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Note on Gagliardo's theorem "$\operatorname{tr} W^{1,1} = L^1$"

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Abstract - We present a short proof of the fact, originally established by Gagliardo, that every function $f \in L^1(\mathbb{R}^n)$ is the trace of a function $u \in W^{1,1}(\mathbb{R}^n \times (0,\infty))$.

Key words and phrases : Sobolev space, trace, Gagliardo.

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1. A short proof

The aim of this note is to give the "proof from the book" of the fact that, given a function $f \in L^1(\mathbb{R}^n)$, there exists some $u \in W^{1,1}(\mathbb{R}^n \times (0,\infty))$ whose trace is f. This classical result goes back to Gagliardo [3]. The impatient reader can go directly to the proof of (1.2), which is the heart of the note.

We start by recalling some definitions and basic facts. Let $\Omega = \mathbb{R}^n \times (0,\infty)$. For $1 \le p < \infty$, we consider the Sobolev space

$$W^{1,p}(\Omega) = \left\{ u \in L^p(\Omega);\ \frac{\partial u}{\partial x_j} \in L^p(\Omega),\ j = 1, \ldots, n+1 \right\},$$

equipped with the standard norm

$$\|u\|_{W^{1,p}} = \|u\|_{L^p} + \sum_{j=1}^{n+1} \left\| \frac{\partial u}{\partial x_j} \right\|_{L^p}.$$

Here, the partial derivatives are generalized derivatives or, equivalently, derivatives in the distributions sense. Before Gagliardo's work [3], the state of the art concerning existence of traces was the following.

Fact 1 $C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ is dense in $W^{1,p}(\Omega)$; see e.g. [2, Corollary 9.8].

Fact 2 For $u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$, set $f(x) = u(x,0)$, $x \in \mathbb{R}^n$. Then we have the estimate

$$\|f\|_{L^p} \le \|u\|_{W^{1,p}};$$

see e.g. [2, Lemma 9.9].

Fact 3 By the first two facts, the mapping $u \mapsto f$ admits a unique linear continuous extension, the *trace operator* tr, from $W^{1,p}(\Omega)$ into $L^p(\mathbb{R}^n)$.

Fact 4 When $p = 2$, Aronszajn [1] discovered that the trace operator is not onto, and that its image is precisely the set

$$\left\{ f \in L^2(\mathbb{R}^n);\ \int_{\mathbb{R}^n} |\xi| \left| \widehat{f}(\xi) \right|^2 d\xi < \infty \right\};$$

here, $\widehat{\ }$ stands for the Fourier transform. The arguments in [1] rely on Fourier transform methods, and are difficult to apply to the case where $p \neq 2$.

Gagliardo achieved the more complicated task of characterizing $\operatorname{tr} W^{1,p}(\Omega)$ for $p \neq 2$. He obtained two results with different flavors and proofs.

Theorem 1.1 (Gagliardo's theorem #1) *For $1 < p < \infty$, we have*

$$\operatorname{tr} W^{1,p}(\Omega) = W^{1-1/p,p}(\mathbb{R}^n). \tag{1.1}$$

Here,

$$W^{1-1/p,p}(\mathbb{R}^n) = \left\{ f \in L^p(\mathbb{R}^n);\ \underbrace{\iint_{\mathbb{R}^n \times \mathbb{R}^n} \frac{|f(x) - f(y)|^p}{|x-y|^{n+p-1}}\, dxdy}_{I(f)} < \infty \right\},$$

endowed with the natural norm

$$\|f\|_{W^{1-1/p,p}} = \|f\|_{L^p} + [I(f)]^{1/p}.$$

For a proof of this result using only elementary arguments, see e.g. [4, Chapter 11]. We just mention here the three steps of the proof.

Step 1 For $u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$ and with $f(x) = u(x,0)$, one proves the "direct" estimate

$$I(f) \leq C\|u\|^p_{W^{1,p}}.$$

Combined with Facts 1 and 2, this leads to the continuity of the trace operator from $W^{1,p}(\Omega)$ into $W^{1-1/p,p}(\mathbb{R}^n)$.

Step 2 One proves that $C^\infty(\mathbb{R}^n) \cap W^{1-1/p,p}(\mathbb{R}^n)$ is dense in $W^{1-1/p,p}(\mathbb{R}^n)$.

Step 3 One proves the existence of a linear map

$$C^\infty(\mathbb{R}^n) \cap W^{1-1/p,p}(\mathbb{R}^n) \ni f \mapsto u \in C^\infty(\overline{\Omega}) \cap W^{1,p}(\Omega)$$

such that $u(x,0) = f(x)$ and we have the "inverse" estimate

$$\|u\|_{W^{1,p}} \leq C\|f\|_{W^{1-1/p,p}}.$$

By Steps 2 and 3 and by Fact 3, each function $f \in W^{1-1/p,p}(\mathbb{R}^n)$ is the trace of some $u \in W^{1,p}(\Omega)$, and in addition we may choose u such that the mapping $f \mapsto u$ is linear continuous.

With moderate work, one may prove that the function

$$f(x) = \begin{cases} 1/(|x|^{n/p}|\ln x|), & \text{if } |x| < 1/2 \\ 0, & \text{if } |x| \geq 1/2 \end{cases}$$

is in $L^p(\mathbb{R}^n)$ but not in $W^{1-1/p,p}(\mathbb{R}^n)$. Thus, for $1 < p < \infty$, the image of the trace operator is a strict subspace of $L^p(\mathbb{R}^n)$.

This contrasts with the next theorem.

Theorem 1.2 (Gagliardo's theorem #2) *We have* $\operatorname{tr} W^{1,1}(\Omega) = L^1(\mathbb{R}^n)$.

Here is our proof of this result.

Proof. In view of Fact 3, we have to prove that for each $f \in L^1(\mathbb{R}^n)$ there exists some $u \in W^{1,1}(\Omega)$ such that $\operatorname{tr} u = f$. We claim that for every $g \in C_c^\infty(\mathbb{R}^n)$, there exists some $v \in C_c^\infty(\overline{\Omega})$ such that $v(x,0) = g(x)$ and

$$\|v\|_{W^{1,1}} \leq C\|g\|_{L^1}. \tag{1.2}$$

Indeed, fix some $\zeta \in C_c^\infty([0,\infty))$ such that $\zeta(0) = 1$, and let

$$v(x, x_{n+1}) = v^\delta(x, x_{n+1}) = g(x)\zeta(x_{n+1}/\delta), \quad \text{with } \delta > 0.$$

By straighforward calculations, we have

$$\lim_{\delta \to 0} \|v^\delta\|_{L^1} = 0, \tag{1.3}$$

$$\lim_{\delta \to 0} \left\| \frac{\partial v^\delta}{\partial x_j} \right\|_{L^1} = 0, \ j = 1, \dots, n, \tag{1.4}$$

and

$$\left\| \frac{\partial v^\delta}{\partial x_{n+1}} \right\|_{L^1} = C\|g\|_{L^1}. \tag{1.5}$$

By (1.3)–(1.5), for sufficiently small δ, v satisfies (1.2).

We complete the proof of the theorem as follows. Let $f \in L^1(\mathbb{R}^n)$. Then there exists a sequence $(f_j) \subset C_c^\infty(R^n)$ such that $\sum_j f_j = f$ in L^1 and

$$\sum_j \|f_j\|_{L^1} \leq 2\|f\|_{L^1}.^{1} \tag{1.6}$$

For each j, consider (in virtue of (1.2)) $v_j \in C_c^\infty(\overline{\Omega})$ satisfy $v_j(x,0) = f_j(x)$ and $\|v_j\|_{W^{1,1}} \leq C\|f_j\|_{L^1}$. Set $v = \sum_j v_j$, so that (by (1.2) and (1.6)) we have $v \in W^{1,1}(\Omega)$ and $\|v\|_{W^{1,1}} \leq C\|f\|_{L^1}$. By Fact 3, we have $\operatorname{tr} v = f$. □

[1] This is a special case of the following trivial fact: if $(X, \| \, \|)$ is a normed space and if Y is a dense linear subspace of X, then for each $x \in X$ there exists a sequence $(y_j) \subset Y$ such that $\sum_j y_j = x$ in X and $\sum_j \|y_j\| \leq 2\|x\|$.

2. Other stories

If we examine the proofs of Theorems 1.1 and 1.2, we see that, in the proof of Theorem 1.1, the map $f \mapsto u$ is linear, while in the case of Theorem 1.2 this is not the case.[2] This is not an artefact of the proof. Peetre [6] proved that there was no linear continuous map

$$L^1(\mathbb{R}^n) \ni f \mapsto u \in W^{1,1}(\Omega)$$

such that $\operatorname{tr} u = f$. For a relatively simple proof of this result, see [7, Section 5].

Another unexpected fact arises when we consider higher order spaces. If we set

$$W^{2,p}(\Omega) = \left\{ u \in L^p(\Omega);\ \frac{\partial u}{\partial x_j} \in W^{1,p}(\Omega),\ j = 1, \dots, n+1 \right\},$$

then (as suggested by Theorem 1.1) for $1 < p < \infty$ we have

$$\operatorname{tr} W^{2,p}(\Omega) = \left\{ f \in L^p(\mathbb{R}^n);\ \frac{\partial u}{\partial x_j} \in W^{1-1/p,p}(\mathbb{R}^n),\ j = 1, \dots, n \right\};$$

see e.g. [4, Chapter 11].

On the other hand, Theorem 1.2 suggests that

$$\operatorname{tr} W^{2,1}(\Omega) = W^{1,1}(\mathbb{R}^n).$$

But this is not true! Uspenskiĭ found the right answer.[3] His discovery was a significant achievement of another type of trace theory, the one of the weighted Sobolev spaces. For a modern treatment of this theory, see [5].

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[2] Indeed, in (1.2) the map v depends not only on g, but also on δ, which need not be the same for every g.

[3] The trace of $W^{2,1}(\Omega)$ is the Besov space $B^1_{1,1}(\mathbb{R}^n)$. But this goes beyond the scope of this note.

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