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Characterization of symmetric extensions of a valuation on a field K to $K(X_1, \ldots, X_n)$

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Abstract - This paper deals with the characterization of the symmetric valuations on $K(X_1, \ldots, X_n)$. Notions as ultrasymmetric extensions and symmetrically-open extensions are defined. Sufficient conditions for extending the symmetry of a valuation are discussed. The main results are a closed-form expression of the r.t.s.-extensions and a complete classification of the symmetrically-open extensions.

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1. Introduction

The classification of the extensions of a valuation, from K to $K(X_1, \ldots, X_n)$ (for $n \geq 2$), is still an open problem in algebra, even if the extensions from K to $K(X)$ have been completely analyzed and described in [4, 7, 10] and [9]. The reason for this is the fact that, when getting with analysis to the second indeterminate (X_2), one has to face the algebraic closure of the field $K(X_1)$, which raises difficult issues in the domain of algebraic geometry (algebraic functions of one or several indeterminates).

In the paper [11], by the same author, it has been defined a special class of extensions of a valuation from K to $K(X_1, \ldots, X_n)$, called *symmetrical valuations*, which treats in an undifferentiated way the n indeterminates and, thanks to this property, allows an analysis that avoids the barrier mentioned above. The main result of that paper was the definition and characterization of the *r.t.s.-extensions*, which will play a crucial role in this study.

This paper continues the work started in [11] by defining the notions of *ultrasymmetry* and *symmetrically-openness*, obtaining a complete classification of the r.t.s.-extensions, discussing the extension of the symmetry to an algebraic closure and finally, using all these, giving a complete classification of the symmetrically-open extensions.

2. General notations and definitions

Let K be a field and v a valuation on K. We will write this pair (K, v). We will denote by k_v the residue field, by G_v the value group, by O_v the valuation ring and by M_v the maximal ideal of v. We will also denote by $\rho_v : O_v \to k_v$ the residual homeomorphism. For $x \in O_v$ we denote by $x^* = \rho_v(x)$, its image in k_v.

Given u and u' two valuations on K, we will say that u is equivalent to u' and write $u \cong u'$, if there exists an isomorphism of order groups $j : G_u \to G'_u$ such that $u' = ju$.

Let K'/K be an extension of fields. We will call a valuation v' on K' an extension of v if $v'(x) = v(x)$ for all x in K. If v' is an extension of v we will canonically identify $k_{v'}$ with a subfield of k_v and G_v with a subgroup of $G_{v'}$.

Let (K, v) be a valued field. If we choose $\bar{K}$ an algebraic closure of K and $\bar{v}$ an extension of v to $\bar{K}$, then the residual field of $\bar{v}$ will be, in fact, an algebraic closure of k_v and the value group of $\bar{v}$ will be $\mathbf{Q}G_v$, namely the smallest divisible group that contains G_v.

We denote by $K(X)$ the field of rational fractions of an indeterminate X over K and with $K[X]$ the ring of polynomials of an indeterminate X over K.

Let u be an extension of v to $K(X)$. We will say that u is a residual-transcendental extension (*r.t.- extension*) if k_u/k_v is a transcendental extension of fields. When not, but we still have $G_u \subseteq \mathbf{Q}G_v$, we will say that u is a residual-algebraic torsion extension (*r.a.t.-extension*) and when $G_u \not\subset \mathbf{Q}G_v$, we will say that u is a residual-algebraic free extension (*r.a.f.-extension*). Additional information to this classification may be found in [4].

In [11] a *symmetric valuation* (with respect to $X_1, \ldots, X_n$) was defined as a valuation w on $K(X_1, \ldots, X_n)$, $n \geq 2$, such that, given any permutation π of $\{1, 2, \ldots, n\}$ and any $f \in K(X_1, \ldots, X_n)$, we have

$$w(f(X_1, X_2, \ldots, X_n)) = w(f(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})).$$

In this case we denote by $\pi f(X_1, X_2, \ldots, X_n) = f(X_{\pi(1)}, X_{\pi(2)}, \ldots, X_{\pi(n)})$, the automorphism $f \to \pi f$ of $K(X_1, \ldots, X_n)$ that leaves the symmetric fractions of polynomials in $K(X_1, \ldots, X_n)$ unchanged.

Let w be a symmetric valuation on $K(X_1, \ldots, X_n)$. Let $\overline{K(X_1, \ldots, X_n)}$ be an algebraic closure of $K(X_1, \ldots, X_n)$ and $\bar{w}$ an extension of w from $K(X_1, \ldots, X_n)$ to $\overline{K(X_1, \ldots, X_n)}$.

We say that $\bar{w}$ *extends the symmetry* of w if, for any partition of $\{1, 2, \ldots, n\} = \{i_1, i_2, \ldots, i_m\} \cup \{j_1, j_2, \ldots, j_{n-m}\}$, with $0 \leq m < n$, the restriction of $\bar{w}$ to $\overline{K(X_{i_1}, \ldots, X_{i_m})}(X_{j_1}, \ldots, X_{j_{n-m}})$ is symmetric with respect to $X_{j_1}, \ldots, X_{j_{n-m}}$, where $\overline{K(X_{i_1}, \ldots X_{i_m})}$ is the closure of $K(X_{i_1}, \ldots X_{i_m})$

in $\overline{K(X_1,\ldots,X_n)}$. For such an extension we denote by:

$$\begin{aligned} \delta_a &:= \bar{w}(X-a), \text{ for any } a\in\bar{K}, \text{ where } X \text{ is arbitrarily} \\ &\quad\ \text{chosen from } X_1,\ldots,X_n; \\ \mathcal{M}_{\bar{w}} &:= \{\delta_a / a\in\bar{K}\}; \end{aligned}$$

and for any i, such that $0\le i\le n$, we denote by:

$$\begin{aligned} K_i &:= K(X_1,\ldots,X_i), \text{ with the convention } K_0=K; \\ u_i &:= \text{the restriction of } w \text{ to } K_i, \text{ with the conventions} \\ &\quad\ u_0=v, u_n=w; \\ O_i, G_i, \text{resp. } k_i &:= \text{the valuation ring, the valuation group,} \\ &\quad\ \text{resp. residual field of } u_i; \\ \mathcal{M}_i &:= \left\{\bar{w}(X_i-\rho)/\rho\in\overline{K(X_1,\ldots,X_{i-1})}\right\}, \text{ for } i\ge 1. \end{aligned}$$

We call the *freedom degree* of the extension w (with respect to v) the quantity

$$\operatorname{freedeg} w = \operatorname{card}\{i\in\{1,\ldots,n\}/G_i\cap \mathbf{Q}G_{i-1}\neq G_i\}.$$

and we notice, due to [4], that $\operatorname{freedeg} w$ represents the number of intermediate extensions from v on K to w on $K(X_1,\ldots,X_n)$ that are residual-algebraic free and this number is independent on the order the indeterminates $X_1,\ldots,X_n$ are taken into account.

Following [11, Theorem 4.3 and Corollary 4.4], we have several equivalent definitions for a *residual-transcendental simple* extension (*r.t.s.-extension*), when speaking about a symmetric extension w, of v from K to $K(X_1,\ldots,X_n)$, a fixed algebraic closure $\overline{K(X_1,\ldots,X_n)}$ and $\bar{w}$ an extension of w from $K(X_1,\ldots,X_n)$ to $\overline{K(X_1,\ldots,X_n)}$ that extends the symmetry of w; namely, we say that w is residual-transcendental simple if and only if any of the following conditions is ensured:

(2.1) u_1 is a r.t.-extension of v to K_1 and $\chi_1,\chi_2,\ldots,\chi_n$ are algebraically independent over k_v, where, for all i, χ_i is a generator of the transcendence of the residue field of $w|_{K(X_i)}$;

(2.2) $\text{tr.deg}(k_w:k_v)=n$ and $\chi_1,\chi_2,\ldots,\chi_n$ are algebraically independent over k_v, where, for all i, χ_i is a generator of the transcendence of the residue field of $w|_{K(X_i)}$;

(2.3) $\operatorname{freedeg}(w)=0$ and $\sup\mathcal{M}_n$ exists and is contained in $\mathcal{M}_1$;

(2.4) there exists $a \in \bar{K}$ and $\delta \in \mathbf{Q}G_v$ such that, for any $F \in \bar{K}[X_1, \ldots, X_n]$ written as $F = \sum_{(i_1,\ldots,i_n)\in I} a_{i_1,\ldots,i_n} \cdot (X_1 - a)^{i_1} \cdot (X_2 - a)^{i_2} \cdot \ldots \cdot (X_n - a)^{i_n}$, with I a finite set of n-tuples of indices, we get

$$\bar{w}(F) = \inf_{(i_1,\ldots,i_n)\in I} \left(\bar{v}(a_{i_1,\ldots,i_n}) + (i_1 + \ldots + i_n) \cdot \delta\right).$$

(2.5) there exists $a \in \bar{K}$ and $\delta \in \mathbf{Q}G_v$ such that the following two conditions are satisfied:

(i) $w(X_i - X_1) = \delta$, for all $i \in \{2, \ldots, n\}$;

(ii) when we denote:

$$g \in K[X] \text{ the minimal monic polynomial of } a;$$
$$v' \text{ an extension of } v \text{ la } K(a);$$
$$\gamma := \sum_{\substack{a' \in \bar{K} \\ g(a')=0}} \inf(\delta, v'(a' - a));$$

then for any $F \in K[X_1, \ldots, X_n]$ written as:

$$F = \sum_{(i_i,\ldots,i_n)\in I} f_{i_1,\ldots,i_n}(X_1) \cdot g(X_1)^{i_1} \cdot (X_2 - X_1)^{i_2} \cdot \ldots \cdot (X_n - X_1)^{i_n}$$

with $\deg f_{i_1,\ldots,i_n} < \deg g$ and I a finite set of n-tuples of indices, we get:

$$w(F) = \inf_{(i_1,\ldots,i_n)\in I} \left(v'(f_{i_1,\ldots i_n}(a)) + i_1 \cdot \gamma + (i_2 + \ldots i_n) \cdot \delta\right).$$

For $n = 1$ we will consider any extension as being, trivially, a r.t.s.-extension.

With the following additional notations:

$e = e(\gamma, K(a))$, the smallest positive integer such that $e \cdot \gamma \in G_v$;
$h \in K[X]$ such that $\deg h < \deg g$ and $v'(h(a)) = e \cdot \gamma$ (X is here generic);
$r_i = g(X_i)^e / h(X_i)$, which is an element $K(X_i)$;
$\chi_i = r_i^*$, the class r_i within the residue field of $w\,|_{K(X_i)}$;

we get, from [11, Corollary 4.5], that:

$G_n = G_{v'} + \mathbf{Z}\gamma \subseteq \mathbf{Q}G_v$;
$k_n = k_{v'}(\chi_1, \ldots, \chi_n)$.

3. Characterization of r.t.s.-extension

Before discussing about the r.t.s.-extensions, we will analyze a simple type of symmetric extensions namely the Gaussian valuation w, which extends an arbitrary valuation v from K to $K(X_1, \ldots, X_n)$ in such a way that, for $F \in K[X_1, \ldots, X_n]$ written as

$$F = \sum_{(i_1,\ldots,i_n)\in I} a_{i_1,\ldots,i_n} \cdot X_1^{i_1} \cdot \ldots \cdot X_n^{i_n}, \text{ with } a_{i_1,\ldots,i_n} \in K$$

where I is a finite set of n-uples of indices, we get:

$$w(F) = \inf_{(i_1,\ldots,i_n)\in I} \left(v(a_{i_1,\ldots,i_n})\right).$$

Proposition 3.1. *The Gaussian valuation w, that extends an arbitrary valuation v from K to $K(X_1, \ldots, X_n)$ has the following properties:*

(P3.1.1) *w is symmetric and $w = 0$;*

(P3.1.2) *w is trivial if and only if v is trivial;*

(P3.1.3) *The restriction w^e of w to $K(e_1^{(n)}, \ldots, e_n^{(n)})$ is also Gaussian so it is itself symmetric and isomorphic with w, as extensions of v to two isomorphic fields.*

Proof. Statements (P3.1.1) and (P3.1.2) are obvious, so we will take care only of (P3.1.3).

Indeed, if we wrote the same symmetric polynomial in the two fields:

$$\begin{aligned} F^e(e_1^{(n)}, \ldots, e_n^{(n)}) &= \sum_{(i_1,\ldots,i_n)\in I} a_{i_1,i_2,\ldots,i_n} \left(e_1^{(n)}\right)^{i_1} \cdot \ldots \cdot \left(e_1^{(n)}\right)^{i_n} \\ &= \sum_{(j_1,\ldots,j_n)\in J} b_{j_1,j_2,\ldots,j_n} X_1^{j_1} \cdot \ldots \cdot X_n^{j_n} \\ &= F(X_1, \ldots, X_n) \end{aligned}$$

then each $a_{i_1,i_2,\ldots,i_n}$ is a linear combination of $b_{j_1,j_2,\ldots,j_n}$, weighted by integer values, but also reversely, so we have:

$$\begin{aligned} w^e(F^e) \geq \inf_{(j_1,\ldots,j_n)\in J} \big(v(b_{j_1,j_2,\ldots,j_n})\big) &= w(F) \\ &\geq \inf_{(i_1,\ldots,i_n)\in I} \big(v(a_{i_1,i_2,\ldots,i_n})\big) = w^e(F^e) \end{aligned}$$

therefore $w^e(F^e)$ is the Gaussian valuation on $K(e_1^{(n)}, \ldots, e_n^{(n)})$, which extends K. □

Now we can move on to the r.t.s.-extensions, which appear as a generalization of the Gaussian ones. However, before a complete characterization of these, we need two preliminary results.

Lemma 3.1. *An extension w on $K(X_1,\dots,X_n)$ of a valuation v on K, with $n\ge 2$, is symmetric if and only if, for each i with $1\le i\le n-1$, w is symmetric with respect to X_i, X_n.*

Proof. "$\Rightarrow$": The assertion is obvious.

"$\Leftarrow$": For $n=2$ the statement is also obvious. Therefore, let's consider $n>2$. Let π be a permutation of the set $\{1,2,\dots,n\}$. By denoting with π_{ij} the inversions (when $i\neq j$) or the identity (when $i=j$), we may write

$$\pi = \mathop{\circ}_{\substack{i=1\\ j_i>i}}^{n-1}(\pi_{i,j_i}) = \mathop{\circ}_{\substack{i=1\\ j_i>i}}^{n-1}(\pi_{n,i}\circ\pi_{n,j_i}\circ\pi_{n,i}) = \mathop{\circ}_{k=1}^{3(n-1)}(\pi_{n,i_k})$$

with $\{j_i\}$ and $\{i_k\}$ two arrays of indices conveniently chosen. From the hypothesis we know that, for each i and any $f\in K(X_1,\dots,X_n)$, we have $w(f)=w(\pi_{n,i}f)$. We conclude that:

$$w(\pi f) = w\left(\left(\mathop{\circ}_{k=1}^{3(n-1)}\pi_{n,i_k}\right)f\right) = w\left(\pi_{n,i_1}\left(\left(\mathop{\circ}_{k=2}^{3(n-1)}\pi_{n,i_k}\right)\right)\right)$$
$$= w\left(\left(\mathop{\circ}_{k=2}^{3(n-1)}\pi_{n,i_k}\right)f\right) = \cdots = w(f)$$

□

Proposition 3.2. *Let w be an extension of v from K to $K(X_1,\dots,X_n)$ such that there exist $a\in\bar K$ and two values $\delta,\epsilon\in\mathbf{Q}G_v$ with $\delta\le\epsilon$, ensuring the following three conditions*

i) (a,δ) *is a minimal pair of definition with respect to* K *and* v*;*

ii) $w(X_i-X_1)=\epsilon$*, for each* $i\in\{2,\dots,n\}$*;*

iii) *when we denote by:*

$$g\in K[X] \text{ the minimal monic polynomial of } a;$$
$$v'\text{—extension of } v \text{ to } K(a);$$
$$\gamma := \sum_{\substack{a'\in\bar K\\ g(a')=0}} \inf(\delta, v'(a'-a));$$

we have that, for all $F\in K[X_1,\dots,X_n]$ *written as:*

$$F=\sum_{(i_1,\dots,i_n)\in I} f_{i_1,\dots,i_n}(X_1)^{i_1}\cdot g(X_1)^{i_1}\cdot(X_2-X_1)^{i_2}\cdot\ldots\cdot(X_n-X_1)^{i_n},$$

with $\deg f_{i_1,\dots,i_n}<\deg g$ *and* I *a finite set of* n*-tuples of indices, we get:*

$$w(F)=\inf_{(i_1,\dots,i_n)\in I}\big(v'(f_{i_1,\dots,i_n}(a))+i_1\cdot\gamma+(i_2+\ldots i_n)\cdot\epsilon\big).$$

In these circumstances, w is a symmetric valuation on $K(X_1,\ldots,X_n)$ and, given $\overline{K(X_1,\ldots,X_n)}$ an algebraic closure of $K(X_1,\ldots,X_n)$ and $\bar{w}$ an extension of w from $K(X_1,\ldots,X_n)$ to $\overline{K(X_1,\ldots,X_n)}$, $\bar{w}$ extends the symmetry of w.

Proof. Let's prove, first, that w is symmetric. According to Lemma 3.1, in order to prove that w is symmetric it is enough to show that w is symmetric with respect to X_1, X_n, because for the rest of the pairs this fact is obvious.

Let, therefore, $F \in K[X_1,\ldots,X_n]$ written as in iii), but let's put

$$g_{i_2,\ldots,i_n}(X_1) = \sum_{i_1 \text{ such that } (i_1,\ldots,i_n)\in I} f_{i_1,\ldots,i_n}(X_1)\cdot g(X_1)^{i_1}.$$

so F becomes:

$$F = \sum_{(\bullet,i_2,\ldots,i_n)\in I} g_{i_2,\ldots,i_{n-1},i_n}(X_1)\cdot(X_2-X_1)^{i_2}\cdot\ldots\cdot(X_n-X_1)^{i_n} \qquad (3.1)$$

and we have:

$$w(F) = \inf_{(\bullet,i_2,\ldots,i_n)\in I}\big(u_1(g_{i_2,\ldots,i_n}) + (i_2+\ldots+i_n)\cdot\epsilon\big).$$

Now let's analyze the polynomial $\pi F \in K[X_1,\ldots,X_n]$, obtained from F by inverting X_n with X_1. Let's consider an arbitrary ω that extends w on $\overline{K(X_1,\ldots,X_n)}$. We have:

$$w(\pi F) = w\left(\sum_{(\bullet,i_2,\ldots,i_{n-1},i_n)\in I} g_{i_2,\ldots,i_{n-1},i_n}(X_n)\cdot(X_2-X_n)^{i_2}\cdot\ldots\right.$$
$$\left.\cdot(X_{n-1}-X_n)^{i_n}\cdot(X_1-X_n)^{i_n}\right)$$

that may be written further, denoting by $J_{i_2,\ldots,i_n}$ the set $\{1,\ldots,\deg g_{i_2,\ldots,i_n}\}$, with $r_{i_2,\ldots,i_n;j}$ being the roots of $g_{i_2,\ldots,i_{n+1}}$, where $j \in J_{i_2,\ldots,i_n}$ and with $a_{i_2,\ldots,i_n}$ being the coefficient of the term with the maximal degree:

$$w(\pi F) = \omega\left(\sum_{(\bullet,i_2,\ldots,i_{n-1},i_n)\in I}\left(a_{i_2,\ldots,i_n}\cdot\left(\prod_{j\in J_{i_2,\ldots,i_n}}(X_n - r_{i_2,\ldots,i_n;j})\right)\right.\right.$$
$$\left.\left.\cdot(X_2-X_n)^{i_2}\cdot\ldots\cdot(X_1-X_n)^{i_n}\right)\right)$$

and from this, having $X_n - r_{i_2,\ldots,i_n;j} = X_n - X_1 + X_1 - r_{i_2,\ldots,i_n;j}$, we get:

$$w(\pi F) = \omega\left(\sum_{(\bullet,i_2,\ldots,i_n)\in I}\ \sum_{H\subset J_{i_2,\ldots,i_n}}\left(a_{i_2,\ldots,i_n}\cdot\left(\prod_{j\in J_{i_2,\ldots,i_n}-H}\big(X_1 - r_{i_2,\ldots,i_n;j}\big)\right)\right.\right.$$
$$\left.\left.\cdot\big(X_2 - X_n\big)^{i_2}\cdot\ldots\cdot\big(X_1 - X_n\big)^{i_n+card(H)}\right)\right) \qquad (3.2)$$

Considering the fact that, for each $i \neq j \in \{1,\ldots,n\}$ and any $r \in \bar{K}$, we get

$$w(X_i - X_j) = w(X_i - X_1 + X_1 - X_j) = \epsilon \geq \delta = \omega(X_1 - r)$$

it may be derived that each term of the double summation in (3.2) has the valuation greater or equal to $w(F)$:

$$\omega\left(a_{i_2,\ldots,i_n}\cdot\left(\prod_{j\in J_{i_2,\ldots,i_n}-H}\big(X_1 - r_{i_2,\ldots,i_n;j}\big)\right)\right.$$
$$\left.\cdot\big(X_2 - X_n\big)^{i_2}\cdot\ldots\cdot\big(X_1 - X_n\big)^{i_n+card(H)}\right) \geq$$
$$\omega\left(a_{i_2,\ldots,i_n}\cdot\left(\prod_{j\in J_{i_2,\ldots,i_n}}\big(X_1 - r_{i_2,\ldots,i_n;j}\big)\right)\right.$$
$$\left.\cdot\big(X_2 - X_n\big)^{i_2}\cdot\ldots\cdot\big(X_1 - X_n\big)^{i_n}\right) =$$
$$w\big(g_{i_2,\ldots,i_n}\cdot(X_2 - X_n)^{i_2}\cdot\ldots\cdot(X_1 - X_n)^{i_n}\big) =$$
$$u_1(g_{i_2,\ldots,i_n} + (i_2 + \ldots + i_n)\cdot\epsilon > w(F)$$

We deduce, therefore, that $w(\pi F) \geq w(F)$. We are left with proving the reverse inequality.

Out of the terms of F, whose valuation is equal to $w(F)$, let's choose one of minimal degree in X_n:

$$g_{l_2,\ldots,l_{n-1},l_n}\cdot(X_2 - X_1)^{l_2}\cdot\ldots\cdot(X_{n-1} - X_1)^{l_{n-1}}\cdot(X_n - X_1)^{l_n},\ \text{with}$$
$$w(F) = u_1(g_{l_2,\ldots,l_{n-1},l_n}) + (l_2 + \ldots + l_{n-1} + l_n)\cdot\epsilon \text{ and}$$
$$l_n \text{ is minimal having this property.}$$

Now we need to write also πF in the form (3.1). In order to do that, we will need to put:

$$X_n - r_{i_2,\ldots,i_n;j} = (X_n - X_1) + (X_1 - r_{i_2,\ldots,i_n;j}) \text{ and}$$
$$X_i - X_n = (X_i - X_1) + (X_1 - X_n) \text{ for } 2 \leq i < n$$

and to perform the replacement in (3.2). It is not necessary to perform all the calculations, as we are interested only in those terms that get summed up for the $(n-1)$-uple $(l_2, \ldots, l_n)$, meaning those that are identified by:

$$F_{l_1,\ldots,l_n,i_2,\ldots,i_n,H} = a_{i_2,\ldots,i_n} \cdot \left(\prod_{j \in J_{i_2,\ldots,i_n} - H} \big(X_1 - r_{i_2,\ldots,i_n;j}\big) \right) \cdot \big(X_2 - X_1\big)^{l_2} \cdot \ldots \cdot \big(X_{n-1} - X_1\big)^{l_{n-1}} \cdot (X_n - X_1)^{l_n} \cdot (-1)^{l_n}$$

with $i_2 \geq l_2, \ldots, i_{n-1} \geq l_{n-1}, i_n \leq l_n$, $H \subseteq J_{i_2,\ldots,i_n}$ and $i_n + i_2 - l_2 + \ldots + i_{n-1} - l_{n-1} + \mathrm{card}(H) = l_n$.

If we denote by $\bar{u}_1$ the restriction of ω to $\bar{K}(X_1)$, then we have:

$$\begin{aligned}
&\omega(F_{l_2,\ldots,l_n,i_2,\ldots,i_n,H}) = \\
&v(a_{i_2,\ldots,i_n}) + \sum_{j \in J_{i_2,\ldots,i_n} - H} \bar{u}_1(X_1 - r_{i_2,\ldots,i_n}) + (i_2 + \ldots + i_n + \mathrm{card}(H)) \cdot \epsilon \geq \\
&v(a_{i_2,\ldots,i_n}) + \sum_{j \in J_{i_2,\ldots,i_n}} \bar{u}_1(X_1 - r_{i_2,\ldots,i_n}) + (i_2 + \ldots + i_n) \cdot \epsilon = \\
&u_1(g_{i_2,\ldots,i_n}) + (i_2 + \ldots + i_n) \cdot \epsilon \geq w(F)
\end{aligned}$$

with the last inequality being strict when $i_n < l_n$. This means that there exists one and only one term equal to $w(F)$ among those that get summed up for the $(n-1)$-uple $(l_2, \ldots, l_n)$, namely $F_{l_2,\ldots,l_n,l_2,\ldots,l_n,\phi}$.

We get, thus, the reverse inequality:

$$w(\pi F) = \inf_{(\bullet,l_2,\ldots,l_n) \in I} \omega \left(\sum_{i_2,\ldots,i_n,H} F_{l_2,\ldots,l_n,i_2,\ldots,i_n,H} \right) = w(F)$$

so w is symmetric with respect to $X_1, \ldots, X_n$.

Now we notice from iii) that (a, δ) is a minimal pair of definition for u_1 (the restriction of w to $K(X_1)$) and, from [5, V-Entiers,§6,10], we get that u_1 is a residual-transcendental extension. Moreover, for each $i \in \{2, \ldots, n\}$, we have $\deg_{X_i} X_1 = 1$, so (X_1, ϵ) is a minimal pair of definition with respect to $K(X_1, \ldots, X_{i-1})$ and u_{i-1} (the restriction of w to $K(X_1, \ldots, X_{i-1})$), which leads to the fact that all the intermediary extensions u_i are residual-transcendental.

Let's fix $L = \overline{K(X_1, \ldots, X_n)}$ an algebraic closure of $K(X_1, \ldots, X_n)$ that extends $\bar{K}$ from the hypothesis. We shall prove, by induction by n, that for any $\bar{w}$, an extension of w from $K(X_1, \ldots, X_n)$ to L, we get $\bar{w}$ extending the symmetry of w. Let $\bar{K}$ be the closure of K in L, $\bar{u}_2$ an extension of u_2 to $\overline{K(X_1, X_2)}$, $\bar{u}_1$ its restriction to $\overline{K(X_1)}$ which, obviously, extends u_1 and

$\bar{v}$ its restriction to $\bar{K}$. As (X_1, ϵ) is a minimal pair of definition of u_2, we derive that, for any $F \in \overline{K(X_1)}[X_2]$ written as

$$F = \sum_{i_2 \in I_2} \rho_{i_2}(X_2 - X_1)^{i_2}, \text{ with } \rho_{i_2} \in \overline{K(X_1)}$$

with I_2 a set of indices, we have

$$\bar{u}_2(F) = \inf_{i_2 \in I_2} \big(\bar{u}_1(\rho_{i_2} + i_2 \cdot \epsilon\big).$$

which means that, for any $F \in \bar{K}[X_1, X_2]$ written as

$$F = \sum_{(i_1,i_2) \in I_{1,2}} a_{i_1,i_2}(X_1 - a)^{i_1}(X_2 - X_1)^{i_2}, \text{ with } a_{i_1,i_2} \in \bar{K}$$

where $I_{1,2}$ is a set of pairs of indices, we get

$$\bar{u}_2(F) = \inf_{(\bullet,i_2) \in I_{1,2}} \left(\bar{u}_1 \left(\sum_{i_1 \text{ such that } (i_1,i_2) \in I_{1,2}} a_{i_1,i_2}(X_1 - a)^{i_1} \right) + i_2 \cdot \epsilon \right)$$

and, since $\bar{u}_1$ extends u_1 which is a r.t.-extension, we have

$$\begin{aligned} \bar{u}_2(F) = \inf_{(\bullet,i_2) \in I_{1,2}} & \left(\inf_{i_1 \text{ such that } (i_1,i_2) \in I_{1,2}} \big(\bar{v}(a_{i_1,i_2}) + i_1 \cdot \delta\big) + i_2 \cdot \epsilon \right) \\ &= \inf_{(i_1,i_2) \in I_{1,2}} \big(\bar{v}(a_{i_1,i_2}) + i_1 \cdot \delta + i_2 \cdot \epsilon\big) \end{aligned}$$

Now let's analyze the polynomial $\pi F \in K[X_1, X_2]$, obtained from F by inverting X_2 with X_1. We have:

$$\begin{aligned} \pi F = & \sum_{(i_1,i_2) \in I_{1,2}} a_{i_1,i_2}(X_2 - a)^{i_1}(X_1 - X_2)^{i_2} = \\ & \sum_{(i_1,i_2) \in I_{1,2}} \sum_{k=0}^{i_1} (-1)^{i_2} a_{i_1,i_2} C_{i_1}^k (X_1 - a)^{i_1-k}(X_2 - X_1)^{k+i_2} = \\ & \sum_{l \geq 0} \left(\sum_{\substack{k,i_2 \geq 0 \\ k+i_2=l}} (-1)^{i_2} \left(\sum_{i_1 \geq k} a_{i_1,i_2} C_{i_1}^k (X_1 - a)^{i_1-k} \right) \right) \cdot (X_2 - X_1)^l = \\ & \sum_{l \geq 0} \left(\sum_{h \geq 0} \left(\sum_{\substack{(i_1,i_2) \in I_{1,2} \\ i_1+i_2=l+h \\ i_1 \geq h}} (-1)^{i_2} a_{i_1,i_2} C_{i_1}^h \right) \cdot (X_1 - a)^h \right) \cdot (X_2 - x_1)^l. \end{aligned}$$

In order to have $(X_1-a)^h(X_2-X_1)^l$ appearing in πF, there must exist a pair $(i_1,i_2)\in I_{1,2}$ featuring $i_1\geq h$ and $i_1+i_2=l+h$, so $i_1\leq l$. Out of these, let's choose the pair (j_1,j_2) for which $\bar{v}(a_{j_1,j_2}C^h_{j_1})$ is minimal. Since $\delta\leq\epsilon$ and $\bar{v}(C^h_{j_1})$ we derive

$$\bar{u}_2\left(\sum_{\substack{(i_1,i_2)\in I_{1,2}\\ i_1+i_2=l+h\\ i_1\geq h}}(-1)^{i_2}a_{i_1,i_2}C^h_{i_1}(X_1-a)^h(X_2-X_1)^l\right)\geq\bar{v}(a_{j_1,j_2})+j_1\cdot\delta+j_2\cdot\epsilon$$

for any l and h, so $\bar{u}_2(\pi F)\geq\bar{u}_2(F)$.

By choosing $(h',l')\in I_{1,2}$ such that $\bar{u}_2(F)=\bar{v}(a_{h',l'})+h'\cdot\delta+l'\cdot\epsilon$ and such that h' is maximal with this property we notice that, among the terms that compose the coefficient of $(X_1-a)^{h'}(X_2-X_1)^{l'}$, there exists one and only one equal to $\bar{u}_2(F)$, namely the one having $i_1=h'$ and $i_2=l'$.

It follows that $\bar{u}_2(\pi F)=\bar{u}_2(F)$, for any $F\in\overline{K(X_1)}[X_2]$, so $\bar{u}_2$ extends the symmetry of u_2.

Let's move on to the induction step and let's consider the target statement true for any $n'<n$. Let $\bar{w}$ be an extension of w from $K(X_1,\ldots,X_n)$ to L, an integer m such that $0\leq m<n$ and a partition of $\{1,2,\ldots,n\}=\{k_1,k_2,\ldots,k_m\}\cup\{l_1,l_2,\ldots,l_{n-m}\}$. Let's denote by $\bar{u}$ the restriction of $\bar{w}$ to $\overline{K(X_{k_1},\ldots,X_{k_m})}(X_{l_1},\ldots,X_{l_{n-m}})$, where $\overline{K(X_{k_1},\ldots,X_{k_m})}$ is the closure of $K(X_{k_1},\ldots,X_{k_m})$ in L. We shall prove that $\bar{u}$ is symmetric with respect to $X_{l_1},\ldots,X_{l_{n-m}}$. There are two cases, depending on the value of m.

If $m>0$, as w is symmetric, we know that, for any $F\in K[X_1,\ldots,X_n]$ written as

$$\sum_{(i_1,\ldots,i_n)\in I}f_{i_1,\ldots,i_n}(X_{k_1})\cdot g(X_{k_1})^{i_1}\cdot(X_{k_2}-X_{k_1})^{i_2}\cdot\ldots\cdot(X_{k_m}-X_{k_1})^{i_m}\cdot$$
$$(X_{l_1}-X_{i_1})^{i_{m+1}}\cdot\ldots\cdot(X_{l_{n-m}}-X_{k_1})^{i_n},$$

with $\deg f_{i_1,\ldots,i_n}<\deg g$ and I a finite set of n-uples of indices, we get:

$$w(F)=\inf_{(i_1,\ldots,i_n)\in I}\big(v'(f_{i_1,\ldots,i_n}(a))+i_1\cdot\gamma+(i_2+\ldots+i_n)\cdot\epsilon\big).$$

Again, all the intermediary extensions are r.t.-extensions so, as above, for any polynomial $G\in\overline{K(X_{k_1},\ldots,X_{k_m})}(X_{l_1},\ldots,X_{l_{n-m}})$ written as

$$\sum_{(i_{m+1},\ldots,i_n)\in J}\eta_{i_{m+1},\ldots,i_n}\cdot(X_{l_1}-X_{k_1})^{i_{m+1}}\cdot\ldots\cdot(X_{l_{n-m}}-X_{k_1})^{i_n},$$

with $\eta_{i_{m+1},\ldots,i_n}\in\overline{K(X_{k_1},\ldots,X_{k_m})}$.

But we are now verifying the conditions of the induction hypothesis, with $n' = n-m < n, \delta' = \epsilon$ and the minimal monic polynomial of X_{k_1} being $g' \in \overline{K(X_{k_1}, \ldots, X_{k_m})}[X]$ with $g'(X) = X - X_{k_1}$ so, applying the induction hypothesis, it follows that $\bar{u}$ is symmetric with respect to $X_{l_1}, \ldots, X_{l_{n-m}}$.

Finally, when $m = 0$, Lemma 3.1 allows us to verify the symmetry, successively, only against two indeterminates, which reduces the analysis of this case to the one above. □

Corollary 3.1. *An extension w, of v from K to $K(X_1, \ldots, X_n)$, is a r.t.s. extension if and only if hypothesis* (2.5) *holds, namely there exists $a \in \bar{K}$ and $\delta \in \mathbf{Q}G_v$ such that the following conditions are true:*

i) $w(X_i - X_1) = \delta$, for all $i \in \{2, \ldots, n\}$;

ii) when we denote by:

$$g \in K[X] \text{ the minimal monic polynomial of } a;$$

$$v' \text{ an extension of } v \text{ to } K(a);$$

$$\gamma := \sum_{\substack{a' \in \bar{K} \\ g(a')=0}} \inf(\delta, v'(a' - a));$$

then, for any $F \in K[X_1, \ldots, X_n]$ written as:

$$F = \sum_{(i_1,\ldots,i_n)\in I} f_{i_1,\ldots,i_n}(X_1) \cdot g(X_1)^{i_1} \cdot (X_2 - X_1)^{i_2} \cdot \ldots \cdot (X_n - X_1)^{i_n},$$

with $\deg f_{i_1,\ldots,i_n} < \deg g$ with I is a finite set of n-tuples of indices, we get:

$$w(F) = \inf_{(i_1,\ldots,i_n)\in I} \big(v'(f_{i_1,\ldots,i_n}(a)) + i_1 \cdot \gamma + (i_2 + \ldots i_n) \cdot \delta\big).$$

In particular, the Gaussian extension verifies the conditions required by Proposition 3.2, *by having $a = 0$ and $\delta = \epsilon = 0$, so it is a particular case of a r.t.s.-extension.*

4. Ultrasymmetric extensions and symmetrically-open extensions

Definition 4.1. *A valuation w on $K(X_1, \ldots, X_n)$, with $n \geq 2$, is called* ultrasymmetric *(with respect to $X_1, \ldots, X_n$) if, for any permutation π of the set $\{1, 2, \ldots, n\}$ and any $f \in K(X_1, \ldots, X_n)$, we have: $w(f) \geq 0 \Leftrightarrow w(\pi f) \geq 0$ and, when both inequalities are strict, we have $f^* = (\pi f)^*$ in k_w.*

Observations:

(D4.1.1) An ultrasymmetric valuation is always symmetric but the reciprocal is not true. Indeed, let's suppose *(reductio ad absurdum)* that w is ultrasymmetric and, at the same time, there exists $f \in K(X_1, \ldots, X_n)$ such that $w(f) < w(\pi f)$. We can assume, without any loss of generality, that $w(f)$ and $w(\pi f)$ are minimal with this property among the permutations of f. Then we have two cases:

(i) $w(f) = w(\pi^{-1} f) < w(\pi f)$, so $w(f/\pi f) < 0 = w(\pi^{-1} f/f)$

(ii) $w(f) < w(\pi f) \leq w(\pi^{-1} f)$, so $w(f/\pi f) < 0 < w(\pi f/f) \leq w(\pi^{-1} f/f)$.

and in both cases the ultrasymmetry of f is invalidated, since $w(\pi^{-1} f/f) = w(\pi^{-1}(f/\pi f))$.

On the other hand, the following example shows that the reciprocal is not true: let w be the trivial valuation on $K(X_1, \ldots, X_n)$, with $n \geq 2$, that extends the trivial valuation on K. In this case, $a = 0$, $\delta = 0$ and k_n is isomorphic with K_n, so we might say that $f^* = f$ for any $f \in K(X_1, \ldots, X_n)$. From:

$$X_1^* = X_1 \neq X_2 = X_2^*$$

we can see immediately that the extension, although symmetric, is not ultrasymmetric.

(D4.1.2) A r.t.s.-extension with respect to $X_1, \ldots, X_n$, with $n \geq 2$, is not ultrasymmetric.

(D4.1.3) The Gaussian valuation, for $n \geq 2$, is not ultrasymmetric. Indeed, $w(X_i - X_j) = 0$, so $X_i^* \neq X_j^*$, for any different i, j in $\{1, 2, \ldots, n\}$.

Definition 4.2. *An extension w, of a valuation v from K to $K(X_1, \ldots, X_n)$, symmetric with respect to $X_1, \ldots, X_n$, is called* symmetrically-open *(with respect to $X_1, \ldots, X_n$) if, adding any number of other indeterminates (elements transcendental and algebraically independent over $K(X_1, \ldots, X_n)$), $X_{n+1}, \ldots, X_{n+r}$, there exists a symmetric extension of it to $K(X_1, \ldots, X_{n+r})$ with respect to $X_1, \ldots, X_{n+r}$.*

Observations:

(D4.2.1) If w is symmetrically-open with respect to $X_1, \ldots, X_n$, with $n \geq 2$, then it is symmetrically-open with respect to $X_1, \ldots, X_i$, for $i < n$. The dual statement will be proved later.

(D4.2.2) Any r.t.s.-extension is symmetrically-open; in particular, any Gaussian extension is symmetrically-open. This means that, if we formally extend the definition above for $n = 0$, we can say that any extension is (trivially) symmetrically-open with respect to the void set.

The next proposition prepares the classification of the symmetrical extensions in a simple way, as it was promised in the introduction. In essence, it states that a symmetrically-open extension cannot have complete freedom in its construction, except for the first intermediary extension, namely the one from K to $K(X_1)$.

But, first, we need an important lemma to regulate the extension of the symmetry to the algebraic closure.

Lemma 4.1. *Let w be an extension of v from K to $K(X_1, \ldots, X_n)$, symmetrically open with respect to $X_1, \ldots, X_n$ and a fixed algebraic closure $\overline{K(X_1, \ldots, X_n)}$ of $K(X_1, \ldots, X_n)$. Consider a partition*

$$\{1, 2, \ldots, n\} = \{i_1, i_2, \ldots, i_m\} \cup \{j_1, j_2, \ldots, j_{n-m}\},$$

with $0 \leq m < n$, then put $L := K(X_{i_1}, \ldots, X_{i_m})$ and denote with $Y_1, \ldots, Y_k$ the indeterminates $X_{j_1}, \ldots X_{j_{n-m}}$ (where $k = n - m$). Let's choose an infinite array of elements, $Y_{k+1}, Y_{k+2}, \ldots$, that are transcendental and algebraic independent over the field $L(Y_1, \ldots, Y_k)$. Then:

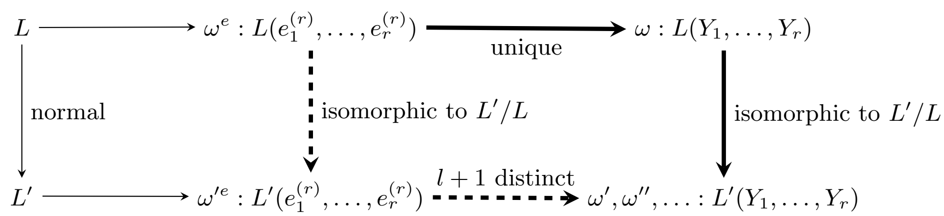
(L4.1.1) *For any L', normal finite extension of L, there exists $r \geq k+1$ and an extension ω of w to $L(Y_1, \ldots, Y_r)$, symmetric with respect to $X_{i_1}, \ldots, X_{i_m}$, $Y_1, \ldots, Y_r$, such that, given any extension ω' of ω to $L'(Y_1, \ldots, Y_r)$, we get ω' symmetric with respect to $Y_1, \ldots, Y_r$.*

(L4.1.2) *Any extension $\bar{w}$ of w to $\overline{K(X_1, \ldots, X_n)}$ also extends the symmetry of w.*

Proof. (L4.1.1) Let's suppose *(reduction ad absurdum)* that for any $r \geq k+1$ and any extension ω of w to $L(Y_1, \ldots, Y_r)$, symmetric with respect to $X_{i_1}, \ldots, X_{i_m}, Y_1, \ldots, Y_r$, there exists ω', an extension of ω to $L'(Y_1, \ldots, Y_r)$, such that ω' is not symmetric with respect to $Y_1, \ldots, Y_r$.

Obviously, the group $\mathrm{Aut}(L'/L)$ is finite and denote by l its order. Let $r := (k+1) \cdot l \geq k+1$. As w is symmetrically-open, we know that there exists ω, an extension of w to $L(Y_1, \ldots, Y_r)$, symmetric with respect to $X_{i_1}, \ldots, X_{i_m}, Y_1, \ldots, Y_r$. Let ω' be an extension of it to $L'(Y_1, \ldots, Y_r)$ which is not symmetric with respect to $Y_1, \ldots, Y_r$ and, moreover, whose restriction to $L'(Y_1, \ldots, Y_{k+1})$ is not symmetric, either. This must exist because, if it hadn't, $r' = k+1$ would invalidate the assumption made. Therefore, there exist $\pi \in S_{k+1}$ and $f \in L'(Y_1, \ldots, Y_{k+1})$ with $\omega'(f) \neq \omega'(\pi f)$.

Let ω^e, respectively ω'^e, be the restriction of ω, respectively ω', to the field generated by the elementary symmetric polynomials $L(e_1^{(r)}, \ldots, e_r^{(r)})$, respectively $L'(e_1^{(r)}, \ldots, e_r^{(r)})$, as it may be seen in the diagram below:



The automorphism groups of the three vertical extensions are isomorphic:

$$\mathrm{Aut}(L'/L) \cong \mathrm{Aut}(L'(e_1^{(r)}, \dots, e_e^{(r)})/L(e_1^{r}, \dots, e_r^{(r)})) \cong \\ \mathrm{Aut}(L'(Y_1, \dots, Y_r)/L(Y_1, \dots, Y_r))$$

the correspondence given by:

$$a \to \sigma(a)$$

$$\sum_{(i_1,\dots,i_r)\in I} a_{i_1,\dots,i_r} \cdot (e_1^{(r)})^{i_1} \cdot \dots \cdot (e_r^{(r)})^{i_r} \to \sum_{(i_1,\dots,i_r)\in I} \sigma(a_{i_1,\dots,i_r}) \cdot (e_1^{(r)})^{i_1} \cdot \dots \\ \cdot (e_r^{(r)})^{i_r}$$

$$\sum_{(i_1,\dots,i_r)\in I} a_{i_1,\dots,i_r} \cdot Y_1^{i_1} \cdot \dots \cdot Y_r^{i_r} \to \sum_{(i_1,\dots,i_r)\in I} \sigma(a_{i_1,\dots,i_r}) \cdot Y_1^{i_1} \cdot \dots \cdot Y_r^{i_r}$$

Let's notice that there must exist at least $l+1$ different extensions of ω'^e to $L'(Y_1, \dots, Y_r)$.

Indeed, $\omega'(f) \neq \omega'(\pi f)$, with $f \in L'(Y_1, \dots, Y_{k+1})$, and let's see π and all the other permutations defined below in S_r. Let's put $f_i \in L'(Y_{i(k+1)+1}, \dots, Y_{(i+1)(k+1)})$, $0 \leq i < l$, obtained from f by translations of its indeterminates, namely $f_i = \tau_i f$ where $\tau_i = \tau_i^{-1}$ inverts the whole group $Y_1, \dots, Y_{k+1}$ with the group $Y_{i(k+1)+1}, \dots, Y_{(i+1)(k+1)}$; in particular, $f_0 = f$. Let's consider all the pairs of extensions of ω'^e that apply the permutation π on the group $Y_{i(k+1)+1}, \dots, Y_{(i+1)(k+1)}$, $0 \leq i < l$, namely $(\omega_i', \omega_i'') = (\tau_i\omega', (\pi \circ \tau_i)\omega')$; in particular, $(\omega_0', \omega_0'') = (\omega', \pi\omega')$. We have $\omega_i'(f_i) \neq \omega_i''(f_i)$, but, since f_i has no common indeterminates with the other $f_j, j < i$, it follows that at least one of ω_i' and ω_i'' is different from all ω_j', ω_j'' with $j < i$. In total, remembering that $\omega_0' \neq \omega_0''$, we have $l+1$ different extensions of ω'^e to $L'(Y_1, \dots, Y_r)$.

In conclusion, the number of extensions of ω^e to $L'(Y_1, \dots, Y_r)$, passing through
$L(Y_1, \dots, Y_r)$ (the path marked by dotted thick arrows), is at least $l+1$. On the other hand, ω, being symmetric, extends in a unique manner ω^e to $L(Y_1, \dots, Y_r)$ ([11, Theorem 3.1]), so the number of extensions of ω^e to $L'(Y_1, \dots, Y_r)$, passing through $L(Y_1, \dots, Y_r)$ (the path marked by continuous thick arrows) is at most l and, thus, we got a contradiction.

(L4.1.2) Let's fix $\bar{w}$ an extension of w to $\overline{K(X_1,\ldots,X_n)}$. Again, we will prove the result by contradiction.

Let's suppose, accordingly, that there exists a partition:

$$\{1,2,\ldots,n\} = \{i_1,i_2,\ldots,i_m\} \cup \{j_1,j_2,\ldots,j_{n-m}\}, \text{ with } 0 \leq m < n,$$

such that the restriction of $\bar{w}$ to $\overline{K(X_{i_1},\ldots,X_{i_m})}(X_{j_1},\ldots,X_{j_{n-m}})$ is not symmetric with respect to $X_{j_1},\ldots,X_{j_{n-m}}$, where $\overline{K(X_{i_1},\ldots,X_{i_m})}$ is the closure of $K(X_{i_1},\ldots,X_{i_m})$ in $\overline{K(X_1,\ldots,X_n)}$.

Denote by $L = K(X_{i_1},\ldots,X_{i_m})$ and by $Y_1,\ldots,Y_k$ the indeterminates $X_{j_1},\ldots,X_{j_{n-m}}$ $(k = n-m)$.

Let's also put $\bar{u} = \bar{w} \mid_{\bar{L}(Y_1,\ldots,Y_k)}$ (we notice that it is an intermediary extension between w and $\bar{w}$).

As $\bar{u}$ is not symmetric, it follows that there exists a polynomial $f \in \bar{L}(Y_1,\ldots,Y_k)$ and a permutation π of $\{1,2,\ldots,k\}$ such that $\bar{u}(f) \neq \bar{u}(\pi f)$. Let $L' \subseteq \bar{L}$ be the normal finite extension of L that contains all the coefficients of f.

According to (L4.1.1) there exists an $r \geq k+1$ and ω an extension of w to $L(Y_1,\ldots,Y_r)$, symmetric with respect to $X_{i_1},\ldots,X_{i_m},Y_1,\ldots,Y_r$, such that given ω', any extension of it to $L'(Y_1,\ldots,Y_r)$, we get that ω' is symmetric with respect to $Y_1,\ldots,Y_r$. But, in particular, ω is symmetric with respect to $Y_1,\ldots,Y_r$ and we know from [11, Lemma 3.4] that there must exist ω', an extension of ω to $L'(Y_1,\ldots,Y_k)$, that extends $\bar{u}$, so we also have $\omega'(f) \neq \omega'(\pi f)$, which leads to a contradiction.

□

We can move on to the announced proposition.

Proposition 4.1. *Let w be a symmetric extension of v, from K to $K(X_1,\ldots,X_n)$, a fixed algebraic closure $\overline{K(X_1,\ldots,X_n)}$ and $\bar{w}$ an extension of w to $\overline{K(X_1,\ldots,X_n)}$.*

Then w is symmetrically-open with respect to $X_1,\ldots,X_n$ if and only if either $n < 2$, or $n \geq 2$ and there exists $\epsilon \in G_2$ an upper bound of the set $\mathcal{M}_1 = \{\bar{w}(X_1 - a)/a \in \bar{K}\}$, such that, for any $F \in K[X_1,\ldots,X_n]$ written as:

$$F = \sum_{(i_2,\ldots,i_n)\in I} f_{i_2,\ldots,i_n} \cdot (X_2 - X_1)^{i_2} \cdot \ldots \cdot (X_n - X_1)^{i_n}, \text{ with } f_{i_2,\ldots,i_n} \in K[X_1]$$

where I is a finite set of $(n-1)$-uples of indices, we get:

$$w(F) = \inf_{(i_2,\ldots,i_n)\in I} \big(u_1(f_{i_2,\ldots,i_n}) + (i_2 + \ldots + i_n) \cdot \epsilon\big).$$

Proof.

"⇒" For $n < 2$ there is nothing to prove. Let's suppose w is symmetrically open and let $n \geq 2$. According to (L4.1.2), $\bar{w}$ extends the symmetry of

w. Let's fix $X_{n+1}, X_{n+2}, \ldots$ an array of elements that are transcendental and algebraically independent over $K(X_1, \ldots, X_n)$. Let $L := K(X_1, \ldots, X_{n-1})$.

For any $\rho \in \bar{L}$ let's denote the closure of $L(\rho)$ in with L_ρ, which is normal finite extension of L. According to (L4.1.1) there exists $r_\rho \geq 1$ and ω_ρ an extension of w to $L(X_1, \ldots, X_n)$, symmetric with respect to $X_1, \ldots, X_{n+r_\rho}$, such that, given any ω', an extension of it to $L_\rho(X_n, \ldots, X_{n+r_\rho})$, we get that ω' is symmetric with respect to $X_1, \ldots, X_{n+r_\rho}$. Let $\bar{w}_\rho$ be the common extension of $\bar{w}$ and ω_ρ to $\overline{K(X_1, \ldots, X_n)}(X_{n+1}, \ldots, X_{n+r_\rho})$, which we know it exists, [11, Lemma 3.4]. Therefore, the restriction of $\bar{\omega}_\rho$ to $L_\rho(X_n, \ldots, X_{n+r_\rho})$, is symmetric with respect to $X_n, \ldots, X_{n+r_\rho}$. We have:

$$\begin{aligned} \bar{w}(X_n - X_1) = \bar{\omega}_\rho(X_{n+1} - X_n) &= \bar{\omega}_\rho(X_{n+1} - \rho + \rho - X_n) \\ &\geq \bar{\omega}_\rho(X_n - \rho) = \bar{w}(X_n - \rho) \end{aligned}$$

and this holds for any $\rho \in \bar{L}$, independently of the choice of r_ρ and ω_ρ.

Let $\mathcal{M}_i = \{\bar{w}(X_i - \rho)/\rho \in \overline{K(X_1, \ldots, X_{i-1})}\}$, with $i \in \{1, \ldots, n\}$. Obviously, $\bar{w}(X_n - X_1) \in \mathcal{M}_n$. From the discussion above, we have $\bar{w}(X_n - X_1) = \sup \mathcal{M}_n$ and let's denote by ϵ this value. Moreover, we have:

$$\epsilon = \bar{w}(X_n - X_1) = \bar{w}(X_2 - X_1) = w(X_2 - X_1) \in \mathcal{M}_2$$

and, since $\mathcal{M}_2 \subseteq \mathcal{M}_n$, it follows that $\epsilon = \sup \mathcal{M}_2$, so ϵ is an upper bound also of $\mathcal{M}_1$. Now, as $\sup \mathcal{M}_2 \in \mathcal{M}_2$, we derive, according to [4], that u_2, the extension of u_1 from $K(X_1)$ to $K(X_1, X_2)$ is either a r.t.-extension, when $\mathbf{Q}G_1 = \mathbf{Q}G_2$, or a r.a.f-extension, when otherwise.

In both cases, the pair (X_1, ϵ) is a definition pair for u_2 and is minimal since $\deg_{X_2} X_1 = 1$.

Consequently, given what we know from [4] and [10], it follows that, for any $F \in K[X_1, X_2]$ written as:

$$F = \sum_{i_2=0}^{s_2} f_{i_2} \cdot (X_2 - X_1)^{i_2}, \text{ with } f_{i_2} \in K[X_1]$$

we get

$$w(F) = \inf_{i_2} \big(u_1(f_{i_2}) + i_2 \cdot \epsilon \big).$$

Now, let $K' = K(X_1, X_2)$ and let's reconsider w and $\bar{w}$ with respect to $X_3, \ldots, X_n, X_{n+1}$. Obviously, w remains symmetric and $\bar{w}$ extends its symmetry.

Furthermore, since

$$\epsilon = \sup \mathcal{M}_n = \sup \mathcal{M}_{n-1} = \ldots = \sup \mathcal{M}_3 \in G_2$$

we deduce that $\mathbf{Q}G_2 = \mathbf{Q}G_3 = \ldots = \mathbf{Q}G_n$ because, if this wasn't true and we took $\mathbf{Q}G_{i-1} \neq \mathbf{Q}G_i$, with the smallest $i \geq 3$ validating this, then there would exist $\rho \in \overline{K(X_1, \ldots, X_{i-1})}$ that would make $\bar{w}(X_i - \rho) \notin \mathbf{Q}G_{i-1}$ and, therefore

$$\bar{w}(X_1 - \rho) = \bar{w}(X_1 - X_i + X_i - \rho) = \bar{w}(X_i - \rho)$$

but this is not possible since $\bar{w}(X_1 - \rho) \in \mathbf{Q}G_{i-1}$.

We have proven, thus, that $\text{freedeg}(w) = 0$, with respect to $X_3, \ldots, X_n$. Using (2.3) we derive that w is a r.t.s.-extension with respect to $X_3, \ldots, X_n$ and, given any $F \in K[X_1, \ldots, X_n]$ written as:

$$F = \sum_{(i_2, \ldots, i_n) \in I} f_{i_2, \ldots, i_n} \cdot (X_2 - X_1)^{i_2} \cdot \ldots \cdot (X_n - X_1)^{i_n}, \text{ with } f_{i_2, \ldots, i_n} \in K[X_1]$$

where I is a finite set of $(n-1)$-uples of indices, we get:

$$w(F) = \inf_{(\cdot, i_2, \ldots, i_n) \in I} \left(u_2 \left(\sum_{i_2 1 \text{ such that } (i_2, i_3, \ldots, i_n) \in I} f_{i_2, \ldots, i_n} \cdot (X_2 - X_1)^{i_2} \right) + (i_3 + \ldots + i_n) \cdot \epsilon \right) = \inf_{(i_2, \ldots, i_n) \in I} \big(u_1(f_{i_2, \ldots, i_n}) + (i_2 + \ldots + i_n) \cdot \epsilon \big).$$

"$\Leftarrow$" If $n = 1$, we are free to choose a value ϵ, upper bound for $\mathcal{M}_1$. This value will be automatically in G_2, once we put $w'(X_2 - X_1) = \epsilon$. So we may consider, directly, the case $n \geq 1$ and let's choose X_{n+1} transcendental over $K(X_1, \ldots, X_n)$. Let's define w' as the extension of w to $K(X_1, \ldots, X_{n+1})$ given by the pair (X_1, ϵ), which is minimal because $\deg_{X_{n+1}} = 1$.

Therefore, for any $F \in K[X_1, \ldots, X_{n+1}]$ written as:

$$F = \sum_{(i_2, \ldots, i_{n+1}) \in I} f_{i_2, \ldots, i_{n+1}} \cdot (X_2 - X_1)^{i_2} \cdot \ldots \cdot (X_{n+1} - X_1)^{i_{n+1}}, \quad \text{with } f_{i_2, \ldots, i_{n+1}} \in K[X_1] \quad (4.1)$$

where I is a finite set of n-uples of indices, we get

$$w'(F) = \inf_{(i_2, \ldots, i_{n+1}) \in I} \big(u_1(f_{i_2, \ldots, i_{n+1}}) + (i_2 + \ldots + i_{n+1}) \cdot \epsilon \big).$$

Let's notice that w', as extension of w, from $K(X_1, \ldots, X_n)$ to $K(X_1, \ldots, X_n)(X_{n+1})$ may be either a r.t.-extension or a r.a.f.-extension, the latter being valid only if $n = 1$ and $\epsilon \notin \mathbf{Q}G_1$. But, in both cases, (see [4] and [10]), ϵ is an upper bound of $\mathcal{M}_n$, which means that, in particular, for any $r \in \bar{K}$, we get:

$$w'(X_{n+1} - X_1) = \epsilon \geq \bar{w}'(X_1 - r)$$

Using the definition of w' we derive that $w'(X_i - X_j) = \epsilon$ for each $i \neq j$ in $\{1, \ldots, n+1\}$.

Further, it is obvious that w' is symmetric with respect to X_i, X_{n+1} for each $i \geq 2$ therefore, in order to check the symmetry of w', it is enough (cf. Lemma 3.1) to check the inversion of X_{n+1} with X_1. Let, thus, $F \in K[X_1, \ldots, X_{n+1}]$ and let's analyze the polynomial $\pi F \in K[X_1, \ldots, X_{n+1}]$ obtained from F by inverting X_{n+1} with X_1. Let's consider $\bar{w}'$ that extends $\bar{w}$ on $\bar{K}(X_1, \ldots, X_{n+1})$.

We have

$$w'(\pi F) = w'\left(\sum_{(i_2,\ldots,i_n,i_{n+1})\in I} f_{i_2,\ldots,i_n,i_{n+1}}(X_{n+1}) \cdot (X_2 - X_{n+1})^{i_2} \cdot \ldots \right.$$
$$\left. \cdot (X_n - X_{n+1})^{i_n} \cdot (X_1 - X_{n+1})^{i_{n+1}}\right)$$

which may be written, further, denoting by $J_{i_2,\ldots,i_{n+1}}$ the set $\{1, \ldots, \deg f_{i_2,\ldots,i_{n+1}}\}$, with $r_{i_2,\ldots,i_{n+1};j}$ being the roots of $f_{i_2,\ldots,i_{n+1}}$, where $j \in J_{i_2,\ldots,i_{n+1}}$ and denoting by $a_{i_2,\ldots,i_{n+1}}$ the coefficient of the term of maximal degree

$$w'(\pi F) = \bar{w}'\left(\sum_{(i_2,\ldots,i_{n+1})\in I}\left(a_{i_2,\ldots,i_{n+1}} \cdot \left(\prod_{j\in J_{i_2,\ldots,i_{n+1}}} (X_{n+1} - r_{i_2,\ldots,i_{n+1};j})\right)\right.\right.$$
$$\left.\left. \cdot (X_2 - X_{n+1})^{i_2} \cdot \ldots \cdot (X_1 - X_{n+1})^{i_{n+1}}\right)\right)$$

which, by replacing $X_{n+1} - r_{i_2,\ldots,i_{n+1};j} = X_{n+1} - X_1 + X_1 - r_{i_2,\ldots,i_{n+1};j}$, becomes:

$$w'(\pi F) =$$
$$\bar{w}'\left(\sum_{(i_2,\ldots,i_{n+1})\in I} \sum_{H\subset J_{i_2,\ldots,i_{n+1}}}\left(a_{i_2,\ldots,i_{n+1}} \cdot \left(\prod_{j\in J_{i_2,\ldots,i_{n+1}}-H} (X_1 - r_{i_2,\ldots,i_{n+1};j})\right)\right.\right.$$
$$\left.\left. \cdot \big(X_2 - X_{n+1}\big)^{i_2} \cdot \ldots \cdot \big(X_1 - X_{n+1}\big)^{i_{n+1}+card(H)}\right)\right) \quad (4.2)$$

Considering the fact that, for any $i \neq j$ in $\{1, \ldots, n+1\}$ and any $r \in \bar{K}$, we get

$$w'(X_i - X_j) = \epsilon \geq \bar{w}'(X_1 - r)$$

it follows that each term of the double summation in (4.2) has its valuation greater or equal to $w'(F)$:

$$\bar{w}'\left(a_{i_2,\ldots,i_{n+1}}\cdot\left(\prod_{j\in J_{i_2,\ldots,i_{n+1}}-H}(X_1-r_{i_2,\ldots,i_{n+1};j})\right)\cdot(X_2-X_{n+1})^{i_2}\cdot\ldots\cdot(X_1-X_{n+1})^{i_{n+1}+card(H)}\right)\geq$$

$$\bar{w}'\left(a_{i_2,\ldots,i_{n+1}}\cdot\left(\prod_{j\in J_{i_2,\ldots,i_{n+1}}}(X_1-r_{i_2,\ldots,i_{n+1};j})\right)\cdot(X_2-X_{n+1})^{i_2}\cdot\ldots\cdot(X_1-X_{n+1})^{i_{n+1}}\right)=$$

$$w'\big(f_{i_2,\ldots,i_{n+1}}\cdot(X_2-X_{n+1})^{i_2}\cdot\ldots\cdot(X_1-X_{n+1})^{i_{n+1}}\big)=$$
$$u_1(f_{i_2,\ldots,i_{n+1}})+(i_2+\ldots+i_{n+1})\cdot\epsilon\geq w'(F)$$

We deduce, thus, that $w'(\pi F)\geq w'(F)$. We are left with the reverse inequality.

Of the terms of F, whose valuation is equal to $w'(F)$, let's chose one of minimal degree in X_{n+1}:

$$f_{l_2,\ldots,l_n,l_{n+1}}\cdot(X_2-X_1)^{l_2}\cdot\ldots\cdot(X_n-X_1)^{l_n}\cdot(X_{n+1}-X_1)^{l_{n+1}},\text{ with}$$
$$w'(F)=u_1(f_{l_2,\ldots,l_n,l_{n+1}})+(l_2+\ldots+l_n+l_{n+1})\cdot\epsilon\text{ and}$$
$$l_{n+1}\text{ is minimal validating this property.}$$

Now, we need to write also πF under the form (4.1). In order to do this, we will need to set:

$$X_{n+1}-r_{i_2,\ldots,i_{n+1};j}=(X_{n+1}-X_1)+(X_1-r_{i_2,\ldots,i_{n+1};j})\text{ and}$$
$$X_i-X_{n+1}=(X_i-X_1)+(X_1-X_{n+1})\text{ for }2\leq i\leq n$$

and to make the replacements in (4.2). It is not necessary to perform all the calculations, because we are interested only in those terms that sum up for the n-uple $(l_2,\ldots,l_{n+1)}$, meaning those of the form:

$$F_{l_2,\ldots,l_{n+1},i_2,\ldots,i_{n+1},H}=$$
$$a_{i_2,\ldots,i_{n+1}}\cdot\left(\prod_{j\in J_{i_2,\ldots,i_{n+1}}-H}(X_1-r_{i_2,\ldots,i_{n+1};j})\right)$$
$$\cdot(X_2-X_1)^{l_2}\cdot\ldots\cdot(X_n-X_1)^{l_n}\cdot(X_{n+1}-X_1)^{l_{n+1}}\cdot(-1)^{l_{n+1}}$$

with $i_2 \geq l_2, \ldots, i_n \geq l_n, i_{n+1} \leq l_{n+1}$, $H \subseteq J_{i_2,\ldots,i_{n+1}}$ and $i_{n+1} + i_2 - l_2 + \ldots + i_n - l_n + \mathrm{card}(H) = l_{n+1}$.

If we denote by $\bar{u}_1$ the restriction of $\bar{w}'$ to $\bar{K}(X_1)$, then we have:

$$\bar{w}'(F_{l_2,\ldots,l_{n+1},i_2,\ldots,i_{n+1},H}) =$$
$$v(a_{i_2,\ldots,i_{n+1}}) + \sum_{j\in J_{i_2,\ldots,i_{n+1}}-H} \bar{u}_1(X_1 - r_{i_2,\ldots,i_{n+1};j}) +$$
$$(i_2 + \ldots + i_{n+1}) + \mathit{card}(H)) \cdot \epsilon \geq$$
$$v(a_{i_2,\ldots,i_{n+1}}) + \sum_{j\in J_{i_2,\ldots,i_{n+1}}} \bar{u}_1(X_1 - r_{i_2,\ldots,i_{n+1};j}) + (i_2 + \ldots + i_{n+1}) \cdot \epsilon =$$
$$u_1(f_{i_2,\ldots,i_{n+1}}) + (i_2 + \ldots + i_{n+1}) \cdot \epsilon \geq w'(F)$$

with the last inequality being strict if $i_{n+1} < l_{n+1}$. This means that there exists one and only one term equal to $w'(F)$ among those that sum up for the n-uple $(l_2, \ldots, l_{n+1})$, namely $F_{l_2,\ldots,l_{n+1},l_2,\ldots,l_{n+1},\phi}$. Thus, we get the reverse inequality:

$$w'(\pi F) = \inf_{(l_2,\ldots,l_{n+1})\in I} \bar{w}'\left(\sum_{i_2,\ldots,i_{n+1},H} F_{l_2,\ldots,l_{n+1},i_2,\ldots,i_{n+1},H}\right) \leq w'(F)$$

We conclude that $w'(\pi F) = w'(F)$, therefore w' is symmetric with respect to $X_1, \ldots, X_{n+1}$. By induction, choosing $X_{n+2}, X_{n+3}, \ldots$ and reasoning similarly, we get a chain of symmetric extensions, leading to the conclusion that w is a symmetrically-open extension with respect to $X_1, \ldots, X_n$.

□

Corollary 4.1. *With the notations above we have:*

(C4.1.1) *The dual statement of* (D4.2.1) *also stands: for any symmetrically-open extension with respect to $X_1, \ldots, X_n$ there exists an extension of it, symmetrically-open with respect to $X_1, \ldots, X_i$, for all $i > n$, with* $\mathrm{tr.deg}(K(X_1, \ldots, X_i) : K) = i$.

(C4.1.2) *For a chain of symmetrically-open extensions, built using* (C4.1.1), *there exists a chain of extensions to the algebraic closures (of the fields each of the extensions in the original chain are defined on), such that their symmetry is also extended.*

(C4.1.3) *A symmetric extension is symmetrically-open if and only if it may be extended to a symmetric valuation on $K(X_1, \ldots, X_{n+1})$ that has an extension further to $\overline{K(X_1, \ldots, X_{n+1})}$ which extends its symmetry.*

(C4.1.4) *If $n \geq 3$, a symmetrically-open extension cannot be ultrasymmetric with respect to $X_1, \ldots, X_n$.*

(C4.1.5) *If w is symmetrically-open with respect to $X_1, \ldots, X_n$ then:*

$$0 \le \operatorname{freedeg} w \le 2;$$
$$n - 2 \le \operatorname{tr.deg}(k_w : k_v) \le n;$$
$$n - 1 \le \operatorname{freedeg} w + \operatorname{tr.deg}(k_w : k_v) \le n.$$

Proof. (C4.1.1), (C4.1.2) The statements are obvious from the closed-form of the symmetrically open extensions, corroborated with Proposition 3.2

(C4.1.3) The implication "⇒" is obvious due to (C4.1.2), so we'll focus on the reverse implication.

Let w' be the extension of w to $K(X_1, \ldots, X_{n+1})$ and $\bar{w}'$ its extension to $\overline{K(X_1, \ldots, X_{n+1})}$. In the proof made for the "⇒" implication in Proposition 4.1 we have, directly:

$$\bar{w}(X_n - X_1) = \bar{w}'(X_{n+1} - X_n) = \bar{w}'(X_{n+1} - \rho + \rho - X_n)$$
$$\ge \bar{w}'(X_n\rho) = \bar{w}(X_n - \rho)$$

for any $\rho \in \overline{K(X_1, \ldots, X_{n-1})}$ wherefrom the proof follows similarly.

(C4.1.4) If we consider w as a valuation on $K(X_1)(X_2, X_3)$, it is symmetrically open with respect to X_2, X_3. From Proposition 4.1 it follows that w might be written as for Corollary 3.1 with:

$$K \to K(X_1);$$
$$a \to X_1;$$
$$g \to X - X_1;$$
$$\delta \to \epsilon = w(X_2 - X_1) = w(X_3 - X_1).$$

Therefore, according to (2.4), w is a r.t.s.-extension with respect to X_2, X_3. Now, using (D4.1.2), we conclude that w is not ultrasymmetric.

(C4.1.5) For $n \le 2$, the first two statements are obvious. If $n \ge 3$, we use the same arguments as above to derive that w, as a valuation on $K(X_1)(X_2, X_3, \ldots, X_n)$, is a r.t.s.-extension with respect to $X_2, X_3, \ldots, X_n$ and, considering (2.2) and (2.3), we conclude that:

$$0 = \operatorname{freedeg}_{X_2,\ldots,X_n} w \ge \operatorname{freedeg} w - 1 \text{ and:}$$
$$n - 1 = \operatorname{tr.deg}(k_w : k_{u_i}) \le \operatorname{tr.deg}(k_w : k_v).$$

We are left with the last inequality. From [4] we know that all the intermediary extensions from K_{i-1} to K_i (with $1 \le i \le n$) may be classified as r.t., r.a.t. or r.a.f. As $\operatorname{tr.deg}(k_w : k_v)$ represents the number of intermediary extensions that are r.t.-extensions and w represents the number of intermediary extensions that are r.a.f.-extensions it remains to be proved that there cannot exist more than one intermediary extension that is r.a.t.-extension, namely the first of the intermediary extensions.

Let's analyze the only intermediary extension that is important not to be a r.a.t.-extension, namely the extension from $K(X_1)$ to $K(X_1, X_2)$. Suppose, by *reduction ad absurdum*, that it is a r.a.t.-extension. Then the set

$$\mathcal{M}_2 = \{\bar{w}(X_2 - \rho)/\rho \in \overline{K(X_1)}\}$$

wouldn't have an upper bound inside.

From Proposition 4.1 we know that there exists $\epsilon \in G_2$, an upper bound for $\mathcal{M}_1 \subseteq \mathcal{M}_2$, such that, for any $F \in K[X_1, X_2]$ written as $F = \sum_{i \in I} f_i \cdot (X_2 - X_1)^i$, with $f_i \in K[X_1]$ where I is a finite set of indices, we get

$$u_2(F) = \inf_{i \in I} \big(u_1(f_i) + i \cdot \epsilon\big).$$

Let $\{\epsilon_j\}_{j \in J}$ be a strictly increasing sequence of elements in $\mathcal{M}_2$, where J is a countable set. As $\mathcal{M}_2$ doesn't have a largest element, we may assume, without any loss of generality, that $\epsilon_0 = \epsilon$. We choose, for each $j \in J$, an element ρ_j in $\overline{K(X_1)}$, of minimal degree over $K(X_1)$, such that $u_2(X_2 - \rho_j) = \epsilon_j$. For $j = 0$ we choose $\rho_0 = X_1$.

Let $\{u'_j\}_{j \in J}$ be the sequence of r.t.-extensions from $K(X_1)$ to $K(X_1, X_2)$ defined by the minimal pairs (ρ_j, ϵ_j). From [4, Theorem 5.1] it follows that this sequence is an ordered system of r.t.-extensions that has u_2 as its limit:

$$u_2(F) = \sup_{j \in J} \big(u'_j(F)\big), \text{ for all } F \in K(X_1, X_2).$$

But this leads to:

$$u_2(F) = u'_0(F) = \sup_{j \in J} \big(u'_j(F)\big)$$

which means that the ordered system of r.t.-extensions is stationary, which contradicts the assertion that $\{\epsilon_j\}_{j \in J}$ is a strictly increasing sequence.

□

5. Characterization of the symmetrically-open extensions

We can now present the main result of this paper, that allows a complete classification of the symmetrically-open extensions in two classes, depending on the existence of a r.a.f.-extension among the intermediary extensions. Additionally, the following theorem states that any extension in the second category (having a r.a.f.-extension among the intermediary ones) may be reduced, in fact, to a sequence of extensions from the first category.

Theorem 5.1. *Let w be a symmetrically-open extension of a valuation v, from K to $K(X_1, \ldots, X_n)$, with $n \geq 2$, a fixed algebraic closure $\overline{K(X_1, \ldots, X_n)}$ and $\bar{w}$ that extends the symmetry of w to $\overline{K(X_1, \ldots, X_n)}$. Then w may be in one of the following possible situations:*

(I) $\operatorname{freedeg} w + \operatorname{tr.deg}(k_w : k_v) = n$ *and, in this case, w is defined by a triplet (a, δ, ϵ), in which we have $a \in \bar{K}$, $\delta \in Z \times \mathbf{Q}G_v$ and $\epsilon \in Z \times Z \times \mathbf{Q}G_v, \epsilon > \delta$ such that, for any $F \in K[X_1, \ldots, X_n]$ written as:*

$$F = \sum_{(i_1,\ldots,i_n)\in I} f_{i_1,\ldots,i_n} \cdot g^{i_1}(X_2 - X_1)^{i_2} \cdot \ldots \cdot (X_n - X_1)^{i_n},$$

$$\text{with } f_{i_1,\ldots,i_n} \in K[X_1], \deg f_{i_1,\ldots,i_n} < \deg g$$

where I is a finite set of n-uples of indices and $g \in K[X_1]$ is the minimal monic polynomial of a over K, we get:

$$w(F) = \inf_{(i_1,\ldots,i_n)\in I} \big(\bar{v}(f_{i_1,\ldots,i_n}(a)) + i_1 \cdot \gamma + (i_2 + \cdot i_n) \cdot \epsilon\big),$$

$$\text{with } \gamma = \sum_{a'\in\bar{K}, g(a')=0} \inf\big(\delta_a, \bar{v}(a' - a)\big)$$

(II) $\operatorname{freedeg} w + \operatorname{tr.deg}(k_w : k_v) = n - 1$ *and, in this case, w is the limit of an ordered system of extensions of type **(I)**, that have in their definition the same value for ϵ.*

Proof. From C4.1.5 we know that $n - 1 \leq \operatorname{freedeg} w + \operatorname{tr.deg}(k_w : k_v) \leq n$ so the cases **(I)** and **(II)** are, indeed, the only possible ones.

In case **(I)** all the intermediary extensions from K_{i-1} to K_i (with $1 \leq i \leq n$) are r.t.-extensions or r.a.f.-extensions. Looking at the first of them, we notice that there exist $a \in \bar{K}$ and $\delta \in Z \times \mathbf{Q}G_v$ such that, for any $f \in K[X_1]$ written as:

$$f = \sum_{i_1\in I_1} f_{i_1} \cdot g^{i_1}, \text{ with } f_{i_1} \in K[X_1], \deg f_{i_1} < \deg g$$

where I_1 is a finite set of indices and $g \in K[X_1]$ is the minimal monic polynomial of a over K, we get:

$$u_1(f) = \inf_{i_1\in I_1} \big(\bar{v}(f_{i_1}(a)) + i_1 \cdot \gamma\big), \text{ with } \gamma = \sum_{a'\in\bar{K}, g(a')=0} \inf\big(\delta_a, \bar{v}(a' - a)\big). \quad (5.1)$$

We also note that:

$$w(X_1 - a) = u_1(X_1 - a) = \delta \in \mathcal{M}_1.$$

From Proposition 4.1 we know that there exists $\epsilon \in G_2$, upper bound of $\mathcal{M}_1$ (so $\epsilon \geq \delta$), such that, for any $F \in K[X_1, \ldots, X_n]$ written as:

$$F = \sum_{(i_2,\ldots,i_n)\in I} f_{i_2,\ldots,i_n} \cdot (X_2 - X_1)^{i_2} \cdot \ldots \cdot (X_n - X_1)^{i_n}, \text{ with } f_{i_2,\ldots,i_n} \in K[X_1]$$

where I is a finite set of $(n-1)$-uples of indices, we get

$$w(F) = \inf_{(i_2,\dots,i_n)\in I} \big(u_1(f_{i_2,\dots,i_n}) + (i_2 + \ldots + i_n)\cdot \epsilon\big).$$

By applying 5.1 for $f_{i_2,\dots,i_n}$ in the parenthesis above, we derive exactly the wanted formula:

$$w(F) = \inf_{(i_1,\dots,i_n)\in I} \big(\bar{v}(f_{i_1,\dots,i_n}(a)) + i_1\cdot\gamma + (i_2 + \ldots + i_n)\cdot \epsilon\big). \tag{5.2}$$

Let's now consider case **(II)**. As we discussed at Corollary 4.1, the extension u_1 of v, from K to $K(X_1)$, is a r.a.t.-extension. Then the set $\mathcal{M}_1$ doesn't have a maximal element.

Let $\{\delta_j\}_{j\in J}$ be an increasing sequence of elements in $\mathcal{M}_1$, where J is a countable set and let's choose, for each $j \in J$, an element a_j in $\bar{K}$, of minimal degree over K, such that we would have $u_1(X_1 - a_j) = \delta_j$. Let's denote by g_j the minimal monic polynomial of a_j. Let $\{u'_j\}_{j\in J}$ be the sequence of the r.t.-extensions from K to $K(X_1)$ defined by the minimal pairs (a_j, δ_j). It follows from [4, Theorem 5.1] that this is an ordered system of r.t.-extensions that has u_1 as limit:

$$u_1(f) = \sum_{j\in J}\big(u'_j(f)\big), \text{ for any } f \in K(X_1).$$

From Proposition 4.1 we know that there exists $\epsilon \in G_2$, an upper bound of $\mathcal{M}_1$ (so $\epsilon \geq \delta_j$ for each $j \in J$), such that, for any $F \in K[X_1,\dots,X_n]$ written as:

$$F = \sum_{(i_2,\dots,i_n)\in I} f_{i_2,\dots,i_n}\cdot (X_2 - X_1)^{i_2}\cdot \ldots \cdot (X_n - X_1)^{i_n}, \text{ with } f_{i_2,\dots,i_n} \in K[X_1]$$

where I is a finite set of $(n-1)$-uples of indices, we have:

$$\begin{aligned} w(F) &= \inf_{(i_2,\dots,i_n)\in I}\big(u_1(f_{i_2,\dots,i_n}) + (i_2+\ldots+i_n)\cdot\big) = \\ &= \inf_{(i_2,\dots,i_n)\in I}\left(\sup_{j\in J}\big(u'_j(f_{i_2,\dots,i_n})\big) + (i_2+\ldots+i_n)\cdot\epsilon\right) = \\ &= \inf_{(i_2,\dots,i_n)\in I}\ \sup_{j\in J}\big(u'_j(f_{i_2,\dots,i_n}) + (i_2+\ldots+i_n)\cdot\epsilon\big). \end{aligned} \tag{5.3}$$

As u'_{j_1} is dominated by u'_{j_2} for any $j_1 < j_2$, the quantity in parenthesis forms an increasing sequence in G_w, so the infimum commutes with supremum and we may rewrite (5.3):

$$w(F) = \sup_{j\in J}\ \inf_{(i_2,\dots,i_n)\in I}\big(u'_j(f_{i_2,\dots,i_n}) + (i_2 + \ldots, i_n)\cdot\epsilon\big).$$

For each $j \in J$ let w_j be the extension of u'_j from $K(X)$ to $K(X_1, \ldots, X_n)$ defined by:

$$w_j(F) = \inf_{(i_2,\ldots,i_n)\in I} \big(u'_j(f_{i_2,\ldots,i_n}) + (i_2 + \ldots + i_n)\cdot \epsilon\big).$$
$$= \inf_{(i_1,\ldots,i_n)\in I_j} \big(\bar{v}(f_{i_1,\ldots,i_n}(a_j)) + i_1 \cdot \gamma_j + (i_2 + \ldots + i_n)\cdot \epsilon\big)$$

where γ_j is given by:

$$\gamma_j = \sum_{a' \in \bar{K}, g_j(a')=0} \inf\big(\delta_j, \bar{v}(a' - a_j)\big) = u'_j(g_j)$$

and the set I_j is defined as

$$I_j = \{(i_1, \ldots, i_n)/(i_2, \ldots, i_n) \in I \text{ and } 0 \le i_1, (i_1 \cdot \deg g_j) \le deg f_{i_2,\ldots,i_n\}}$$

since we wrote each $f_{i_2,\ldots,i_n}$ as

$$f_{i_2,\ldots,i_n} = \sum_{i_1=0}^{k_{i_2,\ldots,i_n,j}} f_{i_1,i_2,\ldots,i_n} \cdot (g_j)^{i_1}, \text{ where } k_{i_2,\ldots,i_n,j} = \left\lfloor \frac{\deg\big(f_{i_2,\ldots,i_n}\big)}{\deg(g_j)} \right\rfloor.$$

We obtained, thus, (5.2) for each w_j and, since $\{u'_j\}_{j\in J}$ is an ordered system of r.t.-extensions that has u_1 as limit, we conclude that $\{w_j\}_{j\in J}$ is an ordered system of extensions of type **(I)** that verifies $w = \sup_{j\in J} w_j$ and all the extensions in the ordered system have the same value for ϵ.

□

The following table describes all the possibilities of definition for a symmetrically open extension of v, from K to $K(X_1, \ldots, X_n)$, avoiding the complex issues with algebraic geometry and specifying the formulas for the valuation group, the residual field and the properties of the extension of each identified type.

#	Parameters	Valuation group	Residual field	Properties
1	a, δ	$G_{v_{a,\delta}} + Z\gamma_{a,\delta}$	$k_{v_{a,\delta}}(\chi_1, \dots, \chi_n)$ $\chi_i = ((X_i - a)/b_\delta)^*$ $b_\delta \in \bar{K}, \bar{v}(b_\delta) = \delta$	r.t.s.-extension tr. deg$(k_w : k_v) = n$ freedeg $w = 0$
2	a, δ, ϵ $\epsilon > \delta, \epsilon \in \mathbf{Q}G_v$	$G_{v_{a,\delta}} +$ $Z\gamma_{a,\delta} + Z\epsilon$	$k_{v_{a,\delta}}(\chi_1, \psi_2, \dots, \psi_n)$ $\chi_1 = ((X_1 - a)/b_\delta)^*$ $b_\delta \in \bar{K}, \bar{v}(b_\delta) = \delta$ $\psi_i = ((X_i - X_1)/b_\epsilon)^*$ $b_\epsilon \in \bar{K}, \bar{v}(b_\epsilon) = \epsilon$	pure r.t.-extension tr. deg$(k_w : k_v) = n$ freedeg $w = 0$
3	a, δ, ϵ $\epsilon > \delta, \epsilon \notin \mathbf{Q}G_v$	$G_{v_{a,\delta}} +$ $Z\gamma_{a,\delta} + Z\epsilon$	$k_{v_{a,\delta}}(\chi_1, \psi_3, \dots, \psi_n)$ $\chi_1 = ((X_1 - a)/b_\delta)^*$ $b_\delta \in \bar{K}, \bar{v}(b_\delta) = \delta$ $\psi_i = ((X_i - X_1)/(X_2 - X_1))^*$	tr. deg$(k_w : k_v) = n - 1$ freedeg $w = 1$
4	$\{a_j\}_j, \{\delta_j\}_j, \epsilon$ $\epsilon > \delta_j, \epsilon \in \mathbf{Q}G_v$	$\bigcup_j (G_{v_{a_j,\delta_j}} +$ $Z\gamma_{a_j,\delta_j} + Z\epsilon)$	$\left(\bigcup k_{v_{a_j,\delta_j}}\right)(\psi_2, \dots, \psi_n)$ $\psi_i = ((X_i - X_1)/b_\epsilon)^*$ $b_\epsilon \in \bar{K}, \bar{v}(b_\epsilon) = \epsilon$	the limit of a #2-sequence tr. deg$(k_w : k_v) = n - 1$ freedeg $w = 0$
5	$\{a_j\}_j, \{\delta_j\}_j, \epsilon$ $\epsilon > \delta_j, \epsilon \notin \mathbf{Q}G_v$	$\bigcup_j (G_{v_{a_j,\delta_j}} +$ $Z\gamma_{a_j,\delta_j} + Z\epsilon)$	$\left(\bigcup k_{v_{a_j,\delta_j}}\right)(\psi_3, \dots, \psi_n)$ $\psi_i = ((X_i - X_1)/(X_2 - X_1))^*$	the limit of a #3-sequence tr. deg$(k_w : k_v) = n - 2$ freedeg $w = 0$
6	a, δ, ϵ $\epsilon \geq \delta, \epsilon \in \mathbf{Q}G_1$ $G_1 = G_{v_{a,\delta}}$ $+Z\gamma_{a,\delta}$	$G_{v_{a,\delta}} +$ $Z\gamma_{a,\delta} + Z\epsilon$	$k_{v_{a,\delta}}(\psi_2, \dots, \psi_n)$ $\psi_i = ((X_i - X_1)/F_\delta)^*$ $F_\epsilon \in K[X_1], u_1(F_\epsilon) = \epsilon$	tr. deg$(k_w : k_v) = n - 1$ freedeg $w = 1$
7	a, δ, ϵ $\epsilon \geq \delta, \epsilon \notin \mathbf{Q}G_1$ $G_1 = G_{v_{a,\delta}}$ $+Z\gamma_{a,\delta}$	$G_{v_{a,\delta}} +$ $Z\gamma_{a,\delta} + Z\epsilon$	$k_{v_{a,\delta}}(\psi_3, \dots, \psi_n)$ $\psi_i = ((X_i - X_1)/(X_2 - X_1))^*$	tr. deg$(k_w : k_v) = n - 2$ freedeg $w = 2$ u_2-ultrasymmetric

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