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Optimality conditions and duality for multiobjective semi-infinite programming problems with generalized (C, α, ρ, d)-convexity

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Abstract - This paper deals with a nonlinear multiobjective semi-infinite programming problem involving generalized (C, α, ρ, d)-convex functions. We obtain sufficient optimality conditions and formulate the Mond-Weir-type dual model for the nonlinear multiobjective semi-infinite programming problem. We also establish weak, strong and strict converse duality theorems relating the problem and the dual problem.

Key words and phrases : multiobjective programming, semi-infinite programming, generalized convexity, optimality, duality.

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1. Introduction

Semi-infinite means that we have finitely many decision variables $x = (x_1, x_2, \ldots, x_n)$ and the feasible set is defined by infinitely many inequality constraints. Due to a growing number of theoretical and practical applications, semi-infinite programming has recently become one of the most substantial research areas in applied mathematics and operations research. There are many applications of semi-infinite programming in different fields such as chebyshev approximation, robotics, mathematical physics, engineering design, optimal control, transportation problems, fuzzy sets, robust optimization etc. For more details on semi-infinite programming, we refer to the survey papers [8-10, 12, 17, 22]. We also refer to the recent works of Canovas et al. [2, 3].

Shapiro (see [21]) gave results on Lagrangian duality for convex semi-infinite programming problems. Kanzi and Nobakhtian (see [13]) have established necessary and sufficient optimality conditions for nonsmooth semi-infinite programming problem under various constraints qualifications. Later, Mishra et al. (see [19]) formulated the Wolfe and Mond-Weir-type dual models and establish duality theorems for the nonsmooth semi-infinite programming problem discussed in [13].

The concept of (F, ρ)-convexity was introduced by Preda [20] as an extension of F-convexity (see [11]) and ρ-convexity (see [23]) to obtain some duality results. Aghezzaf and Hachimi (see [1]) introduced some new classes of generalized (F, ρ)-convexity for vector-valued functions and establish sufficient optimality conditions and duality results for a nonlinear multiobjective programming problem. Liang et al. introduced in [14] the concept of (F, α, ρ, d)-convexity which contains the class of (F, ρ)-convex functions and obtained some optimality conditions and duality results for nonlinear fractional programming problems. Later, Liang et al. (see [15]) extended the results of [14] for a class of multiobjective fractional programming problems. Yuan et al. (see [24]) defined (C, α, ρ, d)-convexity, which is a generalization of (F, α, ρ, d)-convexity and establish optimality conditions and duality results for nondifferentiable minimax fractional programming problems involving the generalized convex functions. Chinchuluun et al. (see [4]) considered optimality conditions and duality results for some multiobjective programming, multiobjective fractional programming and multiobjective variational programming problems with (C, α, ρ, d) type-I functions, see also [5-7, 25, 26]. Recently, Long (see [16]) has obtained optimality conditions and duality results for nondifferentiable multiobjective fractional programming problems using (C, α, ρ, d)-convexity.

In this paper, we obtain sufficient optimality conditions and formulate the Mond-Weir-type dual model for the nonlinear multiobjective semi-infinite programming problem. We establish weak, strong and strict converse duality theorems relating the primal problem and the Mond-Weir-type-dual problem under (generalized) (C, α, ρ, d)-convexity and regularity conditions.

2. Preliminaries

Throughout this paper, let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^n_+$ be the non-negative orthant of $\mathbb{R}^n$. We adopt the following conventions for vectors in the Euclidean space $\mathbb{R}^n$

$$\begin{aligned} x \leqq y &\Leftrightarrow x_i \leqq y_i,\ i = 1, \ldots, n \\ x \leq y &\Leftrightarrow x_i \leqq y_i,\ i = 1, \ldots, n \text{ and } x \neq y \\ x < y &\Leftrightarrow x_i < y_i, i = 1, \ldots, n. \end{aligned}$$

We consider the following nonlinear multiobjective semi-infinite programming problem:

$$\text{(P)} \qquad \begin{aligned} &\text{Min } f(x) \\ &\text{Subject to } g_j(x) \leqq 0, j \in J, \end{aligned}$$

where J is an index set which is possibly infinite, $f(x) = (f_1(x), \ldots, f_p(x))$, $f_i(i \in \{1, 2, \ldots, p\})$ and $g_j(j \in J)$ are differentiable functions from a nonempty open set $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$. We consider the set of feasible solutions for

(P) as follows:

$$S = \{x \in X \mid g_j(x) \leqq 0, j \in J\}.$$

For $x_0 \in S$ the index set of active constraints is denoted by $I = \{j \in J \mid g_j(x_0) = 0\}$. Let us assume that $\alpha : X \times X \to \mathbb{R}_+ \setminus \{0\}$, $\rho \in \mathbb{R}$ and $d : X \times X \to \mathbb{R}_+$ satisfies $d(x, x_0) = 0$ if and only if $x = x_0$. Let $C : X \times X \times \mathbb{R}^n \to \mathbb{R}$ be a function which satisfies $C_{(x,x_0)}(0) = 0$ for any $(x, x_0) \in X \times X$.

The following definitions are taken from [4].

Definition 2.1. *A function $C : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is said to be convex on $\mathbb{R}^n$ if and only if for any fixed $(x, x_0) \in X \times X$ and for any $y_1, y_2 \in \mathbb{R}^n$,*

$$C_{(x,x_0)}(\lambda y_1 + (1-\lambda)y_2) \leqq \lambda C_{(x,x_0)}(y_1) + (1-\lambda)C_{(x,x_0)}(y_2), \quad \forall \lambda \in (0,1).$$

Definition 2.2. *(i) A differentiable function $f_i : X \to \mathbb{R}$ is said to be (strictly) $(C, \alpha_i, \rho_i, d_i)$-convex at $x_0 \in X$ if and only if for any $x \in X$,*

$$\frac{f_i(x) - f_i(x_0)}{\alpha_i(x, x_0)} \geqq (>) C_{(x,x_0)}(\nabla f_i(x_0)) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)}.$$

(ii) The vector-valued function $f : X \to \mathbb{R}^p$ is (C, α, ρ, d)-convex at x_0 if each of its components f_i is $(C, \alpha_i, \rho_i, d_i)$-convex at x_0.

(iii) The function f is said to be (C, α, ρ, d)-convex on X if and only if it is (C, α, ρ, d)-convex at every point in X. In particular, f is said to be strongly (C, α, ρ, d)-convex on X if and only if $\rho > 0$.

The following convention will be used: if f is a p-dimensional vector-valued function, then the vector $\left(C_{(x,x_0)}(\nabla f_1(x_0)), \ldots, C_{(x,x_0)}(\nabla f_p(x_0))\right)$ will be denoted by $C_{(x,x_0)}(\nabla f(x_0))$.

Example 2.1. (see [16]) Let $X = \left\{x \mid \frac{\pi}{4} \leq x \leq \frac{\pi}{2}\right\}$, $\rho = -1$, $\alpha(x, x_0) = 1$, $d(x, x_0) = \sqrt{(x - x_0)^2}$ and $C_{(x,x_0)}(a) = a^2(x - x_0)$ for any $(x, x_0) \in X \times X$. Let $f(x) = \sin^2 x$. Then, it is easy to verify that $f(x)$ is (C, α, ρ, d)-convex at $x_0 = \frac{\pi}{4}$.

Definition 2.3. *(i) A differentiable function $f_i : X \to \mathbb{R}$ is said to be (strictly) $(C, \alpha_i, \rho_i, d_i)$-pseudo-convex at $x_0 \in X$ if and only if for any $x \in X$,*

$$f_i(x) < (\leqq) f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0)) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)} < 0.$$

(ii) The vector-valued function $f : X \to \mathbb{R}^p$ is (C, α, ρ, d)-pseudo-convex at x_0 if each of its components f_i is pseudo-convex at x_0.

Definition 2.4. *(i) A differentiable function $f_i : X \to \mathbb{R}$ is said to be weak strictly $(C, \alpha_i, \rho_i, d_i)$-pseudo-convex at $x_0 \in X$ if and only if for any $x \in X$,*

$$f_i(x) \leq f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0)) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)} < 0.$$

(ii) The vector-valued function $f : X \to \mathbb{R}^p$ is weak strictly (C, α, ρ, d)-pseudo-convex at x_0 if each of its components f_i is weak strictly $(C, \alpha_i, \rho_i, d_i)$-pseudo-convex at x_0.

Definition 2.5. *(i) A differentiable function $f_i : X \to \mathbb{R}$ is said to be strong $(C, \alpha_i, \rho_i, d_i)$-pseudo-convex at $x_0 \in X$ if and only if for any $x \in X$,*

$$f_i(x) \leq f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0)) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)} \leq 0.$$

(ii) The vector-valued function $f : X \to \mathbb{R}^p$ is strong (C, α, ρ, d)-pseudo-convex at x_0 if each of its components f_i is strong pseudo-convex at x_0.

Definition 2.6. *(i) A differentiable function $f_i : X \to \mathbb{R}$ is said to be $(C, \alpha_i, \rho_i, d_i)$-quasi-convex at $x_0 \in X$ if and only if for any $x \in X$,*

$$f_i(x) \leqq f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0)) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)} \leqq 0.$$

(ii) The vector-valued function $f : X \to \mathbb{R}^p$ is (C, α, ρ, d)-quasi-convex at x_0 if each of its components f_i is $(C, \alpha_i, \rho_i, d_i)$-quasi-convex at x_0.

Definition 2.7. *(i) A differentiable function $f_i : X \to \mathbb{R}$ is said to be weak $(C, \alpha_i, \rho_i, d_i)$-quasi-convex at $x_0 \in X$ if and only if for any $x \in X$,*

$$f_i(x) \leq f_i(x_0) \Rightarrow C_{(x,x_0)}(\nabla f_i(x_0)) + \rho_i \frac{d_i(x, x_0)}{\alpha_i(x, x_0)} \leqq 0.$$

(ii) The vector-valued function $f : X \to \mathbb{R}^p$ is weak (C, α, ρ, d)-quasi-convex at x_0 if each of its components f_i is weak $(C, \alpha_i, \rho_i, d_i)$-quasi-convex at x_0.

Definition 2.8. *A feasible point $x_0 \in S$ is said to be an efficient solution for problem* (P) *if and only if there exists no point $x \in S$ such that $f(x) \leq f(x_0)$.*

Definition 2.9. *A feasible point $x_0 \in S$ is said to be a weak efficient solution for problem* (P) *if and only if there exists no point $x \in S$ such that $f(x) < f(x_0)$.*

We note that every efficient solution is weak efficient, however, the converse is not always true.

3. Optimality conditions

We state the following necessary optimality conditions for the nonlinear multiobjective semi-infinite programming problem (P):

Theorem 3.1. (Necessary optimality conditions) *Let x_0 be an efficient solution for* (P) *and $I(x_0) \neq \emptyset$. If* (P) *satisfies a suitable constraint qualification (see* [18]*) at x_0 then there exist $\bar{u} \in \mathbb{R}^p$, $\bar{v} = (\bar{v}_j)_{j\in I}$, such that*

$$\begin{aligned}&\bar{y}^T \nabla f(x_0) + \bar{v}^T \nabla g(x_0) = 0,\\ &\bar{v}^T g(x_0) = 0,\\ &\bar{u} \geqq 0,\ \bar{v} \geqq 0 \text{ and } v_j \neq 0 \text{ for finitely many } j \in I.\end{aligned}$$

Theorem 3.2. *Assume that there exist a feasible solution x_0 for* (P) *and vectors $\bar{u} \in \mathbb{R}^p$ and $\bar{v} = (\bar{v}_j)_{j\in J}$, such that*

$$\begin{aligned}&\bar{u}^T \nabla f(x_0) + \bar{v}^T \nabla g(x_0) = 0, \qquad (3.1)\\ &\bar{v}^T g(x_0) = 0,\\ &\bar{u} > 0,\ \bar{v} \geqq 0 \text{ and } v_j \neq 0 \text{ for finitely many } j \in I.\end{aligned}$$

Let f be strong $(C, \alpha^1, \rho^1, d^1)$-pseudo-convex at x_0 and g_I be $(C, \alpha^2, \rho^2, d^2)$-quasi-convex at x_0 with

$$\sum_{i=1}^{p} \bar{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j\in I} \bar{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqq 0. \qquad (3.2)$$

Then x_0 is an efficient solution for (P)*.*

Proof. Suppose that x_0 is not an efficient solution for (P). Then there exists a feasible solution $x \in S$ such that

$$f(x) \leq f(x_0).$$

As $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is strong $(C, \alpha^1, \rho^1, d^1)$-pseudo-convex at x_0 and g_I is $(C, \alpha^2, \rho^2, d^2)$-quasi-convex at x_0, it follows from the definitions that

$$C_{(x,x_0)}(\nabla f(x_0)) + \rho^1 \frac{d^1(x, x_0)}{\alpha^1(x, x_0)} \leq 0$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0)) + \rho_I^2 \frac{d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} \leqq 0.$$

Using the facts that $\bar{u} > 0$, $\bar{v} \geqq 0$ and C is convex, from the above inequalities, we can conclude that

$$C_{(x,x_0)}\left(\frac{1}{\tau}\bar{u}^T\nabla f(x_0)+\frac{1}{\tau}\bar{v}_I^T\nabla g_I(x_0)\right)+\frac{1}{\tau}\bar{u}^T\rho^1\frac{d^1(x,x_0)}{\alpha^1(x,x_0)}+\frac{1}{\tau}\bar{v}_I^T\rho_I^2\frac{d_I^2(x,x_0)}{\alpha_I^2(x,x_0)}<0,$$

where $\tau = \sum_{i=1}^p \bar{u}_i + \sum_{j\in I} \bar{v}_j$.

Now from equation (3.1) and since $C_{(x,x_0)} = 0$, this implies,

$$\sum_{i=1}^p \bar{u}_i\rho_i^1\frac{d_i^1(x,x_0)}{\alpha_i^1(x,x_0)} + \sum_{j\in I}\bar{v}_j\rho_j^2\frac{d_j^2(x,x_0)}{\alpha_j^2(x,x_0)} < 0,$$

which contradicts (3.2). Hence, x_0 is an efficient solution for (P). □

Example 3.1. Consider the following problem:

$$(\mathrm{P}_1) \qquad \begin{aligned} &\min\ f(x)\\ &\text{Subject to } g_j(x) \leqq 0, j \in J,\\ &x \in \mathbb{R}, \end{aligned}$$

where $F : X(=\mathbb{R}) \to \mathbb{R}^2$ and $g_j : X(=\mathbb{R}) \to \mathbb{R}$, $j \in J$ are the functions defined as:

$$f(x) = (f_1(x), f_2(x)) = (x^2 - 2x, x^3 - x^2)$$

and

$$\begin{aligned} g_1(x) &= x^2(x-2),\\ g_2(x) &= x^3 + x,\\ g_k(x) &= x + \frac{1}{k}, \quad k = 3, 4, \ldots. \end{aligned}$$

Again, let $C : \mathbb{R}\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ be a function defined by $C_{(x,x_0)}(a) = a(x-x_0)$. The set of feasible solutions for the problem (P1) is

$$S = \{x \in \mathbb{R} \mid g_j(x) \leqq 0\} = \{x \in \mathbb{R} \mid x \leqq 0\}.$$

It is easy to verify that $f_i\,(i = 1, 2)$ are strong $(C, \alpha_i, \rho_i, d_i)$-pseudo-convex at $x_0 = 0$ with $\rho_i = 0$, $\alpha(x, x_0) = 1$ and $d_i(x, x_0) = 1$. Also, $g_j (j \in J)$ are $(C, \alpha_j, \rho_j, d_j)$-quasi-convex at $x_0 = 0$, with $\rho_j = 0$, $\alpha_j(x, x_0) = 1$ and $d_j(x, x_0) = 1$.

Clearly, $x_0 = 0$ is a feasible solution for (P_1) and it satisfies the assumptions of Theorem 3.2, where $\bar{u} = \left(\frac{1}{2}, 1\right)$, $\bar{v} = (1, 1, 0, 0, \ldots, 0, \ldots)$. For $x_0 = 0$, the index set of active constraints for (P_1) is $I = \{j \in J \mid g_j(x_0) = 0\} = \{1, 2\}$.

We observe that there exists no point $x \in S$ such that $f(x) \leq f(x_0)$. Hence, $x_0 = 0$ is an efficient solution for (P_1) (Figure 1).

In Theorem 3.2, we require that $\bar{u} > 0$. In order to relax this condition, we need to put some other generalized convexity conditions on f and g_I.

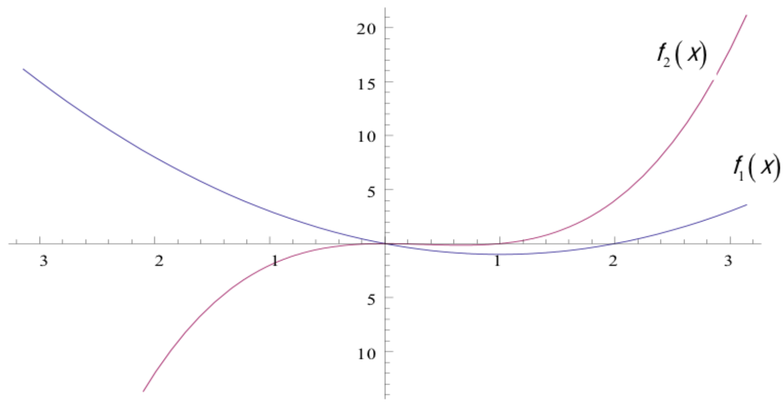

Figure 1. Plot for the objective function $f(x) = (f_1(x), f_2(x))$.

Theorem 3.3. *Assume that there exist a feasible solution x_0 for (P) and vectors $\bar{u} \in \mathbb{R}^p$ and $\bar{v} = (\bar{v}_j)_{j\in J}, \bar{v}_j \in \mathbb{R}$ such that*

$$\begin{aligned}
&\bar{u}^T \nabla f(x_0) + \bar{v}^T \nabla g(x_0) = 0,\\
&\bar{v}^T g(x_0) = 0,\\
&\bar{u} \geq 0m\ \bar{v} \geqq 0 \text{ and } v_j \neq 0 \text{ for finitely many } j \in I.
\end{aligned}$$

Let f be weak strictly $(C, \alpha^1, \rho^1, d^1)$-pseudo-convex at x_0 and g_I be $(C, \alpha^2, \rho^2, d^2)$-quasi-convex at x_0 with

$$\sum_{i=1}^{p} \bar{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j\in I} \bar{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqq 0.$$

Then x_0 is an efficient solution for (P).

Proof. Suppose that x_0 is not an efficient solution for (P). Then there exists a feasible solution $x \in S$ such that

$$f(x) \leq f(x_0).$$

As $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is weak strictly $(C, \alpha^1, \rho^1, d^1)$-pseudo-convex at x_0 and g_I is $(C, \alpha^2, \rho^2, d^2)$-quasi-convex at x_0, it follows from the definitions that

$$C_{(x,x_0}(\nabla f(x_0)) + \rho^1 \frac{d^1(x, x_0)}{\alpha^1(x, x_0)} < 0$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0)) + \rho_I^2 \frac{d_I^2(x,x_0)}{\alpha_I^2(x,x_0)} \leqq 0.$$

The rest of the proof is similar to that of Theorem 3.2. □

Theorem 3.4. *Assume that there exist a feasible solution x_0 for* (P) *and vectors $\bar{u} \in \mathbb{R}^p$ and $\bar{v} = (\bar{v}_j)_{j\in J}, \bar{v}_j \in \mathbb{R}$ such that*

$$\begin{aligned}
&\bar{u}^T \nabla f(x_0) + \bar{v}^T \nabla g(x_0) = 0,\\
&\bar{v}^T g(x_0) = 0,\\
&\bar{u} \geq 0,\ \bar{v} \geq 0 \text{ and } \bar{v}_j \neq 0 \text{ for finitely many } j \in I.
\end{aligned}$$

Let f be weak (C,α^1,ρ^1,d^1)-quasi-convex at x_0 and g_I be strictly (C,α^2,ρ^2,d^2)-pseudo-convex at x_0 with

$$\sum_{i=1}^{p} \bar{u}_i \rho_i^1 \frac{d_i^1(x,x_0)}{\alpha_i^1(x,x_0)} + \sum_{j\in I} \bar{v}_j \rho_j^2 \frac{d_j^2(x,x_0)}{\alpha_j^2(x,x_0)} \geqq 0.$$

Then x_0 is an efficient solution for (P).

Proof. Suppose that x_0 is not an efficient solution for (P). Then there exists a feasible solution $x \in S$ such that

$$f(x) \leq f(x_0).$$

As $g_I(x_0)$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is weak (C,α^1,ρ^1,d^1)-quasi-convex at x_0 and g_I is strictly (C,α^2,ρ^2,d^2)-pseudo-convex at x_0, it follows from the definitions that

$$C_{(x,x_0)}(\nabla f(x_0)) + \rho^1 \frac{d^1(x,x_0)}{\alpha^1(x,x_0)} \leqq 0$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0)) + \rho_I^2 \frac{d_I^2(x,x_0)}{\alpha_I^2(x,x_0)} < 0.$$

The rest of the proof is similar to that of Theorem 3.2. □

Since an efficient solution is also weak efficient solution, Theorem 3.2, Theorem 3.3 and Theorem 3.4 are still valid for weak efficiency. However, we can weaken the convexity assumptions for weak efficient solutions.

Theorem 3.5. *Assume that there exist a feasible solution x_0 for* (P) *and vectors $\bar{u} \in \mathbb{R}^p$ and $\bar{v} = (\bar{v}_j)_{j\in J}, \bar{v}_j \in \mathbb{R}$ such that*

$$\bar{u}^T \nabla f(x_0) + \bar{v}^T \nabla g(x_0) = 0,$$
$$\bar{v}^T g(x_0) = 0,$$
$$\bar{u} \geq 0,\ \bar{v} \geq 0 \text{ and } v_j \neq 0 \text{ for finitely many } j \in I.$$

Let f be $(C, \alpha^1, \rho^1, d^1)$-pseudo-convex at x_0 and g_I be $(C, \alpha^2, \rho^2, d^2)$-quasi-convex at x_0 with

$$\sum_{i=1}^{p} \bar{u}_i \rho_i^1 \frac{d_i^1(x, x_0)}{\alpha_i^1(x, x_0)} + \sum_{j\in I} \bar{v}_j \rho_j^2 \frac{d_j^2(x, x_0)}{\alpha_j^2(x, x_0)} \geqq 0.$$

Then x_0 is a weak efficient solution for (P)*.*

Proof. Suppose that x_0 is not a weak efficient solution for (P). Then there exists a feasible solution $x \in S$ such that

$$f(x) < f(x_0).$$

As $g_I(x_0) = 0$, hence

$$g_I(x) \leqq g_I(x_0).$$

Since f is $(C, \alpha^1, \rho^1, d^1)$-pseudo-convex at x_0 and g_I is $(C, \alpha^2, \rho^2, d^2)$-quasi-convex at x_0, it follows from the definitions that

$$C_{(x,x_0)}(\nabla f(x_0)) + \rho^1 \frac{d^1(x, x_0)}{\alpha^1(x, x_0)} < 0$$

and

$$C_{(x,x_0)}(\nabla g_I(x_0)) + \rho_I^2 \frac{d_I^2(x, x_0)}{\alpha_I^2(x, x_0)} \leqq 0.$$

The rest of the proof is similar to that of Theorem 3.2. □

4. Duality

In this section we formulate the following Mond-Weir-type dual problem (MWD) for the nonlinear multiobjective semi-infinite programming problem (P) and establish weak, strong and strict converse duality theorems.

$$\text{(MWD)} \quad \text{Max} f(y) = (f_1(y), f_2(y), \ldots, f_p(y))$$

$$\text{Subject to } \sum_{i=1}^{p} u_i \nabla f_i(y) + \sum_{j\in J} v_j \nabla g_j(y) = 0, \tag{4.1}$$

$\bar{v}^T g(y) \geqq 0$, $v = (v_j)_{j\in J}$, $v_j \in \mathbb{R}_+$ and $v_j \neq 0$ for finitely many $j \in J$

$$\sum_{i=1}^{p} u_i = 1,\ u_i > 0 (i = 1, 2, \ldots, p),\ y \in X \subseteq \mathbb{R}^n,$$

where f_i, g_j are differentiable functions from a nonempty open set $X \subseteq \mathbb{R}^n$ to $\mathbb{R}$.

Theorem 4.1. (Weak duality) *Let x_0 be a feasible solution for* (P) *and (y_0, u, v) be a feasible solution for* (MWD), *and let any of the following hold:*

(a) *f be $(C, \alpha^1, \rho^1, d^1)$-convex at y_0, g be $(C, \alpha^2, \rho^2, d^2)$-convex at y_0 and*

$$\sum_{i=1}^{p} u_i \rho_i^1 \frac{d_i^1(x_0, y_0)}{\alpha_i^1(x_0, y_0)} + \sum_{j \in J} v_j \rho_j^2 \frac{d_j^2(x_0, y_0)}{\alpha_j^2(x_0, y_0)} \geqq 0. \tag{4.2}$$

(b) *f be strong $(C, \alpha^1, \rho^1, d^1)$-pseudo-convex at y_0 and $v^T g$ be $(C, \alpha^2, \rho^2, d^2)$-quasi-convex at y_0 and*

$$\sum_{i=1}^{p} u_i \rho_i^1 \frac{d_i^1(x_0, y_0)}{\alpha_i^1(x_0, y_0)} + \rho^2 \frac{d^2(x_0, y_0)}{\alpha^2(x_0, y_0)} \geqq 0. \tag{4.3}$$

(c) *$u^T f$ be $(C, \alpha^1, \rho^1, d^1)$-pseudo-convex at y_0 and $v^T g$ be $(C, \alpha^2, \rho^2, d^2)$-quasi-convex at y_0, and*

$$\rho^1 \frac{d^1(x_0, y_0)}{\alpha^1(x_0, y_0)} + \rho^2 \frac{d^2(x_0, y_0)}{\alpha^2(x_0, y_0)} \geqq 0. \tag{4.4}$$

Then the following cannot hold,

$$f(x_0) \leq f(y_0). \tag{4.5}$$

Proof. (a) Let x_0 and (y_0, u, v) be the feasible solutions for (P) and (MWD), respectively. It follows that

$$\sum_{j \in J} v_j g_j(x_0) \leqq 0 \leqq \sum_{j \in J} v_j g_j(y_0).$$

By the $(C, \alpha^2, \rho^2, d^2)$-convexity of g, we get

$$0 \geqq \sum_{j \in J} v_j \frac{g_j(x_0) - g_j(y_0)}{\alpha_j^2(x_0, y_0)} \geqq \sum_{j \in J} v_j C_{(x_0, y_0)}(\nabla g_j(y_0)) + \sum_{j \in J} v_j \rho_j^2 \frac{d_j^2(x_0, y_0)}{\alpha_j^2(x_0, y_0)}. \tag{4.6}$$

Now, suppose that (4.5) holds. Again, by the assumption on $f_i (i = 1, 2, \ldots, p)$, we have

$$\frac{f_i(x_0) - f_i(y_0)}{\alpha_i^1(x_0, y_0)} \geqq C_{(x_0, y_0)}(\nabla f_i(y_0)) + \rho_i^1 \frac{d_i^1(x_0, y_0)}{\alpha_i^1(x_0, y_0)}. \tag{4.7}$$

Let us denote

$$\tau = \sum_{i=1}^{p} u_i + \sum_{j \in J} v_j.$$

It follows from equations (4.1), (4.2), (4.5)−(4.7) and the convexity of $C_{(x_0,y_0)}(\cdot)$ that

$$\begin{aligned}
0 &> \sum_{i=1}^{p} \frac{u_i}{\tau} \frac{f_i(x_0) - f_i(y_0)}{\alpha_i^1(x_0,y_0)} + \sum_{j\in J} \frac{v_j}{\tau} \frac{g_j(x_0) - g_j(y_0)}{\alpha_j^2(x_0,y_0)} \\
&\geqq \sum_{i=1}^{p} \frac{u_i}{\tau} \left(C_{(x_0,y_0)}(\nabla f_i(y_0))\right) + \sum_{j\in J} \frac{v_j}{\tau} \left(C_{(x_0,y_0)}(\nabla g_j(y_0))\right) \\
&\quad + \sum_{i=1}^{p} \frac{u_i}{\tau} \rho_i^1 \frac{d_i^1(x_0,y_0)}{\alpha_i^1(x_0,y_0)} + \sum_{j\in J} \frac{v_j}{\tau} \rho_j^2 \frac{d_j^2(x_0,y_0)}{\alpha_j^2(x_0,y_0)} \\
&\geqq C_{(x_0,y_0)} \left(\frac{1}{\tau} \left(\sum_{i=1}^{p} u_i \nabla f_i(y_0) + \sum_{j\in J} v_j \nabla g_j(y_0) \right) \right) \\
&\quad + \frac{1}{\tau} \left(\sum_{i=1}^{p} u_i \rho_i^1 \frac{d_i^1(x_0,y_0)}{\alpha_i^1(x_0,y_0)} + \sum_{j\in J} v_j \rho_j^2 \frac{d_j^2(x_0,y_0)}{\alpha_j^2(x_0,y_0)} \right), \\
&\geqq 0,
\end{aligned}$$

which gives a contradiction. Hence, the proof of part (a) is complete.

(b) Suppose that (4.5) holds. That is $f(x_0) \leq f(y_0)$, From the feasibility conditions of (P) and (MWD), we have

$$\sum_{j\in J} v_j g_j(x_0) \leqq 0 \leqq \sum_{j\in J} v_j g_j(y_0).$$

Thus, by the assumptions on f and $v^T g$ we get

$$C_{(x_0,y_0)}\left(\nabla f(y_0)\right) + \rho^1 \frac{d^1(x_0,y_0)}{\alpha^2 1(x_0,y_0)} \leqq 0$$

and

$$C_{(x_0,y_0)} \left(\sum_{j\in J} v_j \nabla g_j(y_0) \right) + \rho^2 \frac{d^2(x_0,y_0)}{\alpha^2(x_0,y_0)} \leqq 0.$$

Let us denote

$$\tau = \sum_{i=1}^{p} u_i + 1.$$

It follows from the above two inequalities, equation (4.3) and the convexity of $C_{(x_0,y_0)}(\cdot)$ that

$$\begin{aligned}0 &> \sum_{i=1}^{p} \frac{u_i}{\tau} \left(C_{(x_0,y_0)}(\nabla f_i(y_0))\right) + \frac{1}{\tau} \left(C_{(x_0,y_0)} \left(\sum_{j \in J} v_j \nabla g_j(y_0) \right) \right) \\ &\quad + \sum_{i=1}^{p} \frac{u_i}{\tau} \rho_i^1 \frac{d_i^1(x_0,y_0)}{\alpha_i^1(x_0,y_0)} + \frac{1}{\tau}\rho^2 \frac{d^2(x_0,y_0)}{\alpha^2(x_0,y_0)} \\ &\geqq C_{(x_0,y_0)} \left(\frac{1}{\tau} \left(\sum_{i=1}^{p} u_i \nabla f_i(y_0) + \sum_{j \in J} v_j \nabla g_j(y_0) \right) \right) \\ &\quad + \sum_{i=1}^{p} \frac{u_i}{\tau} \rho_i^1 \frac{d_i^1(x_0,y_0)}{\alpha_i^1(x_0,y_0)} + \frac{1}{\tau}\rho^2 \frac{d^2(x_0,y_0)}{\alpha^2(x_0,y_0)} \\ &\geqq 0,\end{aligned}$$

which gives a contradiction. Hence, the proof of part (b) is complete.

(c) Suppose that (4.5) holds. That is $f(x_0) \leq f(y_0)$. Also, from the feasibility conditions of (P) and (MWD), we have

$$\sum_{j \in J} v_j g_j(x_0) \leqq 0 \leqq \sum_{j \in J} v_j g_j(y_0).$$

Thus, by the assumptions on $u^T f$ and $v^T g$, we get

$$C_{(x_0,y_0)} \left(\sum_{i=1}^{p} u_i \nabla f_i(y_0) \right) + \rho^1 \frac{d^1(x_0,y_0)}{\alpha^1(x_0,y_0)} < 0$$

and

$$C_{(x_0,y_0)} \left(\sum_{j \in J} v_j \nabla g_j(y_0) \right) + \rho^2 \frac{d^2(x_0,y_0)}{\alpha^2(x_0,y_0)} \leqq 0.$$

Let $\tau = 2$. Now, it follows from the above two inequalities, equation (4.4) and the convexity of $C_{(x_0,y_0)}(\cdot)$ that

$$\begin{aligned}0 &> \frac{1}{\tau} C_{(x_0,y_0)} \left(\sum_{i=1}^{p} u_i \nabla f_i(y_0) \right) + \frac{1}{\tau} \left(C_{(x_0,y_0)} \left(\sum_{j \in J} v_j \nabla g_j(y_0) \right) \right) \\ &\quad + \frac{1}{\tau}\rho^1 \frac{d^1(x_0,y_0)}{\alpha^1(x_0,y_0)} + \frac{1}{\tau}\rho^2 \frac{d^2(x_0,y_0)}{\alpha^2(x_0,y_0)} \\ &\geqq C_{(x_0,y_0)} \left(\frac{1}{\tau} \left(\sum_{i=1}^{p} u_i \nabla f_i(y_0) + \sum_{j \in J} v_j \nabla g_j(y_0) \right) \right) \\ &\quad + \frac{1}{\tau} \left(\rho^1 \frac{d^1(x_0,y_0)}{\alpha^1(x_0,y_0)} + \rho^2 \frac{d^2(x_0,y_0)}{\alpha^2(x_0,y_0)} \right) \\ &\geqq 0,\end{aligned}$$

which gives a contradiction. Hence, the proof of part (c) is complete. □

Theorem 4.2. (Strong duality) *Let $x_0 \in S$ be an efficient solution for* (P) *and assume that* (P) *satisfies a suitable constraint qualification (see* [18]*). Then there exist $\bar{u} \in \mathbb{R}^p_+, \bar{u} > 0$, $\bar{v} = (\bar{v}_j)_{j \in J}, v_j \in \mathbb{R}_+$, such that $(x_0, \bar{u}, \bar{v})$ is a feasible solution for* (MWD)*. Furthermore, if the assumptions of Theorem* 4.1 *are satisfied, then $(x_0, \bar{u}, \bar{v})$ is an efficient solution for* (MWD) *and the objective values of* (P) *and* (MWD) *are equal.*

Proof. As $x_0 \in S$ is an efficient solution for (P) and a suitable constraint qualification (see [18]) is satisfied, by Theorem 3.1, there exist $\bar{u} \in \mathbb{R}^p_+$, $\bar{u} > 0$, $\bar{v} = (\bar{v}_j)_{j \in J}$, $v_j \in \mathbb{R}_+$, such that $(x_0, \bar{u}, \bar{v})$ is a feasible solution for (MWD). If $(x_0, \bar{u}, \bar{v})$ is not an efficient solution for (MWD), then there exists a feasible solution (x^*, u^*, v^*) for (MWD), such that

$$(f_1(x_0), \ldots, f_p(x_0)) \leq (f_1(x^*), \ldots, f_p(x^*)),$$

which contradicts Theorem 4.1. Hence, the proof is complete. □

Theorem 4.3. (Strict converse duality) *Let x_0 and (x^*, u^*, v^*) be the efficient solutions for* (P) *and* (MWD)*, respectively. If the assumptions of Theorem* 4.2 *are satisfied and f is strictly $(C, \alpha^1, \rho^1, d^1)$-convex at x^*, then $x_0 = x^*$.*

Proof. We proceed by contradiction. Let $x_0 \neq x^*$. By strong duality theorem, there exist $\bar{u} \in \mathbb{R}^p_+$, $\bar{u} > 0$, $\bar{v} = (\bar{v}_j)_{j \in J}$, $\bar{v}_j \in \mathbb{R}_+$ such that $(x_0, \bar{u}, \bar{v})$ is an efficient solution for (MWD). Hence,

$$f(x_0) = f(x^*). \tag{4.8}$$

Since x_0 and (x^*, u^*, v^*) are the feasible solutions for (P) and (MWD), respectively, it follows that

$$\sum_{j \in J} v_j^* g_j(x_0) \leqq 0 \leqq \sum_{j \in J} v_j^* g_j(x^*). \tag{4.9}$$

By the $(C, \alpha^2, \rho^2, d^2)$-convexity of g we get

$$0 \geq q \sum_{j \in J} v_j^* \frac{g_j(x_0) - g_j(x^*)}{\alpha_j^2(x_0, x^*)} \geqq \sum_{j \in J} v_j^* C_{(x_0, x^*)}(\nabla g_j(x^*)) + \sum_{j \in J} v_j^* \rho_j^2 \frac{d_j^2(x_0, x^*)}{\alpha_j^2(x_0, x^*)}. \tag{4.10}$$

Again, by the assumption on $f_i\,(i = 1, 2, \ldots, p)$, we have

$$\frac{f_i(x_0) - f_i(x^*)}{\alpha_i^1(x_0, x^*)} > C_{(x_0, x^*)}(\nabla f_i(x^*)) + \rho_i^1 \frac{d_i^1(x_0, x^*)}{\alpha_i^1(x_0, x^*)}. \tag{4.11}$$

Let us denote

$$\tau = \sum_{i=1}^{p} u_i^* + \sum_{j \in J} v_j^*.$$

It follows from equations (4.8)−(4.11) and the convexity of $C_{(x_0,x^*)}(\cdot)$ that

$$\begin{aligned}
0 > & \sum_{i=1}^{p} \frac{u_i^*}{\tau} \left(C_{(x_0,x^*)}(\nabla f_i(x^*)) \right) + \sum_{j\in J} \frac{v_j^*}{\tau} \left(C_{(x_0,x^*)}(\nabla g_j(x^*)) \right) \\
& + \sum_{i=1}^{p} \frac{u_i^*}{\tau} \rho_i^1 \frac{d_i^1(x_0,x^*)}{\alpha_i^1(x_0,x^*)} + \sum_{j\in J} \frac{v_j^*}{\tau} \rho_j^2 \frac{d_j^2(x_0,x^*)}{\alpha_j^2(x_0,x^*)} \\
\geqq & \, C_{(x_0,x^*)} \left(\frac{1}{\tau} \left(\sum_{i=1}^{p} u_i^* \nabla f_i(x^*) + \sum_{j\in J} v_j^* \nabla g_j(x^*) \right) \right) \\
& + \frac{1}{\tau} \left(\sum_{i=1} 6pu_i^* \rho_i^1 \frac{d_i^1(x_0,x^*)}{\alpha_i^1(x_0,x^*)} + \sum_{j\in J} v_j^* \rho_j^2 \frac{d_j^2(x_0,x^*)}{\alpha_j^2(x_0,x^*)} \right) \\
\geqq & \, 0,
\end{aligned}$$

which gives a contradiction. Therefore, $x_0 = x^*$. □

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