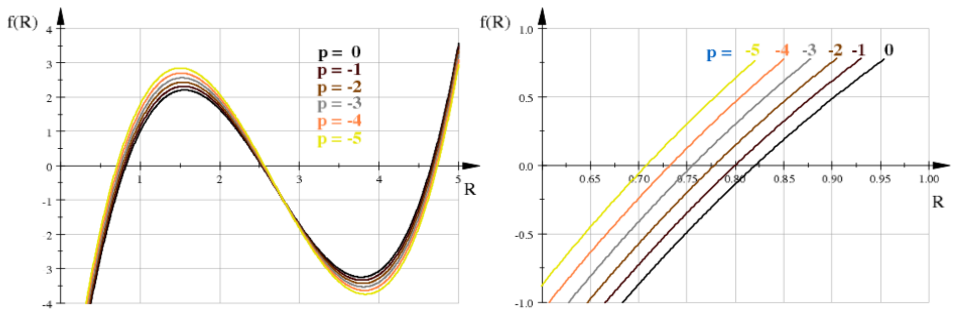

Fig. 3. The graph of the function f for $p \in \{-5, -4 \ldots, 0\}$ (compression case)

It is important to observe that, *the increasing of the traction leads to the increasing of the velocity, and the increasing of the compression leads to the decreasing of the velocity.* At the same time, *in the traction case the obtained velocity is greater than that obtained in the classical case, and in the compression case the obtained velocity is smaller than in the classical one.* However, we observe *the graphics asymmetry for* $p = \pm\varepsilon$, $\varepsilon \in \{1, \ldots, 5\}$ (see Figure 2-3).

9. The generalized velocity equation. Special case $\mathring{S}_{12} \neq 0$

In this case, the equation (4.13.2) is equivalent with

$$\left\{(4\mathring{C}_{66} + \mathring{S}_{11} + \mathring{S}_{22} - \mathring{\rho}\, V^2) - \frac{\left[\mathring{S}_{12}^{\,2} + (2\mathring{C}_{66} + \mathring{S}_{22})^2\right]}{(\mathring{C}_{11} + \mathring{S}_{22})(\mathring{C}_{66} + \mathring{S}_{22})}\mathring{S}_{22}\right\} \cdot$$

$$\cdot (Re\mathcal{X}_{(2)} + Re\mathcal{X}_{(1)}) = \tag{9.1}$$

$$= \frac{\left[\mathring{S}_{12}^{\,2} + (2\mathring{C}_{66} + \mathring{S}_{22})^2\right]}{(\mathring{C}_{11} + \mathring{S}_{22})(\mathring{C}_{66} + \mathring{S}_{22})}(Re\mathcal{X}_{(2)}\mathring{C}_{66} + Re\mathcal{X}_{(1)}\mathring{C}_{11}) \; .$$

Taking into account that both members of this equality are positive, and the following notation

$$
\begin{aligned}
&\overset{\circ}{S}{}_{12}^{2} + (2\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})^{2} = b\,,\ \ \overset{\circ}{\rho}(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22}) = c\,,\\
&\overset{\circ}{S}_{22}b - (4\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11} + \overset{\circ}{S}_{22})(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22}) = d\,,\\
&(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22}) = e\,,\ \ (\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})\,\overset{\circ}{\rho} e = g\,,\\
&(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})^{2}\big[(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11}) - \overset{\circ}{S}{}_{12}^{2}\big] = f\,,\\
&(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})^{2}\big[(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{22})(\overset{\circ}{C}_{11} + \overset{\circ}{S}_{11}) - \overset{\circ}{S}{}_{12}^{2}\big] = h\,,\\
&(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22})\,\overset{\circ}{\rho} e = j\,,\ \ (\overset{\circ}{C}_{66} + \overset{\circ}{S}_{11})(\overset{\circ}{C}_{66} + \overset{\circ}{S}_{22}) - \overset{\circ}{S}{}_{12}^{2} = s\,,
\end{aligned}
\tag{9.2}
$$

and substituting $Re\mathcal{X}_{(2)}$ and $Re\mathcal{X}_{(1)}$ from (4.4) and (4.6), the equality (9.1) has the form:

$$
\begin{aligned}
&\big[Re\mathcal{X}_{(2)}\,Re\mathcal{X}_{(1)}\big]\Big\{(2e^{2}c^{2})V^{4} + (4cde^{2})V^{2} + (2e^{2})(d^{2} - b^{2}\overset{\circ}{C}_{11}\overset{\circ}{C}_{66})\Big\} =\\
&\quad = \Big[c^{2}(g+j)\Big]V^{6} + \Big[2cd\,(g+j) - c^{2}(f+h)\Big]V^{4} +\\
&\quad + \Big[-2cd\,(f+h) + d^{2}(g+j) - b^{2}(g\overset{\circ}{C}{}_{66}^{2} + j\overset{\circ}{C}{}_{11}^{2})\Big]V^{2} +\\
&\quad + \Big[b^{2}(f\overset{\circ}{C}{}_{66}^{2} + h\overset{\circ}{C}{}_{11}^{2}) - d^{2}(f+h)\Big]\,.
\end{aligned}
\tag{9.3}
$$

Using the notations (4.14) and (9.2), the system (4.13) is equivalent with the system

$$
\begin{aligned}
&l\big[Re\mathcal{X}_{(1)}\,Re\mathcal{X}_{(2)}\big] = mV^{4} + nV^{2} + p\,,\\
&\Big\{(2e^{2}c^{2})V^{4} + (4cde^{2})V^{2} + (2e^{2})(d^{2} - b^{2}\overset{\circ}{C}_{11}\overset{\circ}{C}_{66})\Big\}\cdot\big[Re\mathcal{X}_{(2)}\,Re\mathcal{X}_{(1)}\big] =\\
&\quad = \Big[c^{2}(g+j)\Big]V^{6} + \Big[2cd\,(g+j) - c^{2}(f+h)\Big]V^{4} +\\
&\quad + \Big[-2cd\,(f+h) + d^{2}(g+j) - b^{2}(g\overset{\circ}{C}{}_{66}^{2} + j\overset{\circ}{C}{}_{11}^{2})\Big]V^{2} +\\
&\quad + \Big[b^{2}(f\overset{\circ}{C}{}_{66}^{2} + h\overset{\circ}{C}{}_{11}^{2}) - d^{2}(f+h)\Big]\,.
\end{aligned}
\tag{9.4}
$$

Multiplying by l both members of the equation (9.4.2), and taking into account (9.4.1), we obtain the equation

$$
\alpha V^{8} + \beta V^{6} + \gamma V^{4} + \delta V^{2} + \varepsilon = 0\,,
\tag{9.5}
$$

where

$$
\begin{aligned}
&2e^2c^2m \overset{not}{=} \alpha\,,\ 2e^2c^2n + 4cde^2m - lc^2(g+j) \overset{not}{=} \beta\,,\\
&2e^2c^2p + 4cde^2n + 2e^2d^2m - 2e^2b^2m\overset{\circ}{C}_{11}\overset{\circ}{C}_{66} - 2cdl(g+j) +\\
&\qquad\qquad + c^2l(f+h) \overset{not}{=} \gamma\,,\\
&4cde^2p + 2e^2d^2n - 2e^2b^2n\overset{\circ}{C}_{11}\overset{\circ}{C}_{66} + 2cdl(f+h) - d^2l(g+h) +\\
&\qquad\qquad + b^2l(g\overset{\circ}{C}{}_{66}^{2} + j\overset{\circ}{C}{}_{11}^{2}) \overset{not}{=} \delta\,,\\
&2e^2d^2p - 2e^2b^2p\overset{\circ}{C}_{11}\overset{\circ}{C}_{66} - b^2l(f\overset{\circ}{C}{}_{66}^{2} + h\overset{\circ}{C}{}_{11}^{2}) + d^2l(f+h) \overset{not}{=} \varepsilon\,,
\end{aligned}
\tag{9.6}
$$

with the other constants defined by (4.14) and (9.2).
In addition V satisfies the condition

$$V^2\overset{\circ}{\rho}r < s\ . \tag{9.7}$$

Concerning the polarization directions, the system (4.10) of the boundary conditions, gives us A_2 as a function depending on A_1, and following (4.8), (4.4) and (4.6), we can find the polarization direction $\overrightarrow{a_\circ}(x_2)$ and the according displacement vector described by (4.9).

10. Numerical example. Special case $\overset{\circ}{S}_{12} \neq 0$

We shall consider now that the isotropic material is SiO_2 (silica).

In the absence of the initial fields, the classical Rayleigh equation (7.4) has the solution $V = V_T\sqrt{\widetilde{R}}$, where $V_T = 3763.3105$ m/s, $\widetilde{R} = 0.8204850$, so $V = 3408.8302$ m/s.

In the case of active initial fields, considering $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = (p\%)\overset{\circ}{C}_{11}$ and $\overset{\circ}{S}_{12} = (u\%)\overset{\circ}{C}_{12}$, where $p \in \{-2,-1,0,1,2\}$, $u \in \{-2,-1,1,2\}$, we shall establish the admissible solutions of generalized Rayleigh equation (7.1), for every situation concerning p, u respectively.

In this way, if we take

-the material constants:

$$\overset{\circ}{C}_{11} = 7.85\cdot 10^{10}\ N/m^2\ ,\ \overset{\circ}{C}_{12} = 1.61\cdot 10^{10}\ N/m^2\ ,$$

$$\overset{\circ}{C}_{66} = 3.12\cdot 10^{10}\ N/m^2\ ,\ \overset{\circ}{\rho} = 2203\ kg/m^3\ ,$$

-the parametric initial stress field:

$$\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = (p\%)\overset{\circ}{C}_{11}\ ,\ p\in\mathbb{Z}\ ,\ |p|\le 2;\ \overset{\circ}{S}_{12} = (u\%)\overset{\circ}{C}_{12}\ ,\ u\in\mathbb{Z}^*,\ |u|\le 2\ ,$$

-the limit velocity: $V_l = \sqrt{s/\overset{\circ}{\rho}r}$, *and the admissibility condition:* $0 < \overset{\circ}{V} < V_l$,

-the relative variation: $\Delta_r = (\overset{\circ}{V} - V)/V$, where V is the classical solution,

Table 2. Numerical data obtained in the special case $\overset{\circ}{S}_{12} \neq 0$

$\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = (p\%)\overset{\circ}{C}_{11}$	$\overset{\circ}{S}_{12} = (u\%)\overset{\circ}{C}_{12}$	The limit velocity	The generalized velocity solution	The relative variation
p	u	V_l	$\overset{\circ}{V}$	Δ_r
-2	0	3667.4025	3277.8479	$\simeq -3.84\%$
-1	0	3715.6659	3344.2905	$\simeq -1.89\%$
0	**0**	**3763.3105**	**3408.8302**	$=$ **0.0%**
1	0	3810.3593	3471.6145	$\simeq 1.84\%$
2	0	3856.8342	3532.7742	$\simeq 3.63\%$
-2	± 1	3667.3000	3277.8000	$\simeq -3.84\%$
-1	± 1	3715.6139	3344.2355	$\simeq -1.89\%$
0	± 1	3763.2604	3408.7775	$\simeq -0.001\%$
1	± 1	3810.3110	3471.5639	$\simeq 1.84\%$
2	± 1	3856.8000	3532.7000	$\simeq 3.63\%$
-2	± 2	3667.1859	3277.6176	$\simeq -3.84\%$
-1	± 2	3715.4577	3344.0703	$\simeq -1.89\%$
0	± 2	3763.1100	3408.6192	$\simeq -0.006\%$
1	± 2	3810.1662	3471.4121	$\simeq 1.83\%$
2	± 2	3856.6480	3532.5797	$\simeq 3.63\%$

we obtain the numerical data of Table 2.

Therefore, following these numerical data, we observe that, in realistic assumptions concerning the parametric initial stress field, the solution $\overset{\circ}{V}$ of the generalized Rayleigh equation is close to the classical solution, in the sense that $|\Delta_r| \leq 3.84\%$. At the same time, in the case of $p = 0$, $u \to 0$, $u \neq 0$, the generalized velocity solution, and classical velocity solution coincide.

It is important to observe that, the generalized equation (9.5) was obtained in the case $\overset{\circ}{S}_{12} \neq 0$ explicitly. Related to this equation, if we consider that $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = 0$ and $\overset{\circ}{S}_{12} \to 0$, then the new obtained equation and the classical Rayleigh's equation, *do not coincide*. Thus, the equation (9.5) is a *new equation* which generalizes *not in form* the classical equation. However, the admissible solution of this new equation proves to be an *approximation* of the classical solution, and in the case $\overset{\circ}{S}_{11} = \overset{\circ}{S}_{22} = 0$ and $\overset{\circ}{S}_{12} \to 0$ these solutions *coincide*.

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