

## Best proximity points for $\mathcal{K}$ –rational proximal contraction of first and second kind

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**Abstract** - In this paper, we present best proximity point theorems for new class of  $\mathcal{K}$ –rational proximal contraction, in the setting of metric spaces. Some illustrative example are also given.

**Key words and phrases** : Best proximity point, proximal contraction, rational contraction.

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### 1. Introduction and preliminaries

Fixed point theory focusses on the strategies for solving non-linear equations of the kind  $Tx = x$  in which  $T$  is a self mapping defined on a subset of a metric space, a normed linear space, a topological vector space or some pertinent framework.

Let  $A$  and  $B$  be two nonempty subsets of a metric space. In general for nonself mapping  $T : A \rightarrow B$ , the fixed point equation  $Tx = x$  may not have a solution. In this case it is focused on the possibility of finding an element  $x$  that is in closed proximity to  $Tx$  in some sense, i.e., to find an approximate solution  $x \in A$  such that error  $d(x, Tx)$  is minimum, possibly  $d(x, Tx) = \text{dist}(A, B)$ .

A point  $p \in A$  is called *best proximity point* of  $T : A \rightarrow B$  if  $d(p, Tp) = \text{dist}(A, B)$ , where

$$\text{dist}(A, B) := \inf \{d(x, y) : (x, y) \in A \times B\}.$$

In fact, best proximity point theorems have been studied to find necessary conditions such that the minimization problem  $\min_{x \in A} d(x, Tx)$ , has at least one solution.

A best proximity point becomes a fixed point if the underlying mapping is a self-mapping. Therefore, it can be concluded that best proximity point theorems generalize fixed point theorems in a natural way. In recent years, existence of best proximity points of various nonself contractive mappings

have been studied by several authors (cf. [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21]).

Let  $A$  and  $B$  be two nonempty subsets of a metric space  $(X, d)$ . S. Basha introduced in [16] the concept of proximal contraction of first and second kind as follows:

**Definition 1.1.** *A mapping  $T : A \rightarrow B$  is said to be a proximal contraction of first kind if there exists a non-negative number  $\alpha < 1$  such that, for all  $u_1, u_2, x_1, x_2 \in A$*

$$\left. \begin{aligned} d(u_1, Tx_1) &= \text{dist}(A, B) \\ d(u_2, Tx_2) &= \text{dist}(A, B) \end{aligned} \right\} \Rightarrow d(u_1, u_2) \leq \alpha d(x_1, x_2).$$

It can be observed that a self-mapping that is a proximal contraction of the first kind reduces to a contraction.

**Definition 1.2.** *A mapping  $T : A \rightarrow B$  is said to be a proximal contraction of second kind if there exists a non-negative number  $\alpha < 1$  such that, for all  $u_1, u_2, x_1, x_2$  in  $A$*

$$\left. \begin{aligned} d(u_1, Tx_1) &= \text{dist}(A, B) \\ d(u_2, Tx_2) &= \text{dist}(A, B) \end{aligned} \right\} \Rightarrow d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2).$$

The requirement for a self-mapping  $T$  to be a proximal contraction of second kind is that

$$d(T^2x_1, T^2x_2) \leq \alpha d(Tx_1, Tx_2)$$

for all  $x_1, x_2$  in the domain of  $T$ . Conversely, any contraction self-mapping is a proximal contraction of the second kind but converse is not true, it can be seen in the following example.

**Example 1.1.** (see [16]) Consider  $\mathbb{R}$  endowed with the Euclidean metric. Let the self-mapping  $T : [0, 1] \rightarrow [0, 1]$  be defined as

$$Tx = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{otherwise.} \end{cases}$$

Then,  $T$  is a proximal contraction of the second kind but not a contraction. Further, the above example exhibit that a self-mapping that is a proximal contraction of second kind is not necessarily continuous.

Basha and Shahzad (see [17]) extended the above definitions to generalized proximal contraction of first and second kind.

**Definition 1.3.** *A mapping  $T : A \rightarrow B$  is said to be a generalized proximal contraction of the first kind, if there exists non-negative numbers  $\alpha, \beta, \gamma, \delta$  with  $\alpha + \beta + \gamma + 2\delta < 1$  such that the conditions*

$$d(u_1, Tx_1) = \text{dist}(A, B) \quad \text{and} \quad d(u_2, Tx_2) = \text{dist}(A, B)$$

imply the inequality that

$$d(u_1, u_2) \leq \alpha d(x_1, x_2) + \beta d(x_1, u_1) + \gamma d(x_2, u_2) + \delta [d(x_1, u_2) + d(x_2, u_1)]$$

for all  $u_1, u_2, x_1, x_2 \in A$ .

If  $T$  is a self-mapping on  $A$ , then the requirement in the preceding definition reduces to the following condition

$$d(Tx_1, Tx_2) \leq \alpha d(x_1, x_2) + \beta d(x_1, Tx_1) + \gamma d(x_2, Tx_2) + \delta [d(x_1, Tx_2) + d(x_2, Tx_1)].$$

**Definition 1.4.** A mapping  $T : A \rightarrow B$  is said to be a generalized proximal contraction of the second kind if there exists non-negative numbers  $\alpha, \beta, \gamma, \delta$  with  $\alpha + \beta + \gamma + 2\delta < 1$  such that the conditions

$$d(u_1, Tx_1) = \text{dist}(A, B) \quad \text{and} \quad d(u_2, Tx_2) = \text{dist}(A, B)$$

imply the inequality that

$$d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2) + \beta d(Tx_1, Tu_1) + \gamma d(Tx_2, Tu_2) + \delta [d(Tx_1, Tu_2) + d(Tx_2, Tu_1)]$$

for all  $u_1, u_2, x_1, x_2 \in A$ .

A mapping that is generalized proximal contraction of the second kind is not necessarily a generalized proximal contraction of the first kind, as illustrated in the next example given by [17].

**Example 1.2.** Consider the space  $\mathbb{R}^2$  with Euclidean metric. Let  $A = \{(-1, x) : x \in \mathbb{R}\}$  and  $B = \{(1, x) : x \in \mathbb{R}\}$ . Let  $T : A \rightarrow B$  be defined as

$$T((-1, x)) = \begin{cases} (1, 1) & \text{if } x \text{ is rational} \\ (1, -1) & \text{otherwise} \end{cases}$$

Then,  $T$  is a generalized proximal contraction of the second kind but not a generalized proximal contraction of the first kind. Further, it can be observed that the generalized proximal contractions are not necessarily continuous.

Motivated by the above studies, we now define notion of  $\mathcal{K}$ -rational proximal contraction of first and second kind:

**Definition 1.5.** A mapping  $T : A \rightarrow B$  is said to be a  $\mathcal{K}$ -rational proximal contraction of the first kind if there exist nonnegative real numbers  $\alpha, \beta, \gamma, \delta$  and  $\omega$  with  $\alpha + \beta + \delta + \gamma + 2\omega < 1$ , such that the conditions

$$d(u_1, Tx_1) = \text{dist}(A, B) \quad \text{and} \quad d(u_2, Tx_2) = \text{dist}(A, B)$$

imply that

$$d(u_1, u_2) \leq \alpha d(x_1, x_2) + \beta \frac{d(x_1, u_1)d(x_1, u_2) + d(x_2, u_2)d(x_2, u_1)}{d(x_1, u_2) + d(x_2, u_1)} + \gamma d(x_1, u_1) \\ + \delta d(x_2, u_2) + \omega [d(x_1, u_2) + d(x_2, u_1)]$$

for all  $u_1, u_2, x_1, x_2 \in A$ , and  $d(x_1, u_2) + d(x_2, u_1) \neq 0$ .

Note that, if  $\beta = 0$ , then from the Definition 1.5 we get the definition of the generalized proximal contraction of the first kind.

**Definition 1.6.** A mapping  $T : A \rightarrow B$  is said to be a  $\mathcal{K}$ -rational proximal contraction of the second kind if there exist nonnegative real numbers  $\alpha, \beta, \gamma, \delta$  and  $\omega$  with  $\alpha + \beta + \gamma + \delta + 2\omega < 1$ , such that the conditions

$$d(u_1, Tx_1) = \text{dist}(A, B) \text{ and } d(u_2, Tx_2) = \text{dist}(A, B)$$

imply the inequality

$$d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2) \\ + \beta \frac{d(Tx_1, Tu_1)d(Tx_1, Tu_2) + d(Tx_2, Tu_2)d(Tx_2, Tu_1)}{d(Tx_1, Tu_2) + d(Tx_2, Tu_1)} \\ + \gamma d(Tx_1, Tu_1) + \delta d(Tx_2, Tu_2) + \omega [d(Tx_1, Tu_2) + d(Tx_2, Tu_1)]$$

for all  $u_1, u_2, x_1, x_2 \in A$  and  $d(Tx_1, Tu_2) + d(Tx_2, Tu_1) \neq 0$ .

Note that, if  $\beta = 0$ , then from the Definition 1.6 we get the definition of the generalized proximal contraction of the second kind with.

## 2. Main results

We now establish existence, uniqueness and convergence theorems for  $\mathcal{K}$ -rational proximal contraction of the first kind.

**Theorem 2.1.** Let  $A$  and  $B$  be nonempty, closed subsets of a complete metric space  $(X, d)$  such that  $B$  is approximatively compact with respect to  $A$ . Suppose that  $A_0$  and  $B_0$  are non-empty and  $T : A \rightarrow B$  is a mapping satisfying the following conditions:

- (i)  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the first kind.
- (ii)  $T(A_0) \subseteq B_0$ .

Then there exists a unique element  $x \in A$  such that  $d(x, Tx) = \text{dist}(A, B)$ . Further, for each fixed  $x_0 \in A_0$ , there is a sequence  $\{x_n\}$ , defined by the relation  $d(x_{n+1}, Tx_n) = \text{dist}(A, B)$ , converges to the best proximity point  $x$  of the mapping  $T$ .

**Proof.** Let  $x_0$  be a fixed element in  $A_0$ . Since  $T(A_0) \subseteq B_0$ ,  $Tx_0 \in B_0$  so, there exists an element  $x_1 \in A_0$  such that

$$d(x_1, Tx_0) = \text{dist}(A, B).$$

Again, since  $Tx_1 \in B_0$ , it follows that there is  $x_2 \in A_0$  such that

$$d(x_2, Tx_1) = \text{dist}(A, B).$$

Continuing this process, we can derive a sequence  $\{x_n\}$  in  $A_0$ , such that

$$d(x_{n+1}, Tx_n) = \text{dist}(A, B),$$

for every nonnegative integer  $n$ .

Since  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the first kind, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha d(x_{n-1}, x_n) \\ &\quad + \beta \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})d(x_n, x_n)}{d(x_{n-1}, x_{n+1}) + d(x_n, x_n)} \\ &\quad + \gamma d(x_{n-1}, x_n) + \delta d(x_n, x_{n+1}) + \omega [d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] \\ &\leq \alpha d(x_{n-1}, x_n) + \beta \frac{d(x_{n-1}, x_n)d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_{n+1})} \\ &\quad + \gamma d(x_{n-1}, x_n) + \delta d(x_n, x_{n+1}) + \omega [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= \alpha d(x_{n-1}, x_n) + \beta d(x_{n-1}, x_n) + \gamma d(x_{n-1}, x_n) + \delta d(x_n, x_{n+1}) \\ &\quad + \omega [d(x_{n-1}, x_n) + d(x_n, x_{n+1})] \\ &= \frac{\alpha + \beta + \gamma + \omega}{1 - \delta - \omega} d(x_{n-1}, x_n). \end{aligned}$$

It then follows that

$$d(x_n, x_{n+1}) \leq \lambda d(x_{n-1}, x_n),$$

where  $\lambda = \frac{\alpha + \beta + \gamma + \omega}{1 - \delta - \omega} < 1$ . So that,  $\{x_n\}$  is a Cauchy sequence and since space is complete and  $A$  is closed, the sequence  $\{x_n\}$  converges to some  $x \in A$ .

Further,

$$\begin{aligned} d(x, B) &\leq d(x, Tx_n) \\ &\leq d(x, x_{n+1}) + d(x_{n+1}, Tx_n) \\ &= d(x, x_{n+1}) + \text{dist}(A, B) \\ &\leq d(x, x_{n+1}) + d(x, B). \end{aligned}$$

Taking as  $n \rightarrow \infty$ , we have

$$d(x, B) \leq \lim_{n \rightarrow \infty} d(x, Tx_n) \leq d(x, B),$$

hence  $d(x, Tx_n) \rightarrow d(x, B)$ . Since  $B$  is approximatively compact with respect to  $A$ , the sequence  $\{Tx_n\}$  has a subsequence  $\{Tx_{n_k}\}$  converging to some element  $y \in B$ . It follows that,

$$d(x, y) = \lim_{k \rightarrow \infty} d(x_{n_k+1}, Tx_{n_k}) = \text{dist}(A, B),$$

hence  $x$  must be a member of  $A_0$ . Since  $T(A_0) \subseteq B_0$ , hence  $Tx \in B_0$ , then by definition of  $x_n$  there exists  $v \in A$  such that

$$d(v, Tx) = \text{dist}(A, B). \quad (2.1)$$

Next, we prove that  $x = v$ . Since  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the first kind, we get

$$\begin{aligned} d(v, x_{n+1}) &\leq \alpha d(x, x_n) + \beta \frac{d(x, v)d(x, x_{n+1}) + d(x_n, x_{n+1})d(x_n, v)}{d(x, x_{n+1}) + d(x_n, v)} \\ &\quad + \gamma d(x, v) + \delta d(x_n, x_{n+1}) + \omega [d(x, x_{n+1}) + d(x_n, v)]. \end{aligned} \quad (2.2)$$

Letting  $n \rightarrow \infty$  in (2.2), we get

$$d(v, x) \leq (\gamma + \omega)d(v, x),$$

which implies  $d(v, x) = 0$ , that is  $x = v$ . Then it follows from (2.1), that

$$d(x, Tx) = d(v, Tx) = \text{dist}(A, B).$$

Now, to prove the uniqueness of the best proximity point, assume that  $u$  is another best proximity point of  $T$  so that  $d(u, Tu) = \text{dist}(A, B)$ . Since  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the first kind, we have

$$\begin{aligned} d(x, u) &\leq \alpha d(x, u) + \beta \frac{d(x, x)d(x, u) + d(u, u)d(u, x)}{d(x, u) + d(u, x)} + \gamma d(x, x) + \delta d(u, u) \\ &\quad + \omega [d(x, u) + d(u, x)] \\ &\leq (\alpha + 2\omega)d(x, u). \end{aligned}$$

Since  $\alpha + 2\omega < 1$ , above inequality implies that  $d(x, u) = 0$ , that is  $x = u$ . Hence,  $T$  has a unique best proximity point. This completes the proof.  $\square$

By taking  $\beta = 0$  in Theorem 2.1, we obtain the following result, due to Basha and Shahzed [17, Theorem 3.1].

**Corollary 2.1.** *Let  $A$  and  $B$  be two nonempty, closed subsets of a complete metric space  $(X, d)$  such that  $B$  is approximatively compact with respect to  $A$ . Suppose that  $A_0$  and  $B_0$  are non-empty and  $T : A \rightarrow B$  satisfying the following conditions:*

- (i)  $T$  is a generalized proximal contraction of the first kind.
- (ii)  $T(A_0) \subseteq B_0$ .

*Then there exists a unique element  $x \in A$  such that  $d(x, Tx) = \text{dist}(A, B)$ . Further, for any fixed  $x_0 \in A_0$ , there is a sequence  $\{x_n\}$ , defined by the relation  $d(x_{n+1}, Tx_n) = \text{dist}(A, B)$ , converges to the best proximity point  $x$ .*

The preceding best proximity point theorem subsumes the following result which serves as a non self mapping analogue of famous Banach contraction principle.

**Corollary 2.2.** (see [17]) *Let  $A$  and  $B$  be two nonempty, closed subsets of a complete metric space  $(X, d)$  such that  $B$  is approximately compact with respect to  $A$ . Suppose that  $A_0$  and  $B_0$  are non-empty and  $T : A \rightarrow B$  satisfying the following conditions:*

- (i) *There exists a nonnegative real number  $\alpha < 1$  such that, for all  $u_1, u_2, x_1, x_2 \in A$ , the condition*

$$d(u_1, Tx_1) = \text{dist}(A, B) \text{ and } d(u_2, Tx_2) = \text{dist}(A, B)$$

*imply that  $d(u_1, u_2) \leq \alpha d(x_1, x_2)$ ;*

- (ii)  $T(A_0) \subseteq B_0$ .

*Then there exists a unique element  $x \in A$  such that  $d(x, Tx) = \text{dist}(A, B)$ . Further, for any fixed  $x_0 \in A_0$ , there is a sequence  $\{x_n\}$ , defined by the relation  $d(x_{n+1}, Tx_n) = \text{dist}(A, B)$ , converges to the best proximity point  $x$  of the mapping  $T$ .*

We now establish a result for  $\mathcal{K}$ -rational proximal contraction of the second kind.

**Theorem 2.2.** *Let  $A$  and  $B$  be nonempty, closed subsets of a complete metric space  $(X, d)$  such that  $A$  is approximately compact with respect to  $B$ . Suppose that  $A_0$  and  $B_0$  are non-empty and  $T : A \rightarrow B$  is a satisfying the following conditions:*

- (i)  *$T$  is a continuous  $\mathcal{K}$ -rational proximal contraction of the second kind;*

- (ii)  $T(A_0) \subseteq B_0$ .

*Then there exists a unique element  $x \in A$  such that  $d(x, Tx) = \text{dist}(A, B)$ . Further, for any fixed  $x_0 \in A_0$ , there is a sequence  $\{x_n\}$ , defined by the relation  $d(x_{n+1}, Tx_n) = \text{dist}(A, B)$ , converges to the best proximity point  $x$  of the mapping  $T$ .*

*If  $p$  is another best proximity point of  $T$ , then  $Tx = Tp$ , and so  $T$  is constant on the set of all best proximity points of  $T$ .*

**Proof.** As in the proof of Theorem 2.1, we can find a sequence  $\{x_n\}$  in  $A_0$  such that

$$d(x_{n+1}, Tx_n) = \text{dist}(A, B),$$

for all non-negative integer  $n$ . Since  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the second kind, we get

$$\begin{aligned}
d(Tx_n, Tx_{n+1}) &\leq \alpha d(Tx_{n-1}, Tx_n) \\
&+ \beta \frac{d(Tx_{n-1}, Tx_n)d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_{n+1})d(Tx_n, Tx_n)}{d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)} \\
&+ \gamma d(Tx_{n-1}, Tx_n) + \delta d(Tx_n, Tx_{n+1}) \\
&+ \omega [d(Tx_{n-1}, Tx_{n+1}) + d(Tx_n, Tx_n)] \\
&\leq \alpha d(Tx_{n-1}, Tx_n) + \beta \frac{d(Tx_{n-1}, Tx_n)d(Tx_{n-1}, Tx_{n+1})}{d(Tx_{n-1}, Tx_{n+1})} \\
&+ \gamma d(Tx_{n-1}, Tx_n) + \delta d(Tx_n, Tx_{n+1}) \\
&+ \omega [d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\
&= \alpha d(Tx_{n-1}, Tx_n) + \beta d(Tx_{n-1}, Tx_n) + \gamma d(Tx_{n-1}, Tx_n) \\
&+ \delta d(Tx_n, Tx_{n+1}) + \omega [d(Tx_{n-1}, Tx_n) + d(Tx_n, Tx_{n+1})] \\
&= \frac{\alpha + \beta + \gamma + \omega}{1 - \delta - \omega} d(Tx_{n-1}, Tx_n).
\end{aligned}$$

Set  $\lambda = \frac{\alpha + \beta + \gamma + \omega}{1 - \delta - \omega}$ , then  $\lambda < 1$  and it follows that

$$d(Tx_n, Tx_{n+1}) \leq \lambda d(Tx_{n-1}, Tx_n).$$

So that,  $\{Tx_n\}$  is a Cauchy sequence and, since  $(X, d)$  is complete and  $B$  is closed, the sequence  $\{Tx_n\}$  converges to some  $y \in B$ .

Further,

$$\begin{aligned}
d(y, A) &\leq d(y, x_{n+1}) \leq d(y, Tx_n) + d(Tx_n, x_{n+1}) \\
&\leq d(y, Tx_n) + \text{dist}(A, B) \\
&= d(y, Tx_n) + d(y, A).
\end{aligned}$$

Letting  $n \rightarrow \infty$ , we obtain

$$d(y, A) \leq \lim_{n \rightarrow \infty} d(y, x_{n+1}) \leq d(y, A),$$

hence  $d(y, x_n) \rightarrow d(y, A)$ . Since  $A$  is approximatively compact with respect to  $B$ , the sequence  $\{x_n\}$  has a subsequence  $\{Tx_{n_k}\}$  converging to some element  $x \in A$ . By continuity of  $T$ , we have

$$d(x, Tx) = \lim_{k \rightarrow \infty} d(x_{n_k+1}, Tx_{n_k}) = \text{dist}(A, B),$$

i.e.,  $x$  is a proximity point of  $T$ . Let us assume that  $p$  is another best proximity point of  $T$  then

$$d(p, Tp) = \text{dist}(A, B).$$



Since  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the second kind, we get

$$\begin{aligned} d(Tx, Tp) \leq & \alpha d(Tx, Tp) + \beta \frac{d(Tx, Tx)d(Tx, Tp) + d(Tp, Tp)d(Tx, Tp)}{d(Tx, Tp) + d(Tx, Tp)} \\ & + \gamma d(Tx, Tx) + \delta d(Tp, Tp) + \omega [d(Tx, Tp) + d(Tx, Tp)]. \end{aligned} \quad (2.3)$$

From (2.3), we get

$$d(Tx, Tp) \leq (\alpha + 2\omega)d(Tx, Tp),$$

Since  $\alpha + 2\omega < 1$ , above inequality implies that  $d(Tx, Tp) = 0$ , that is  $Tx = Tp$ . This completes the proof.  $\square$

Taking  $\beta = 0$  in Theorem 2.2, we get following.

**Corollary 2.3.** (see [17, Theorem 3.4]) *Let  $A$  and  $B$  be two nonempty, closed subsets of a complete metric space  $(X, d)$  such that  $A$  is approximatively compact with respect to  $B$ . Also suppose that  $A_0$  and  $B_0$  are non-empty and  $T : A \rightarrow B$  is a mapping satisfying the following conditions:*

- (i)  $T$  is a continuous generalized proximal contraction of the second kind.
- (ii)  $T(A_0) \subseteq B_0$ .

*Then there exists a unique element  $x \in A$  such that  $d(x, Tx) = \text{dist}(A, B)$ . Further, for any fixed  $x_0 \in A_0$ , there is a sequence  $\{x_n\}$ , defined by the relation  $d(x_{n+1}, Tx_n) = \text{dist}(A, B)$ , converges to the best proximity point  $x$  of the mapping  $T$ , and  $T$  is constant on the set of all best proximity points of  $T$ .*

Taking  $\beta = \gamma = \delta = 0$  in the Theorem 2.1, we get following result.

**Corollary 2.4.** *Let  $A$  and  $B$  be two nonempty, closed subsets of a complete metric space  $(X, d)$  such that  $A$  is approximatively compact with respect to  $B$ . Suppose that  $A_0$  and  $B_0$  are non-empty and  $T : A \rightarrow B$  is a mapping satisfying the following conditions:*

- (i) *There exists a nonnegative real number  $\alpha < 1$  such that, for all  $u_1, u_2, x_1, x_2 \in A$ , the condition*

$$d(u_1, Tx_1) = \text{dist}(A, B) \text{ and } d(u_2, Tx_2) = \text{dist}(A, B)$$

*imply that  $d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2)$ .*

- (ii)  $T(A_0) \subseteq B_0$ .

*Then there exists a unique element  $x \in A$  such that  $d(x, Tx) = \text{dist}(A, B)$ . Further, for any fixed  $x_0 \in A_0$ , there is a sequence  $\{x_n\}$ , defined by the relation  $d(x_{n+1}, Tx_n) = \text{dist}(A, B)$ , converges to the best proximity point  $x$  and  $T$  is constant on the set of all best proximity points of  $T$ .*

If we consider  $T$  to be  $\mathcal{K}$ -rational proximal contraction of the first kind as well as  $\mathcal{K}$ -rational proximal contraction of a second kind, then the assumption of approximately compactness of the domain or the co-domain of the mapping can be dropped.

**Theorem 2.3.** *Let  $A$  and  $B$  be nonempty, closed subsets of a complete metric space  $(X, d)$ . Suppose that  $A_0$  and  $B_0$  are non-empty and  $T : A \rightarrow B$  is a mapping satisfying the following conditions:*

- (i)  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the first as well as a  $\mathcal{K}$ -rational proximal contraction of a second kind;
- (ii)  $T(A_0) \subseteq B_0$ .

Then there exists a unique element  $x \in A$  such that

$$d(x, Tx) = \text{dist}(A, B),$$

and the sequence  $\{x_n\}$  converges to the best proximity point  $x$ , where  $x_0$  is any fixed element in  $A$  and  $d(x_{n+1}, Tx_n) = \text{dist}(A, B)$  for all  $n \geq 0$ .

**Proof.** Proceeding as in the point of the Theorem 2.1, we find a sequence  $\{x_n\}$  in  $A_0$  such that

$$d(x_{n+1}, Tx_n) = \text{dist}(A, B),$$

for all nonnegative integer  $n$ . As in Theorem 2.1, we can show that the sequence  $\{x_n\}$  is a Cauchy sequence, and hence converges to some  $x \in A$ . As in Theorem 2.2, it can be claimed that the sequence  $\{Tx_n\}$  is a Cauchy sequence and hence converges to some  $y \in B$ . So, we get

$$d(x, y) = \lim_{n \rightarrow \infty} d(x_{n+1}, Tx_n) = \text{dist}(A, B).$$

Therefore,  $x$  becomes an element of  $A_0$ . Since  $T(A_0) \subseteq B_0$ , we have  $d(u, Tx) = \text{dist}(A, B)$  for some  $u \in A$ .

Since  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the first kind, we obtain

$$\begin{aligned} d(u, x_{n+1}) &\leq \alpha d(x, x_n) + \beta \frac{d(x, u)d(x, x_{n+1}) + d(x_n, x_{n+1})d(x_n, u)}{d(x, x_{n+1}) + d(x_n, u)} \\ &\quad + \gamma d(x, u) + \delta d(x_n, x_{n+1}) + \omega [d(x, x_{n+1}) + d(x_n, u)]. \end{aligned}$$

Letting  $n \rightarrow \infty$ , we get

$$d(u, x) \leq (\gamma + \omega)d(u, x),$$

since  $(\gamma + \omega) < 1$ , it implies that  $d(u, x) = 0$ , i.e.  $u = x$ . Thus it follows that

$$d(x, Tx) = d(u, Tx) = \text{dist}(A, B),$$

hence  $x$  is a best proximity point of  $T$ . We can prove the uniqueness of the best proximity point of the mapping  $T$  as in Theorem 2.1.

This completes the proof.  $\square$

**Example 2.1.** Consider the complete metric space  $X = \mathbb{R}^2$  with the Euclidean metric. Suppose that

$$A = \{(x, y) : x \in [-2, -1] \cup [1, 2], y \in [-1, 1]\}$$

and

$$B = \{(0, y) : y \in [-1, 1]\}.$$

Now define a mapping  $T : A \rightarrow B$  as below:

$$T(x, y) = \begin{cases} (0, x + 1) & \text{if } x \leq -1, \\ (0, x - 1) & \text{if } x \geq 1, \end{cases}$$

for all  $(x, y) \in A$ .

Then

$$\begin{aligned} \text{dist}(A, B) &= 1, \\ A_0 &= \{(\pm 1, a) : a \in [-1, 1]\} \\ B_0 &= \{(0, 0)\}, \end{aligned}$$

also  $T(A_0) = \{(0, 0)\} \subseteq B_0$ .

Set  $u_1 = (1, -1)$ ,  $u_2 = (-1, 1)$ ,  $x_1 = (-2, 0)$  and  $x_2 = (2, 0)$ . Then, we have

$$Tu_1 = (0, 0), Tu_2 = (0, 0), Tx_1 = (0, -1) \text{ and } Tx_2 = (0, 1).$$

Further,

$$d(x_1, Tu_1) = 1, d(x_2, Tu_2) = 1$$

hence,

$$d(x_1, Tu_1) = d(x_2, Tu_2) = \text{dist}(A, B).$$

Set  $\alpha = \frac{1}{4}$ ,  $\beta = \frac{1}{8}$ ,  $\gamma = \frac{1}{16}$ ,  $\delta = \frac{1}{10}$ , and  $\omega = \frac{1}{20}$ , then

$$\alpha + \beta + \gamma + \delta + 2\omega = \frac{51}{80} < 1.$$

After a simple calculations, we can see that

$$d(u_1, u_2) \not\leq \alpha d(x_1, x_2) + \beta \frac{d(x_1, u_1)d(x_1, u_2) + d(x_2, u_2)d(x_2, u_1)}{d(x_1, u_2) + d(x_2, u_1)}.$$

Hence,  $T$  is not a  $\mathcal{K}$ -rational proximal contraction of the first kind.

Again,

$$\begin{aligned} d(Tx_1, Tx_2) &= 2, & d(Tu_1, Tu_2) &= 0, & d(Tx_1, Tu_1) &= 1, \\ d(Tx_2, Tu_2) &= 1, & d(Tx_1, Tu_2) &= 1, & d(Tx_2, Tu_1) &= 1, \end{aligned}$$

hence

$$d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2) + \beta \frac{d(Tx_1, Tu_1)d(Tx_1, Tu_2) + d(Tx_2, Tu_2)d(Tx_2, Tu_1)}{d(Tx_1, Tu_2) + d(Tx_2, Tu_1)},$$

i.e.,  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the second kind.

We now give an example of  $\mathcal{K}$ -rational proximal contraction which is not a generalized proximal contraction studied by Basha and Shehzad in [17].

**Example 2.2.** Let  $X = \mathbb{R}^2$  with the metric

$$d((x, y), (x', y')) = \max\{|x - x'|, |y - y'|\}$$

and let  $A = \{(x, 0) : 0 \leq x \leq 1\}$  and  $B = \{(x, 1) : 0 \leq x \leq 1\}$ . Then  $\text{dist}(A, B) = 1$ ,  $A_0 = A$  and  $B_0 = B$ . Define a mapping  $T : A \rightarrow B$  by  $T(x, 0) = (\frac{x}{2}, 1)$ . We can see that  $T(A_0) \subset B_0$ . Set  $\alpha = \frac{1}{60}$ ,  $\beta = \frac{1}{2}$ ,  $\gamma = \frac{1}{30}$ ,  $\delta = \frac{1}{30}$ , and  $\omega = \frac{1}{6}$  then  $\alpha + \beta + \delta + \gamma + 2 \cdot \omega = \frac{55}{60} < 1$ . Pick  $u_1 = (0, 0)$ ,  $u_2 = (\frac{1}{2}, 0)$ ,  $x_1 = (1, 0)$  and  $x_2 = (1, 0)$ , then

$$\begin{aligned} Tu_1 &= (0, 1), & Tu_2 &= \left(\frac{1}{4}, 1\right), & Tx_1 &= \left(\frac{1}{2}, 1\right) \\ Tx_2 &= \left(\frac{1}{2}, 1\right), & d(x_1, Tu_1) &= d(x_2, Tu_2) = \text{dist}(A, B) = 1. \end{aligned}$$

Also,

$$\begin{aligned} d(x_1, x_2) &= 0, & d(u_1, u_2) &= \frac{1}{2}, & d(x_1, u_1) &= 1, \\ d(x_2, u_2) &= \frac{1}{2}, & d(x_1, u_2) &= \frac{1}{2}, & d(x_2, u_1) &= 1. \end{aligned}$$

We can see that

$$d(u_1, u_2) \leq \alpha d(x_1, x_2) + \frac{\beta d(x_1, u_1)d(x_1, u_2) + d(x_2, u_2)d(x_2, u_1)}{d(x_1, u_2) + d(x_2, u_1)},$$

hence  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the first kind. But,

$$d(u_1, u_2) \not\leq \alpha d(x_1, x_2) + \gamma d(x_1, u_1) + \delta d(x_2, u_2) + \omega [d(x_1, u_2) + d(x_2, u_1)],$$

hence  $T$  is not a generalized proximal contraction of the first kind. Now turn to the proximal contraction of second kind.

$$\begin{aligned} d(Tx_1, Tx_2) &= 0, & d(Tu_1, Tu_2) &= \frac{1}{4}, & d(Tx_1, Tu_1) &= \frac{1}{2}, \\ d(Tx_2, Tu_2) &= \frac{1}{4}, & d(Tx_1, Tu_2) &= \frac{1}{4}, & d(Tx_2, Tu_1) &= \frac{1}{2}. \end{aligned}$$

Then, we have

$$d(Tu_1, Tu_2) \leq \alpha d(Tx_1, Tx_2) + \frac{\beta d(Tx_1, Tu_1)d(Tx_1, Tu_2) + d(Tx_2, Tu_2)d(Tx_2, Tu_1)}{d(Tx_1, Tu_2) + d(Tx_2, Tu_1)}.$$

Hence  $T$  is a  $\mathcal{K}$ -rational proximal contraction of the second kind, also

$$d(Tu_1, Tu_2) \not\leq \alpha d(Tx_1, Tx_2) + \gamma d(Tx_1, Tu_1) + \delta d(Tx_2, Tu_2) + \omega [d(Tx_1, Tu_2) + d(Tx_2, Tu_1)].$$

Hence  $T$  is not a generalized proximal contraction of the second kind.

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