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A new approach to ductile damage

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Abstract - Generally, modelling of the plastic response of porous solids is done using stress-based plastic potentials. However, in order to gain understanding of the combined effects of all invariants for general three-dimensional loadings, a strain-rate based approach is required. In this paper, this approach is used to investigate the dilatational response for porous Tresca and von Mises solids for both compressive and tensile loadings. It is demonstrated that the presence of voids in the respective matrices induces dependence on all invariants, the noteworthy finding being the key role played by the plastic flow of the matrix. If the matrix is governed by the von Mises criterion, the shape of the cross-sections of the strain-rate surface with the octahedral plane deviates slightly from a circle, and changes very little as the absolute value of the mean strain rate increases. In contrast, if the matrix obeys Tresca's criterion, the cross-section evolves from a regular hexagon to a smooth triangle with rounded corners. It is shown that the very specific couplings between invariants dramatically affect damage evolution in the respective porous materials.

Key words and phrases : strain-rate potentials; porous Tresca solid; porous Mises solid; porosity evolution

Mathematics Subject Classification (2010) **:** 74Qxx, 74Rxx.

1. Introduction

Ziegler has shown in [28] that a strain-rate potential (SRP) can be associated to any convex stress-potential. A strain-rate based approach is generally adopted in crystal plasticity because it is much easier to calculate numerically a crystallographic SRP than to compute a crystallographic stress-based plastic potential (e.g. see [25]).

It is worth noting that analytic expressions for strain-rate potentials that are exact duals (i.e. work-equivalent) to stress-based potentials are known only for classical isotropic yield criteria such as von Mises, Tresca, or Drucker-Prager (see [22]), the orthotropic Hill (see [12]) criterion (see [14]), the orthotropic criterion of Cazacu et al. (see [2]) (see [4]; [26]).

However, all these strain-rate based potentials apply only to fully-dense materials for which the plastic flow can be considered to be incompressible.

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For real materials, which contain defects (e.g. cracks, voids), the hypothesis of plastic incompressibility does not apply any more. Based on rigourous limit-analysis theorems, Rice and Tracey (see [21]) and Gurson (see [11]) have demonstrated that the presence of voids induces a dependence of the mechanical response on the mean stress and as such the plastic flow is accompanied by volume changes.

Stress-based potentials have been developed to capture the characteristics of such materials. However, most of these models are based on the hypothesis that the matrix (void-free material) obeys von Mises yield criterion. Examples include the classical Gurson (see [11]) model and its various extensions (e.g. Tvergaard in [24], Gologanu in [9]). It is to be noted that in all these models the effects of the mean stress and shear stresses are decoupled. However, finite-element (FE) cell model calculations as well as very recent full-field calculations for porous polycrystals with constituent grains deforming by crystallographic slip have shown that there is a very specific dependence of yielding with the signs of the mean stress and the third-invariant of the stress deviator, J_3^{Σ}. Specifically, for tensile loadings the response corresponding to $J_3^{\Sigma} \geq 0$ is softer than that corresponding to $J_3^{\Sigma} \leq 0$ while for compressive loadings, the opposite holds true (e.g. Richelsen and Tvergaard in [20]; Cazacu and Stewart in [3]; Lebensohn and Cazacu in [15]; Alves et al. [1]).

For axisymmetric loadings, using micromechanical considerations Cazacu et al. (see [5, 6]) developed analytical yield criteria for porous materials with von Mises and Tresca matrix, respectively. These stress-based criteria account for the combined effects of the sign of the mean stress and of the third-invariant of the stress deviator on the dilatational response. Most importantly, it was explained by Revil-Baudard and Cazacu in [18] the role of the third-invariant on void growth and void collapse. An excellent agreement between the predictions of these models and and FE unit-cell model calculations were reported in Alves et al. (see [1]) and Cazacu et al. (see [7]), respectively.

However, to gain understanding of the combined effects of all invariants for general three-dimensional loadings, a strain-rate based approach appears is most appropriate. Revil-Baudard and Cazacu in [19], very recently developed strain-rate potentials for porous solids with von Mises and Tresca matrix.

In this paper, the dilatational response according to these new SRPs is investigated. The structure of the paper is as follows. We begin by presenting the modeling framework. Next, the strain-rate potentials for porous von Mises and Tresca solids are given. New and intriguing features of the response of the porous material are revealed by analyzing the SRPs projections in various planes and the predicted void evolution. Furthermore, it is shown that the level of porosity in the material strongly influences the couplings between the invariants. Next, we revisit some aspects of Gurson's treatment. The implications of the approximations made by Gurson (see [10, 11]) are discussed in relation to the exact strain-rate potentials for porous solids with Tresca and von Mises matrix obtained by Revil-Baudard and Cazacu in [19]. We conclude by summarizing the main findings of this study.

Regarding notations, vector and tensors are denoted by boldface characters. If $\mathbf{A}$ and $\mathbf{B}$ are second-order tensors, the contracted tensor product between such tensors are defined as: $\mathbf{A}:\mathbf{B} = A_{ij}B_{ij}, \quad i,j = 1\ldots 3$.

2. General framework

2.1. Strain-rate potentials for plastic incompressible materials

In the framework of the mathematical theory of plasticity the onset of plastic flow is generally described by specifying a convex yield function, $\varphi(\boldsymbol{\sigma})$, in the stress space. Assuming associated flow rule, the plastic strain rate tensor $\mathbf{d}$ is obtained by differentiation of $\varphi(\boldsymbol{\sigma})$, i.e.

$$\mathbf{d} = \dot{\lambda}\frac{\partial\varphi}{\partial\boldsymbol{\sigma}}, \tag{2.1}$$

where $\boldsymbol{\sigma}$ is the Cauchy stress tensor, and $\dot{\lambda} \geq 0$ stands for the plastic multiplier. The yield surface is defined as $\varphi(\boldsymbol{\sigma}) = \sigma_T$, where σ_T is the uniaxial yield in tension. Alternatively, a dual potential in the strain-rate space can be defined (see [28, 14]) :

$$\psi(\mathbf{d}) = \dot{\lambda}, \tag{2.2}$$

and the stresses are derived from $\psi(\mathbf{d})$ as:

$$\boldsymbol{\sigma} = \sigma_T\frac{\partial\psi}{\partial\mathbf{d}}. \tag{2.3}$$

The yield function $\varphi(\boldsymbol{\sigma})$ is generally taken homogeneous of degree one with respect to positive multipliers, so the plastic dissipation is:

$$\pi(\mathbf{d}) = \sup_{\boldsymbol{\sigma}}\left(\sigma_{ij}d_{ij}\right) = \dot{\lambda}\sigma_T, \qquad i,j = 1,\ldots,3, \tag{2.4}$$

where C is the convex domain delimited by the yield surface. Note that the functions $\psi(\mathbf{d})$ and $\varphi(\boldsymbol{\sigma})$ are dual potentials. If the plastic behaviour is described by the von Mises criterion, i.e.

$$\varphi_{\text{Mises}}(\boldsymbol{\sigma}) = \sqrt{(3/2)\boldsymbol{\sigma'}:\boldsymbol{\sigma'}},$$

the associated strain-rate potential is: $\psi_{\text{Mises}}(\mathbf{d}) = \sqrt{(2/3)\mathbf{d}:\mathbf{d}} = \dot{\bar{\varepsilon}}$, where $\dot{\bar{\varepsilon}}$ denotes the von Mises equivalent strain rate and $\boldsymbol{\sigma'}$ the stress deviator. Using Eq.(2.4), the plastic dissipation of a von Mises material is

$$\pi_{\text{Mises}}(\mathbf{d}) = \sigma_T\sqrt{(2/3)\mathbf{d}:\mathbf{d}}. \tag{2.5}$$

If the plastic behaviour is described by Tresca's criterion potential, then

$$\varphi(\boldsymbol{\sigma}) = \max\left(|\sigma_1 - \sigma_2|, |\sigma_2 - \sigma_3|, |\sigma_3 - \sigma_1|\right),$$

with σ_1, σ_2, and σ_3 being the principal values of $\boldsymbol{\sigma}$, and the associated strain-rate potential is: $\psi_{\text{Tresca}}(\mathbf{d}) = \left(|d_1| + |d_2| + |d_3|\right)/2$, with d_1, d_2 and d_3 being the principal values of $\mathbf{d}$. Using Eq.(2.4), the Tresca plastic dissipation is:

$$\pi_{\text{Tresca}}(\mathbf{d}) = \frac{\sigma_T}{2}\left(|d_1| + |d_2| + |d_3|\right). \tag{2.6}$$

For an isotropic material (which means that $\varphi(\boldsymbol{\sigma})$ of equation (2.1) is an isotropic function), the principal directions of $\mathbf{d}$ and $\boldsymbol{\sigma}$ coincide. The projections of the von Mises SRP (Eq. 2.5) and that of the Tresca SRP (Eq. 2.6) on the octahedral plane are shown in Fig. 1.

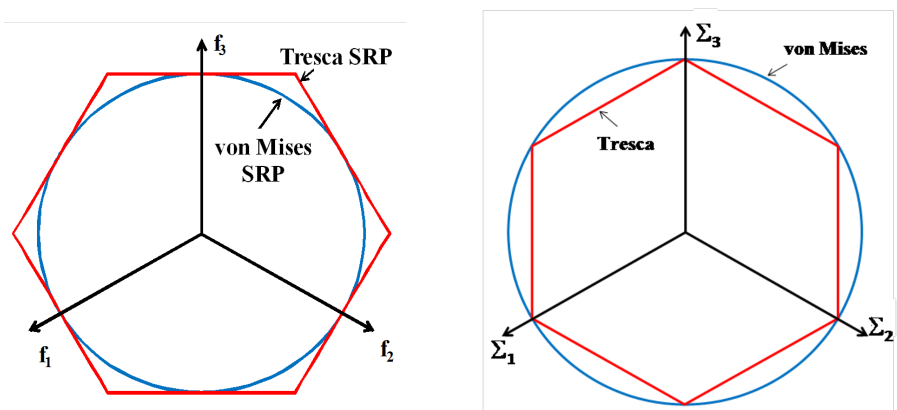

Figure 1. (a) Section of the von Mises strain-rate potential (SRP) (equation (2.5)) and Tresca's SRP (equation (2.6)) with the octahedral plane; (b) representation of their respective duals in the stress space, i.e. the normalized von Mises and Tresca yield surfaces, respectively.

2.2. Limit analysis framework for development of plastic potentials for porous metallic materials

The kinematic approach of limit analysis of Hill-Mandel (see [13, 17]) offers a rigorous framework for the development of plastic potentials for porous solids. If the matrix (void-free material) is rigid-plastic, it has been shown (see [23]) that there exists a strain-rate potential $\Pi = \Pi(\mathbf{D}, f)$ such that the stress at any point in the porous solid is given by:

$$\boldsymbol{\Sigma} = \frac{\partial \Pi(\mathbf{D}, f)}{\partial \mathbf{D}} \text{ with } \Pi(\mathbf{D}, f) = \inf_{d \in K(\mathbf{D})} \langle \pi(\mathbf{d}) \rangle_\Omega, \tag{2.7}$$

where Ω is a representative volume element composed of the matrix and a traction-free void, while $\langle\ \rangle$ denotes the average value over Ω; f is the porosity (ratio between the volume of the void and the volume of Ω); $\pi(\mathbf{d})$ is the matrix's plastic dissipation with $\mathbf{d}$ being the local plastic strain rate tensor (see Eq. 2.4). Minimization is done over $K(\mathbf{D})$, which is the set of incompressible velocity fields compatible with homogeneous strain-rate boundary conditions, i.e.

$$\mathbf{v} = \mathbf{D}\mathbf{x}, \quad \text{for any } \mathbf{x} \in \partial\Omega. \tag{2.8}$$

Revil-Baudard and Cazacu (in [19]) used this kinematic homogenization approach to obtain 3-D plastic potentials for porous solids. The limit analysis was conducted for general 3-D states, i.e.

$$\mathbf{d} = D_1\mathbf{e}_1 \otimes \mathbf{e}_1 + D_2\mathbf{e}_2 \otimes \mathbf{e}_2 + D_3\mathbf{e}_3 \otimes \mathbf{e}_3, \tag{2.9}$$

with D_1, D_2, D_3 being the eigenvalues (unordered) of $\mathbf{D}$ and $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ its eigenvectors. It was assumed that the voids are spherical and randomly distributed in the matrix, so the strain-rate potential of the porous solid, $\Pi(\mathbf{D}, f)$, is isotropic.

Thus, $\Pi(\mathbf{D}, f)$ depends on the strain-rate tensor $\mathbf{D}$ only through its invariants: $D_m = (D_1 + D_2 + D_3)/3$, and the second and third-invariant of the deviator of $\mathbf{D}'$, respectively, i.e.

$$\Pi(\mathbf{D}, f) = \Pi(D_m, J_{2D}, J_{3D}), \tag{2.10}$$

where $J_{2D} = \sqrt{(D_1'^2 + D_2'^2 + D_3'^2)/2}$ and $J_{3D} = D_1' D_2' D_3'$, with $D_i' = D_i - D_m$, $i = 1, \ldots, 3$.

To analyze the role played by the mean strain rate, D_m, on the dilatational response, for a fixed value of the porosity f, the shape of the cross-sections of the SRP with the deviatoric planes $D_m = constant$, need to be determined. For this purpose it is convenient to introduce the $Oxyz$ frame, which is related to the principal frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ by the following relations:

$$\mathbf{e}_x = \frac{1}{\sqrt{3}}(\mathbf{e}_1 + \mathbf{e}_2 + \mathbf{e}_3), \quad \mathbf{e}_y = -\frac{1}{\sqrt{2}}(\mathbf{e}_1 + \mathbf{e}_2), \quad \mathbf{e}_z = \frac{1}{\sqrt{6}}(2\mathbf{e}_3 - \mathbf{e}_1 - \mathbf{e}_2). \tag{2.11}$$

Consider an arbitrary state represented by a point $P(D_1, D_2, D_3)$ belonging to the SRP isosurface $\Pi(\mathbf{D}, f) = constant$. Since the Ox-axis coincides with the hydrostatic axis, the plane that contains the state P and is parallel to the Oyz-plane contains all the states belonging to the SRP with the same D_m. Thus, the intersection of the SRP with the deviatoric plane $D_m = constant$ is obtained by expressing the SRP in the (xyz) coordinates and then imposing $D_x = constant$. Indeed, the SRP of a porous solid can be expressed as:

$$\Pi(\mathbf{D}, f) = \Pi(D_m, D_1', D_2', D_3', f) = \Pi(D_x, D_y, D_z, f),$$

$$
\begin{aligned}
D_x &= \sqrt{3} D_m \\
D_y &= (D'_2 - D'_1)\frac{\sqrt{2}}{2} \\
D_z &= \sqrt{\frac{3}{2}} D'_3.
\end{aligned}
\tag{2.12}
$$

Thus, any point $P(D_1, D_2, D_3)$ belonging to the intersection of the SRP locus with any deviatoric plane is characterized by the polar coordinates, (R, γ) (see also Figure 2(a)):

$$
R = |OP| = \sqrt{D_1'^2 + D_2'^2 + D_3'^2} = \sqrt{2J_{2D}}, \tag{2.13a}
$$

while γ denotes the angle between $\mathbf{e}_y$ and OP, so

$$
\tan(\gamma) = \frac{D_z}{D_y} = \sqrt{3}\frac{D'_3}{D'_2 - D'_1}, \tag{2.13b}
$$

and any state D belonging to the SRP surface is solely defined by (D_m, R, γ).

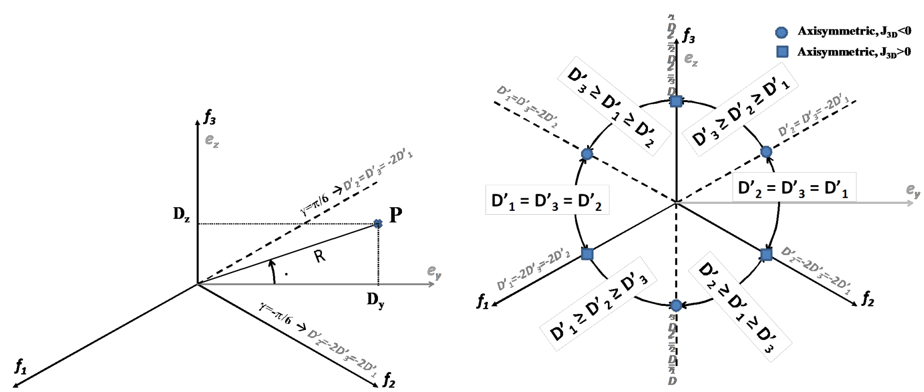

Figure 2. (a) Definition of the polar coordinates (R, γ), representing any state $P(D_1, D_2, D_3)$ belonging to the intersection of any strain-rate potential isosurface with any deviatoric plane (plane of normal the hydrostatic axis) (b) General symmetry properties of the cross-section of the strain-rate potential of an isotropic material.

Let $\mathbf{f}_i$ be the projections of the eigenvectors $\mathbf{e}_i$, $i = 1, \ldots, 3$ on a deviatoric plane. Obviously, $\mathbf{f}_3 = \mathbf{e}_z$ (see equation (2.11) and Figure 2(a)). For a porous solid that is isotropic, the SRP has three-fold symmetry so it is sufficient to determine $R = R(\gamma)$ only in the sector $-\frac{\pi}{6} \le \gamma \le \frac{\pi}{6}$. The sector $-\frac{\pi}{6} \le \gamma \le \frac{\pi}{6}$ corresponds to the following ordering of the principal values of D': $D'_2 \ge D'_3 \ge D'_1$ (see equation (2.13) and Figure 1(b)). In particular, the sub-sector $-\frac{\pi}{6} \le \gamma \le 0$ corresponds to states on the SRP for which $(D'_2 \ge 0, D'_3 \le 0, D'_1 \le 0)$, so the third-invariant $J_{3D} > 0$ while the sub-sector $0 \le \gamma \le \frac{\pi}{6}$ corresponds to states for which $(D'_2 \ge 0, D'_3 \ge 0, D'_1 \le 0)$

so $J_{3D} < 0$ (see also Figure 2(b)). Axisymmetric states correspond to either $\gamma = -\frac{\pi}{6}$ ($D_1' = D_3' < D_2'$) or $\gamma = \frac{\pi}{6}$ ($D_2' = D_3' > D_1'$). Note that for $-\frac{\pi}{6} \leq \gamma \leq \frac{\pi}{6}$, D_1', D_2', D_3' can be expressed as:

$$\begin{aligned} D_1' &= -\frac{R(\gamma)}{\sqrt{6}}\left(\sqrt{3}\cos\gamma + \sin\gamma\right) \\ D_2' &= \frac{R(\gamma)}{\sqrt{6}}\left(\sqrt{3}\cos\gamma - \sin\gamma\right) \\ D_3' &= \frac{2R(\gamma)}{\sqrt{6}}\sin\gamma \end{aligned} \tag{2.14}$$

with $\sin 3\gamma = -\dfrac{27}{2} \cdot \dfrac{J_{3D}}{(J_{2D})^{\frac{3}{2}}}$ (see equation (2.13)).

The angle γ is related to the dimensionless parameter ν introduced by Drucker (see [8]),

$$\nu = \frac{D'_{\text{int}}}{D'_{\text{min}} - D'_{\text{max}}}, \tag{2.15}$$

where $D_{\text{min}} = \min(D_1', D_2', D_3')$, $D_{\text{max}} = \max(D_1', D_2', D_3')$ while D_{int} is the intermediate principal value.

3. Three-dimensional strain-rate potentials for porous solids with von Mises and Tresca matrices containing spherical voids

For spherical void geometry an appropriate representative volume element (RVE) is a hollow sphere. Let a denote its inner radius and $b = af^{-\frac{1}{3}}$ its outer radius. In Revil-Baudard and Cazacu ([19]) the limit analysis was conducted for 3-D conditions for both tensile and compressive states. Use was made of the trial velocity field $\mathbf{v}$, deduced by Rice and Tracey in [21],

$$\mathbf{v} = \mathbf{v}^{\text{v}} + \mathbf{v}^{\text{S}}, \tag{3.1}$$

where $\mathbf{v}^{\text{v}}$ describes the expansion of the cavity while $\mathbf{v}^{\text{S}}$ is associated to changes in the shape of the cavity. Imposing the boundary conditions and the constraint of matrix incompressibility, i.e.:

$$\mathbf{v}(\mathbf{x} = b\mathbf{e}_r) = \mathbf{D}\mathbf{x} \text{ and } \operatorname{div}(\mathbf{v}) = 0,$$

where $\mathbf{x}$ is the Cartesian position vector that denotes the current position in the RVE and $\mathbf{e}_r$ is the radial unit vector, it follows that:

$$\mathbf{v}^{\text{v}} = \left(\frac{b^3}{r^2}\right) D_m \mathbf{e}_r \text{ and } \mathbf{v}^{\text{S}} = \mathbf{D}'\mathbf{x}, \tag{3.2}$$

where $r = \sqrt{x_1^2 + x_2^2 + x_3^2}$ is the radial coordinate.

If the plastic flow in the matrix is governed by either von Mises or Tresca criterion, the exact solution of the problem of a hollow sphere subjected to hydrostatic states is the same (see [16]). This solution is the term $\mathbf{v}^{\mathrm{v}} = \left(\frac{b^3}{r^2}\right) D_m \mathbf{e}_r$ given by equation (3.2). Thus, for purely hydrostatic states, the mechanical response of a porous Tresca or a porous Mises solid is also the same.

Since the velocity $\mathbf{v}$ is incompressible and compatible with homogeneous strain rate boundary conditions (equation (3.2)), Hill-Mandel lemma applies. Thus, an upper-bound estimate of the exact plastic potential of porous solid with von Mises matrix is:

$$\Pi^+_{\text{Mises}}(\mathbf{D}, f) = \frac{\sigma_{\mathrm{T}}}{V} \int_\Omega \pi_{\text{Mises}}(\mathbf{d}) dV, \tag{3.3}$$

with $V = 4\pi b^3/3$, Ω is the domain occupied by the matrix, and $\pi_{\text{Mises}}(\mathbf{d})$ is the local plastic dissipation associated to the von Mises criterion (see equation (2.5)) for $\mathbf{d} = \frac{1}{2}(\nabla \mathbf{v} + \nabla \mathbf{v}^T)$ and $\mathbf{v}$ given by equation (3.1), i.e.

$$\pi_{\text{Mises}}(\mathbf{d}) = \sigma_{\mathrm{T}} \sqrt{\begin{array}{l} (2/3)\left(D_1'^2 + D_2'^2 + D_3'^2\right) + \\ 4D_m^2 (b/r)^6 - 4D_m (b/r)^3 \left(D_1'^2 x_1^2 + D_2'^2 x_2^2 + D_3'^2 x_3^2\right) \end{array}}. \tag{3.4}$$

For general 3-D states, the integral given by equation (3.3) cannot be amenable to an exact analytic calculation.

However, very recently Cazacu et al. have shown in [5] that for axisymmetric states the integrals expressing the SRP can be calculated explicitly, without making the approximations considered by Gurson (see [10, 11]).

For all other loadings, numerical integration methods need to be used. As already mentioned, it is sufficient to evaluate this SRP in the sector $-\frac{\pi}{6} \leq \gamma \leq \frac{\pi}{6}$.

The integral expressing the von Mises SRP can be put in the form:

$$\Pi^+_{\text{Mises}}(\mathbf{D}, f) = \frac{\sigma_T}{V} \int_\Omega 2\sigma_T \sqrt{(R^2/6) + D_m^2 (b/r)^2 - 4RD_m (b/r)^3 F(\gamma, x_i^2)/(r^2/\sqrt{6})} dV \tag{3.5}$$

with $F(\gamma, x_1^2, x_2^2, x_3^2) = \sqrt{3}(x_2^2 - x_1^2)\cos\gamma + (2x_3^2 - x_1^2 - x_2^2)\sin\gamma$.

The integration is further simplified by making a change of coordinates from the coordinate system $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ of the eigenvectors of $\mathbf{D}$ and Cartesian coordinates (x_1, x_2, x_3) to spherical coordinates. For axisymmetric loadings, the integral estimated numerically was compared to the exact results, the differences being negligible (less than 10^{-7}).

As an example, in Figure 3 is shown a 3-D isosurface of the von Mises porous solid corresponding to a porosity $f = 5\%$ for both tensile ($D_m = \operatorname{tr}(D) > 0$) and compressive ($D_m < 0$) states. First, let us note that the presence of voids induces a strong influence of the mean strain rate D_m. In

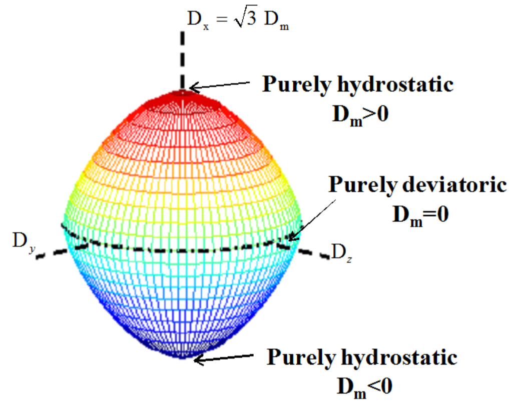

Figure 3. The 3-D surface for a porous solid with von Mises matrix according to equation (3.3) for both tensile mean strain rate ($D_m = \mathrm{tr}(\mathbf{D}) > 0$) and compressive ($D_m < 0$) states. Note that this convex surface contains all the points (D_m, R, γ) that produce the same plastic dissipation for the porous solid. Initial porosity: $f = 0.05$.

contrast to the SRP for the fully dense material, the SRP for $f = 5\%$ is closed on the hydrostatic axis.

To fully assess the effects of all invariants of the strain-rate, $\mathbf{D}$, on the plastic response of the porous solid, the cross-sections of the same 3-D iso-surface with several deviatoric planes $D_m = constant$ are considered (see Figure 3).

Note that the intersection with the plane $D_m = 0$, is a circle. This is to be expected since states for which $D_m = 0$ correspond to purely deviatoric loadings for which the plastic dissipation of the porous solid coincides with that of the matrix (von Mises behavior). The cross-sections with all the other deviatoric planes $D_m = constant$, have three-fold symmetry with respect to the origin, and deviate from a circle. This indicates that the third-invariant $J_{3D} = D_1' D_2' D_3'$ affects the plastic response of a porous solid with von Mises matrix. As an example, in Figure 4 is plotted $R(\gamma)$ (normalized by $R(\gamma = -\frac{\pi}{6})$) for the cross-section corresponding to $D_m = 6.0 \cdot 10^{-4}\mathrm{s}^{-1}$ and $D_m = 0$ (matrix behavior), respectively. The cross-section $D_m = 0$ is a circle, so: $R(\gamma) = R(\gamma) = -\frac{\pi}{6}$ i.e. it is a straight line. As concerns the cross-section $D_m = 6.0 \cdot 10^{-4}\mathrm{s}^{-1}$, note the influence of the third-invariant J_{3D} (or γ) as evidenced by the deviation of $R(\gamma)/R(\gamma = -\frac{\pi}{6})$ from a straight line. The most pronounced difference is between the axisymmetric states, i.e. between $R(\gamma = -\frac{\pi}{6})$ and $R(\gamma = \frac{\pi}{6})$. The noteworthy result is that this holds true irrespective of the level of D_m i.e. the shape of the cross-sections are similar and the most pronounced deviation from a circle is for axisymmetric states (see also Figure 4).

A remarkable property of the exact plastic potentials (stress-based and

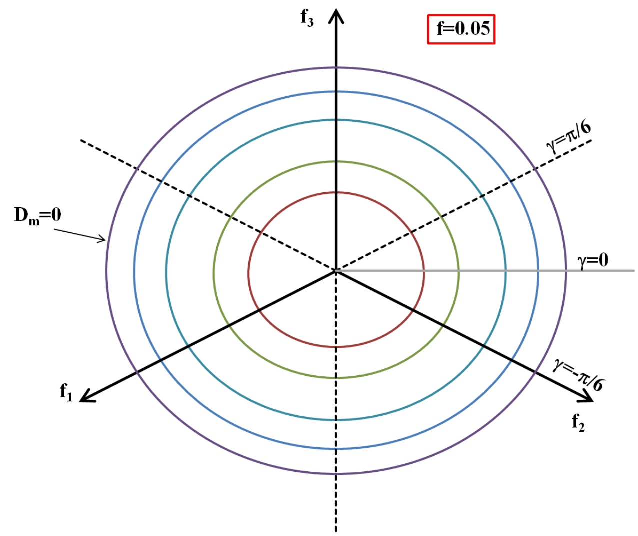

Figure 4. Cross-sections of the 3-D isosurface of a porous von Mises material with several deviatoric planes $D_m = constant$: outer cross-section represents the intersection with the plane $D_m = 0$ while the inner cross-section corresponds to $D_m = 9 \cdot 10^{-4}\text{s}^{-1}$. Initial porosity: $f = 0.05$.

strain-rate based) of a porous solid with von Mises matrix is their centro-symmetry. This property is preserved by $\Pi^+_{\text{Mises}}(\mathbf{D}, f)$. This means that for any porosity f: $\Pi^+_{\text{Mises}}(D_m, R, \gamma) = \Pi^+_{\text{Mises}}(-D_m, R, -\gamma)$, i.e. the surface is symmetric with respect to the origin (see also equation (3.5) and Figure 3). To further illustrate this noteworthy property, in Figure 5 are shown the cross-sections of the same 3-D isosurface $\Pi^+_{\text{Mises}}(\mathbf{D}, f) = 6.0 \cdot 10^{-3}$ ($f = 0.05$) with a deviatoric plane corresponding to a positive mean strain rate ($D_m = 6 \cdot 10^{-4}\text{s}^{-1}$, interrupted line) and a compressive mean strain rate ($D_m = -6 \cdot 10^{-4}\text{s}^{-1}$, solid line), respectively. The symmetry of the respective cross-sections with respect to the origin is clearly seen. For example, for loadings corresponding to $J_{3D} > 0$ (i.e. $-\frac{\pi}{6} < \gamma < 0$) to produce the same plastic dissipation $R(\gamma)$ (or $\sqrt{2J_{2D}}$) must be higher for compressive states ($D_m < 0$ - interrupted line) than for tensile states ($D_m > 0$ - solid line). The reverse holds true for loadings corresponding to $J_{3D} < 0$ ($0 < \gamma < \frac{\pi}{6}$).

For a given D_m, in order to reach the same plastic dissipation in the porous solid: for $D_m > 0, R(\gamma)$ is a monotonically decreasing function of γ (see also Figure 4) while for $D_m < 0$, $R(\gamma)$ must be a monotonically increasing function of γ.

An upper-bound estimate of the overall plastic potential of porous solid with Tresca matrix is:

$$\Pi^+_{\text{Tresca}}(\mathbf{D}, f) = \frac{\sigma_T}{V} \int_\Omega \pi_{\text{Tresca}}(\mathbf{d})\, dV, \tag{3.6}$$

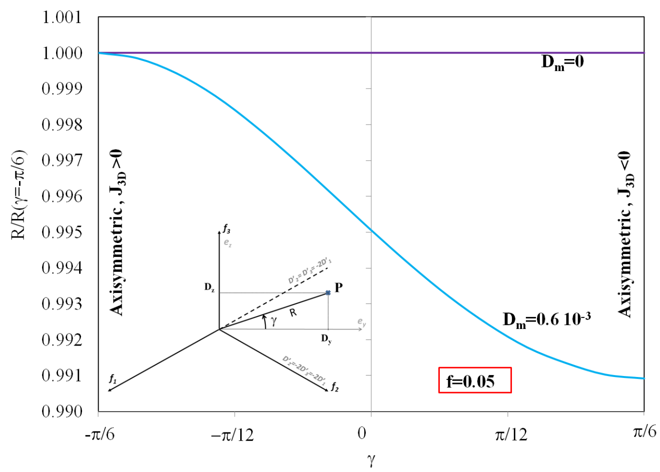

Figure 5. Evolution of $R(\gamma)$ (normalized by $R = R(-\frac{\pi}{6})$ for the cross-section of the surface of the porous Mises material with the deviatoric planes $D_m = 6 \cdot 10^{-4}\mathrm{s}^{-1}$ and $D_m = 0$ (von Mises behavior), respectively. Initial porosity: $f = 0.05$.

where $\pi_{\text{Tresca}}(\mathbf{d})$ is the local plastic dissipation associated to the Tresca criterion,

$$\pi_{\text{Tresca}}(\mathbf{d}) = \sigma_T(|d_1| + |d_2| + |d_3|) \tag{3.7}$$

where d_1, d_2, d_3 are the principal values (unordered) of the strain-rate field $\mathbf{d}$, corresponding to the velocity field given by equation (3.2).

A major difficulty in obtaining a closed-form expression of the Tresca SRP is that $\pi_{\text{Tresca}}(\mathbf{d})$ depends on the sign of each of the principal values of the local strain-rate tensor, $\mathbf{d}$ (see equation (3.7)). This is a direct consequence of the Tresca's criterion being dependent on the third-invariant of the stress deviator.

Only for axisymmetric loadings, the signs of the principal values d_1, d_2, d_3 for $\mathbf{d}$ can be determined analytically. For these loadings, very recently Cazacu et al. showed in [6] that the integrals expressing the overall plastic dissipation $\Pi^+_{\text{Tresca}}(\mathbf{D}, f)$ could be calculated explicitly, without any approximation (for more details about the calculations, the reader is referred to [6]). As already mentioned, for purely hydrostatic loadings (i.e. $\mathbf{D} = D_m\mathbf{I}$) $\Pi^+_{\text{Tresca}}(\mathbf{D}, f) = \Pi^+_{\text{Mises}}(\mathbf{D}, f) = 2\sigma_T|D_m|\ln f$.

For general 3-D states the overall plastic dissipation $\Pi^+_{\text{Tresca}}(\mathbf{D}, f)$ (equation (3.6)) can be estimated only numerically. For axisymmetric loadings, the numerical values are very close to the analytical ones (error less than 10^{-7}).

As an example, in Figure 6 is shown a normalized ($\sigma_T = 1$) 3-D isosurface of the porous Tresca solid corresponding to a porosity $f = 5\%$ for states characterized by ($D_m > 0$) and ($D_m < 0$), respectively. Specifically, this

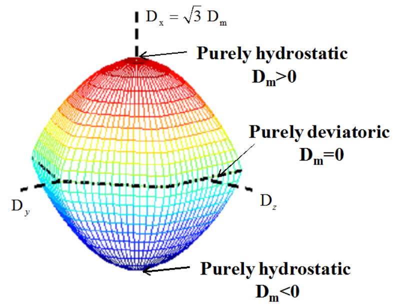

Figure 6. Cross-sections of the surface of the porous von Mises material, $\Pi^{+}_{\text{Mises}}(\mathbf{D}, f) = 6.0 \cdot 10^{-3}$ ($f = 0.05$) with the deviatoric planes $D_m = 6 \cdot 10^{-4}\text{s}^{-1}$ (interrupted line) and $D_m = -6 \cdot 10^{-4}\text{s}^{-1}$ (solid line). Note the centro-symmetry of the cross-sections due to the invariance of the plastic response to the transformation $(D_m, \mathbf{D}') \longrightarrow (-D_m, -\mathbf{D}')$. Initial porosity: $f = 0.01$.

convex surface contains all states $D_m, R, \gamma)$ that produce the same plastic dissipation $\Pi^{+}_{\text{Tresca}}(\mathbf{D}, f) = 6.0 \cdot 10^{-3}$ for the porous solid. The presence of voids induces a strong influence of the mean strain rate D_m on the plastic dissipation, the SRP for the porous Tresca material being closed on the hydrostatic axis.

To investigate the effects of all invariants on the response of the porous Tresca solid, the cross-sections of the same 3-D isosurface with deviatoric planes $D_m = \textit{constant}$ are considered (see Figure 7). Note that the intersection with the plane $D_m = 0$ is a regular hexagon. This is to be expected since states for which $D_m = 0$ correspond to purely deviatoric loadings for which the plastic dissipation of the porous solid coincides with that of the matrix.

It is very interesting to note the very strong influence of D_m on the plastic behavior of the porous Tresca material, the shape of the cross-sections changing drastically with the level of $D_m = \textit{constant}$. Due to the presence of voids, all cross-sections are smoothed out, their shape evolving from a hexagon ($D_m = 0$) to a triangle with rounded corners (e.g. the innermost cross-section). Note that the evolution of R with γ is very specific and depends strongly on D_m as shown in Figure 7.

To better assess the importance of this coupling between all invariants, in Figure 8 is plotted $R(\gamma)$ (normalized by $R(\gamma = -\frac{\pi}{6})$) for cross-sections corresponding to $D_m = \textit{constant}$ (in the range $D_m = 0$ to $D_m = 9.0 \cdot 10^{-4}$). Since Tresca's criterion depends on both J_{2D} and J_{3D}, even the cross-section corresponding to $D_m = 0$ is not a circle.

Furthermore, for $D_m = 0$, the analysis of the evolution of R with γ shows that it has a maximum at $\gamma = 0$ (i.e. states corresponding to $J_{3D} = 0$) while the minima correspond to the axisymmetric states, i.e. $\gamma = -\frac{\pi}{6}$ and $\gamma = \frac{\pi}{6}$.

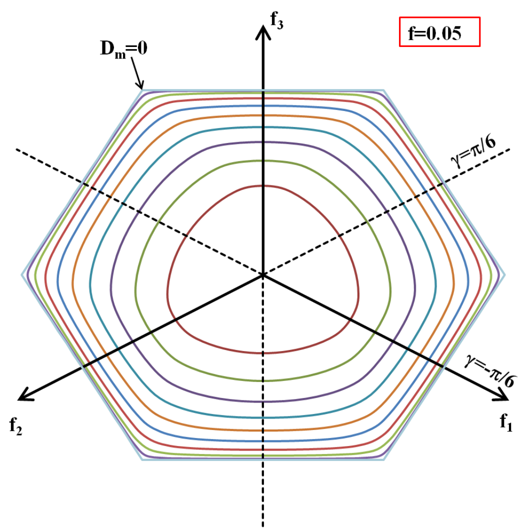

Figure 7. The 3-D surface for a porous solid with Tresca matrix according to equation (3.6) for both tensile ($D_m = \mathrm{tr}(\mathbf{D}) > 0$) and compressive ($D_m < 0$) states. Note that this convex surface contains all the points (D_m, R, γ) corresponding to the same plastic dissipation $\Pi^+_{\text{Tresca}}(\mathbf{D}, f) = 6.0 \cdot 10^{-3}$ for the porous solid. Initial porosity: $f = 0.05$.

Note also that only for $D_m = 0$ (i.e. matrix behavior) $R(\gamma = -\frac{\pi}{6}) = R(\gamma = \frac{\pi}{6})$, i.e. the plastic dissipation is the same for the axisymmetric state corresponding to $J_{3D} > 0$ and the axisymmetric state corresponding to $J_{3D} < 0$. Note the strong influence of D_m on the variation of R with γ.

Indeed, with increasing D_m the maximum of $R(\gamma)$ is no longer at $\gamma = 0$, but shifts towards the axisymmetric case corresponding to $\gamma = -\frac{\pi}{6}$ ($D_1 = D_3 < D_2$ and $J_{3D} > 0$); on the other hand, the minimum of $R(\gamma)$ is always obtained for $\gamma = -\frac{\pi}{6}$ (axisymmetric state corresponding to $J_{3D} < 0$). Another specificity of the dilatational response of a porous Tresca solid is that irrespective of the cross-section $D_m = \textit{constant}$, there are two states with the same R: the axisymmetric state $\gamma = -\frac{\pi}{6}$ and another state say $\gamma = \gamma_1$; the value of γ_1 depending on D_m (e.g. for $D_m = 0$, $\gamma_1 = \frac{\pi}{6}$, the higher D_m the lower is γ_1).

As already mentioned, $\Pi^+_{\text{Tresca}}(\mathbf{D}, f)$ is centro-symmetric. This means that for any porosity f: $\Pi^+_{\text{Tresca}}(D_m, R, \gamma, f) = \Pi^+_{\text{Tresca}}(-D_m, R, -\gamma, f)$.

To illustrate this remarkable property, in Figure 9 are shown the cross-sections of the same 3-D isosurface $\Pi^+_{\text{Tresca}}(\mathbf{D}, f) = 6.0 \cdot 10^{-3}$ ($f = 0.51$) with the deviatoric planes $D_m = 7 \cdot 10^{-4}\mathrm{s}^{-1}$ and $D_m = 9 \cdot 10^{-4}\mathrm{s}^{-1}$, respectively (interrupted lines) as well as the cross-sections with the planes $D_m = -7 \cdot 10^{-4}\mathrm{s}^{-1}$ and $D_m = -9 \cdot 10^{-4}\mathrm{s}^{-1}$, respectively (solid lines). The symmetry of all cross-sections with respect to the origin is clearly seen. For example, for states corresponding to $J_{3D} > 0$ ($-\frac{\pi}{6} < \gamma < 0$) to produce the same plastic

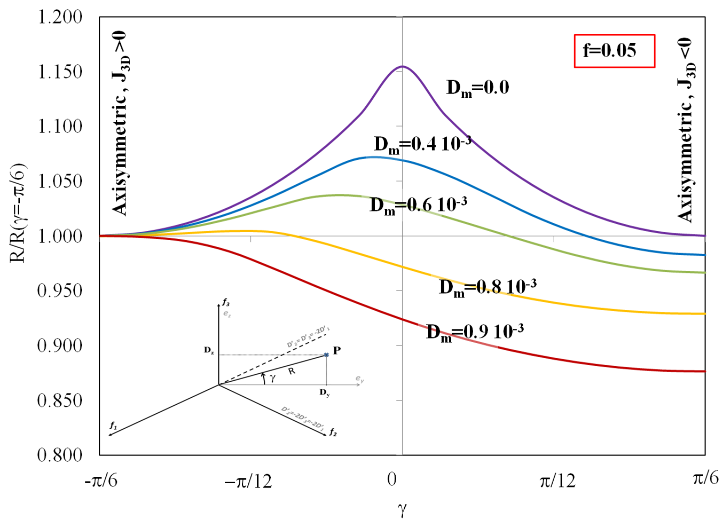

Figure 8. Cross-sections of the 3-D isosurface of porous Tresca material (equation (3.7)) with several deviatoric planes $D_m = constant$: Outer cross-section represents $D_m = 0$ (matrix behavior) Initial porosity: $f = 0.05$. Note the drastic change in the shape of the cross-section from a regular hexagon to a smooth triangle.

dissipation, R (or J_{2D}) must be higher for compressive states ($D_m < 0$ - interrupted line) than for tensile states ($D_m > 0$ - solid line). The reverse holds true for loadings corresponding to $J_{3D} < 0$ ($0 < \gamma < \frac{\pi}{6}$).

It is also very interesting to compare the behaviour of porous solids with Mises and Tresca matrix, respectively. For a porous Tresca solid the shapes of the cross-sections with deviatoric planes are strongly dependent on the level of D_m, whereas for a porous Mises the shape of the cross-sections are similar irrespective of the level of D_m.

4. Discussion on the role of the plastic flow of the matrix on the dilatational response

The most widely used plastic potential for isotropic porous solids containing randomly distributed spherical voids was proposed by Gurson in [11]. This yield criterion was derived by conducting limit analysis on a hollow sphere made of a rigid-plastic material obeying von Mises yield criterion using the trial velocity field deduced by Rice and Tracey in [21] (i.e. equation (3.2)). In his analysis, Gurson (see [10, 11]) assumed that the coupled effects between the mean strain rate D_m and D' (i.e. the cross-term $D_m(b/r)^3\left(D'_1x_1^2 + D'_2x_2^2 + D'_3x_3^2\right)$ in the expression of $\pi_{\text{Mises}}(\mathbf{d})$ given by equation (3.4)) can be neglected, i.e.

$$\pi(\mathbf{d})^{\text{Mises}} \simeq \sigma_T\sqrt{4D_m^2(b/r)^6 + (2/3)R^2}.$$

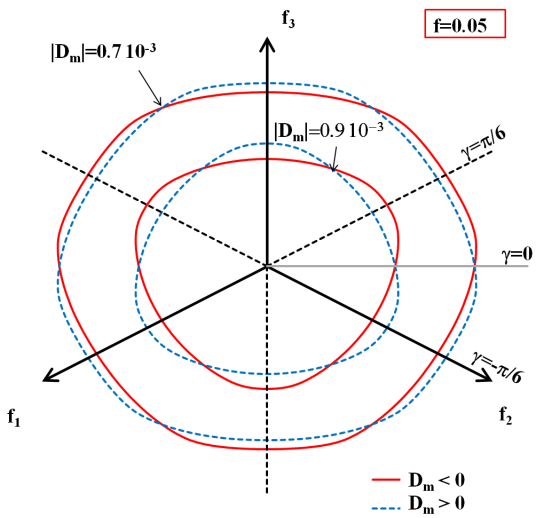

Figure 9. Shape of the cross-sections revealed by the evolution of $R(\gamma)$ (normalized by $R(-\frac{\pi}{6})$ corresponding to several deviatoric planes $D_m = constant$ for the porous Tresca material. Initial porosity: $f = 0.05$.

With this approximation, the SRP of a porous von Mises solid takes the following expression:

$$\Pi_{\text{Gurson}}(\mathbf{D}, f) = 2|D_m| \left[\begin{array}{l} \dfrac{\sqrt{1+6D_m^2/R^2} - \sqrt{f^2+6D_m^2/R^2}}{(D_m/R)\sqrt{6}} + \\ + \ln\left(\dfrac{1}{f} \cdot \dfrac{(D_m/R)\sqrt{6} + \sqrt{f^2+6D_m^2/R^2}}{(D_m/R)\sqrt{6} + \sqrt{1+6D_m^2/R^2}} \right) \end{array} \right]. \tag{4.1}$$

Note that $\Pi_{\text{Gurson}}(\mathbf{D}, f)$ is the exact dual of the classic stress-based potential of Gurson (see [11]).

For axisymmetric states, Cazacu et al. have shown in [6] that if the same simplifying hypothesis considered by Gurson in [10, 11] is made when evaluating the local plastic dissipation associated to Tresca's criterion given by equation (3.7), the truncated expression of the overall plastic dissipation at which one arrives coincides with $\Pi_{\text{Gurson}}(\mathbf{D}, f)$ given by equation (4.1). This means that neglecting couplings between the mean strain rate and D' (normal and shear effects) the specificities of the plastic flow of the matrix are erased. It is worth comparing the strain-rate potential for a porous Mises material obtained by Gurson in [10] (i.e. equation (4.1)) with the exact 3-D strain-rate potential for a porous Mises material given by equation (3.3), and the strain-rate potential for a porous Tresca material given by equation (3.6). Note that:

Since Gurson's SRP was obtained by truncating the overall plastic dis-

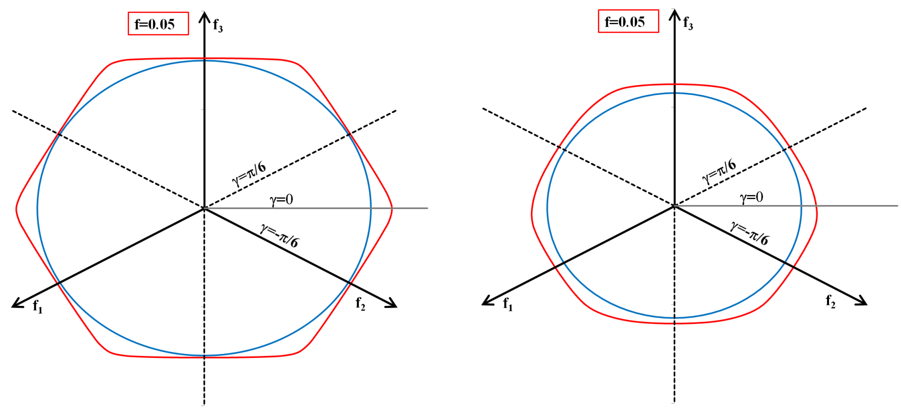

Figure 10. Comparison between the shapes of the cross-sections of the exact SRP for a porous von Mises material (equation (3.3), blue solid line), and the SRP for a porous Tresca material (equation (3.7)) (red solid line) corresponding to the same porosity $f = 0.05$ and the same level of plastic energy $6.0 \cdot 10^{-3}$. The cross-sections correspond to: (a) $D_m = 2 \cdot 10^{-4}\mathrm{s}^{-1}$; (b) $D_m = 6 \cdot 10^{-4}\mathrm{s}^{-1}$. Initial porosity: $f = 0.05$. Note that of the two potentials, the porous Tresca SRP is the least dissipative (red solid line).

sipation (see equation (4.1)), it is necessarily interior to the exact SRP, which is Π^{+}_{Mises} (given by equation (3.3)). Irrespective of the level of D_m, Tresca's SRP is exterior to the surfaces corresponding to von Mises matrix. This means that Gurson's SRP is the most dissipative of the three SRP's, since in order to reach the same value of the plastic dissipation, the norm of the loading, $R(\gamma)$, is lower than for a porous Mises (equation (3.3)) or porous Tresca (equation (3.6)). On the other hand, Tresca's SRP is the least dissipative potential.

The noteworthy result revealed is the very strong influence of the plastic flow of the matrix on the response of a porous solid. If the matrix obeys the von Mises criterion the shape of the cross-sections of the porous solid changes very little as D_m increases, but if the matrix behavior is described by Tresca's criterion the shape of the cross-section evolves from a hexagon to a triangle with rounded corners. Although the difference between the yield surfaces of a porous solid with von Mises matrix and that with a Tresca matrix becomes less important with increasing D_m (see Fig.10), it strongly affects void evolution. This will be further examined in the next section.

5. Role of plastic flow of the matrix on void evolution

In this section, the influence of the yield criterion describing the plastic flow of the matrix on void evolution in the porous solid is investigated by comparing the predictions of the porous Tresca criterion and the porous von Mises criterion. Figs.11 show the predicted void evolution as a function of

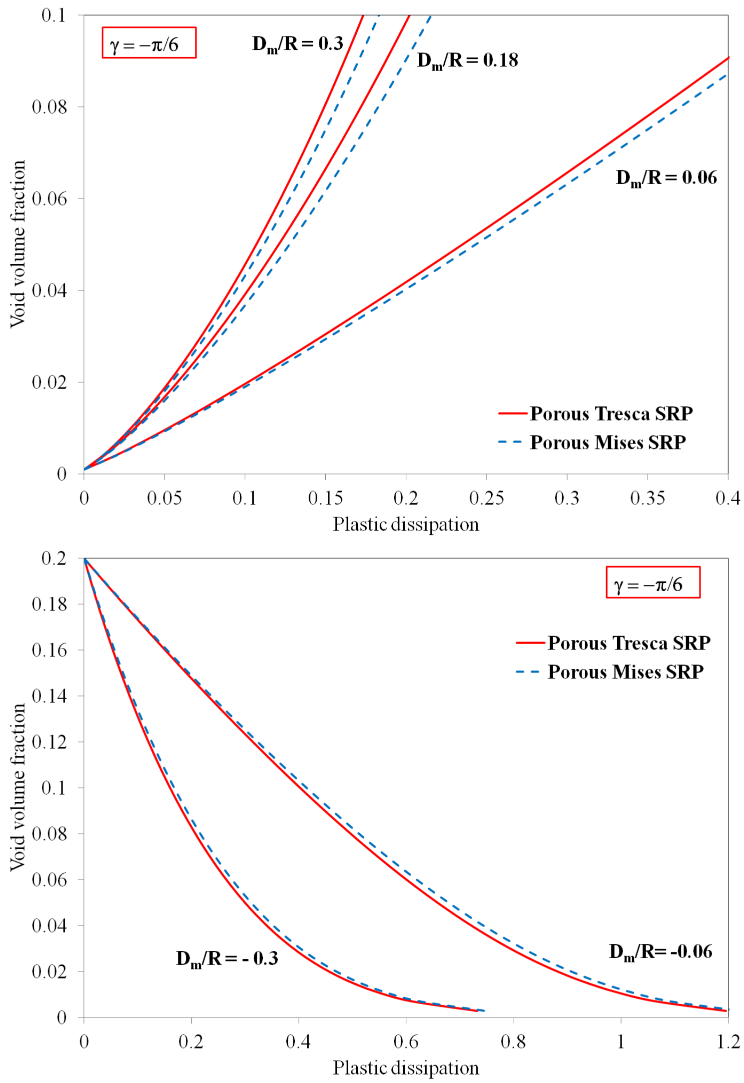

Figure 11. Void volume fraction evolution as a function of the plastic dissipation for axisymmetric loadings ($\gamma = -\pi/6$) such that the ratio between the mean strain rate D_m and $R(\gamma)$ is fixed: (a) loading paths at positive mean strain rate $D_m > 0$; initial porosity: f=0.001, (b) loading paths at negative mean strain rate $D_m > 0$; initial porosity: f=0.2.

the plastic dissipation corresponding to axisymmetric loadings such that the ratio $D_m/R(\gamma = -\pi/6)$ is fixed. Calculations are shown for both positive and negative mean strain-rate. The initial porosity is $f_0 = 0.001$. The rate of void growth (which corresponds to loadings at positive mean strain rate D_m) or collapse (which corresponds to loadings at negative mean strain rate D_m) is much faster in a porous solid with Tresca matrix than in a porous solid with von Mises matrix. The differences in the rate of void growth are significant. For example, at a plastic dissipation egal to 0.15, for a strain rate loading such that $D_m/R(\gamma = -\pi/6) = 0.18$, according to the porous von Mises f = 0.062 while according to the porous Tresca SRP, f =0.067. For a strain rate loading path defined by $D_m/R(\gamma = -\pi/6) = 0.18$, according to the porous von Mises f = 0.076 while according to the porous Tresca SRP, f =0.081.

Both the porous Mises SRP and the porous Tresca SRP involve a very specific dependence on the third invariant, J_{3D}, (or on the loading parameter γ), as shown in Fig. 5 and Fig. 8, respectively. As already mentionned, for a porous Mises solid, the maxiamal influence of the parameter γ is that between axisymmetric states, while for a porous Tresca solid, the manner in which the third invariant, J_{3D} influences the dilational response depends on both the level of the mean strain rate and that of the void volume fraction (see also Fig.13 and Table 2). As an example. Fig. 12 shows the evolution of the void volume fraction as a function of the plastic dissipation for a porous Tresca solid for loadings at a fixed strain ratio $D_m/R(\gamma) = 0.06$ for different values of the parameter $\gamma = \{-\pi/6, -\pi/12, 0, \pi/12, \pi/6\}$. For this type of loading, the rate of void growth is the fastest for shear strain-loading ($\gamma = 0$) while the lowest rate of void growth is obtained for $\gamma = \pi/6$ (that is axisymmetric loadings at $J_{3D} < 0$).

To further illustrate the dependence on the third invariant, J_{3D} of the dilational response of a porous Mises solid and a porous Tresca solid, Fig. 13 shows the plastic dissapation needed to reach a given void volume fraction for loadings at fixed strain ratio $D_m/R(\gamma)$ corresponding to different values of $\gamma = \{-\pi/6, -\pi/12, 0, \pi/12, \pi/6\}$ at fixed strain-rate loading D_m/R. If the mean-strain rate is positive (i.e. void growth), for a porous Mises solid, the plastic work that must be dissipated to reach a given void volume fraction wil be smallest for axisymmetric loadings with $J_{3D} > 0$ (i.e. $\gamma = -\pi/6$) and the largest one for axisymmetric loadings with $J_{3D} < 0$ (i.e. $\gamma = \pi/6$) (see also Table 1). This conclusion is consistent with that drawn previously (see Fig. 5). It is to be noted that the very specific couplings between the third invariant and the mean strain-rate (i.e. the centro-symmetry of the SRP) has strong consequences on void evolution. For negative mean strain-rate, for a porous Mises solid, the plastic dissipation necessary to reach the same void volume fraction will be the smallest for axisymmetric loadings with $J_{3D} > 0$ ($\gamma = -\pi/6$) and the largest for axisymmetric loadings with

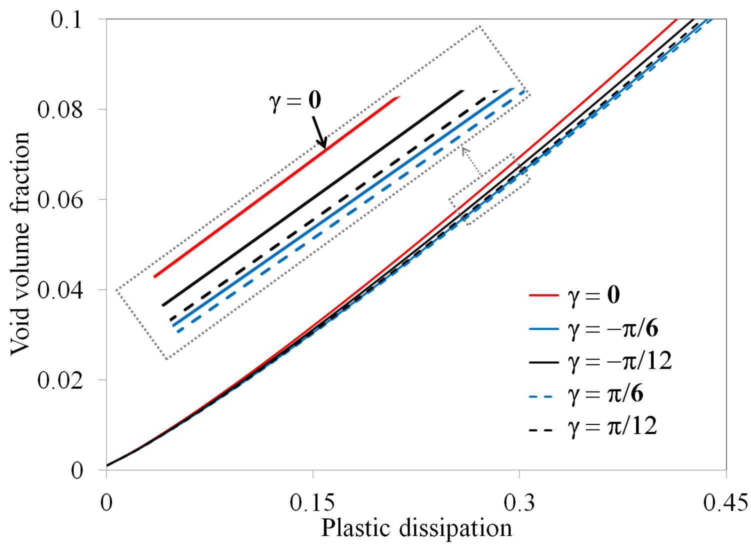

Figure 12. Evolution of the void volume fraction as a function of the plastic dissipation for a porous Tresca solid subjected to loadings at fixed strain ratio $D_m/R(\gamma) = 0.06$ for different values of the parameter $\gamma = \{-\pi/6, -\pi/12, 0, \pi/12, \pi/6\}$

$J_{3D} < 0$ $(\gamma = -\pi/6)$.

The same studies have been conducted for a porous Tresca solid (see Table 2). As already mentionned, for a porous Tresca solid, the influence of the third invariant J_3^D on the dilatational response depends on the initial void volume fraction and on the mean strain-rate D_m. As a example, for low mean strain-rate $D_m/R(\gamma) = 0.06$, the smallest and largest plastic dissipation necessary to reach a given void volume fraction are obtained for shear loading ($\gamma = 0$) and for axisymmetric loadings with $J_3^D < 0$ $(\gamma = \pi/6)$, respectively. On the other hand, for $D_m/R(\gamma) = 0.18$, the smallest and largest plastic dissipation necessary to reach the same given void volume fraction are obtained for axisymmetric loadings at $\gamma = -\pi/6$ and for loading at $\gamma = \pi/12$, respectively. It is to be noted that the centro-symmetry property of the porous Tresca SRP implies that if the sign of the mean strain-rate changes (i.e. $D_m/R(\gamma) = -0.18$) , the maximal and minimal plastic dissipation is now obtained for $\gamma = -\pi/12$ and $\gamma = \pi/6$, respectively.

6. Summary and Conclusions

The aim of this paper was to investigate the properties of the 3-D plastic potentials for porous solids with Tresca and von Mises, matrices respectively.

The role of the plastic flow of the matrix on the dilatational response was analyzed for general 3-D conditions for both compressive and tensile states.

Loading	Porous Mises SRP				
	$\gamma = -\pi/6$	$\gamma = -\pi/12$	$\gamma = 0$	$\gamma = \pi/12$	$\gamma = \pi/6$
$D_m/R = 0.06$	0.4533	0.4536	0.4542	0.4548	0.4550
$D_m/R = 0.18$	0.2370	0.2373	0.2380	0.2386	0.2389
$D_m/R = -0.06$	1.2229	1.2225	1.2214	1.2201	1.2195
$D_m/R = -0.18$	0.8059	0.8056	0.8046	0.8036	0.8031

Table 1. Plastic dissipation necessary to reach a given void volume fraction f for loadings at fixed strain ratio $D_m/R(\gamma)$ and for differents values of γ according to the porous von Mises SRP. For loadings at positive mean strain-rate, $f = 100f_0$ (with $f_0 = 0.001$); for loadings at negative mean strain-rate, $f = 0.01f_0$ (with $f_0 = 0.2$.

Loading	Porous Tresca SRP				
	$\gamma = -\pi/6$	$\gamma = -\pi/12$	$\gamma = 0$	$\gamma = \pi/12$	$\gamma = \pi/6$
$D_m/R = 0.06$	0.4384	0.4284	0.4163	0.4352	0.4417
$D_m/R = 0.18$	0.2217	0.2220	0.2258	0.2292	0.2254
$D_m/R = -0.06$	1.1950	1.1875	1.1546	1.1752	1.1887
$D_m/R = -0.18$	0.7850	0.7944	0.7893	0.7835	0.7794

Table 2. Plastic dissipation necessary to reach a given void volume fraction f for loadings at fixed strain ratio $D_m/R(\gamma)$ and for differents values of γ according to the porous Tresca SRP. For loadings at positive mean strain-rate, $f = 100f_0$ (with $f_0 = 0.001$); for loadings at negative mean strain-rate, $f = 0.01f_0$ (with $f_0 = 0.2$.

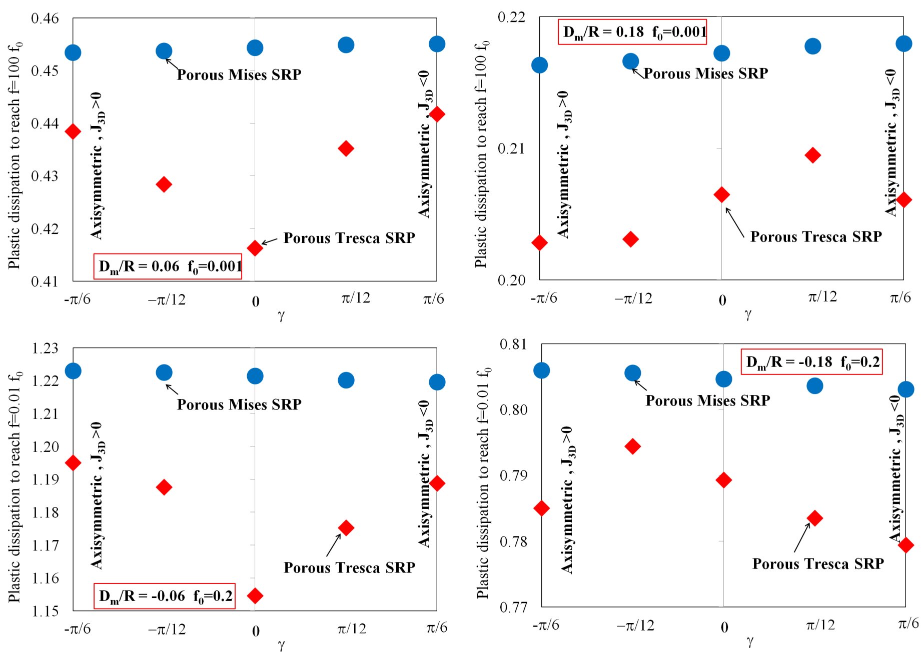

Figure 13. Plastic dissipation necessary to reach a given void volume fraction f for loadings at fixed strain ratio $D_m/R(\gamma)$ and for differents values of γ according to the porous von Mises SRP and the porous Tresca SRP. For loadings at positive mean strain-rate, $f = 100f_0$ (with $f_0 = 0.001$); for loadings at negative mean strain-rate, $f = 0.01f_0$ (with $f_0 = 0.2$.

It has been shown that if the matrix is described by the von Mises criterion:

- The plastic response of the porous solid depends on all three invariants of the strain-rate tensor D.
- The coupling between the invariants of $\mathbf{D'}$ i.e. between $R(=\sqrt{2J_{2D}})$ and γ (measure of $\sqrt{2J_{2D}}$ and J_{3D}) is very specific:
  - for $D_m > 0$, $R(\gamma)$ is a monotonically decreasing function of γ
  - for $D_m < 0$, $R(\gamma)$ is a monotonically increasing function of γ
- The strongest effect of the third-invariant is for axisymmetric states i.e. between $R(\gamma = \frac{\pi}{6})$ and $R(\gamma = -\frac{\pi}{6})$.

Therefore, the most influence of the parameter γ (or J_{3D}) on void growth or void collapse) occurs for axisymmetric states. It is very worth noting that the same conclusions concerning the influence of the third-invariant on void evolution were drawn by Rice and Tracey in [21] for the case of large hydrostatic stresses.

It has been shown that if the plastic behavior of the matrix is described by Tresca's criterion:

- The shapes of the cross-sections of the 3-D surfaces with deviatoric planes are strongly dependent on the level of D_m.

  As the absolute value of D_m increases, the shape changes from a regular hexagon ($D_m = 0$) to a triangle with rounded corners.
- The level of porosity is key in how fast the shape changes with the mean strain rate D_m. If the level of porosity is small, the cross-sections smooth out slower than in the case when the level of porosity in the matrix is higher.
- For $D_m = 0$ (i.e. Tresca behavior) the maximum of$R(\gamma)$ is at $\gamma = 0$ ($J_{3D} = 0$), the minima being for axisymmetric states.

  For $D_m > 0$: the maximum of $R(\gamma)$ is not at $\gamma = 0$ anymore, but shifts toward the axisymmetric case corresponding to $\gamma = -\frac{\pi}{6}$ ($D_1 = D_3 < D_2$ and $J_{3D} > 0$); on the other hand, the minimum of $R(\gamma)$ is always obtained for $\gamma = \frac{\pi}{6}$(axisymmetric state corresponding to $J_{3D} < 0$). The reverse holds true for $D_m < 0$.
- While in the case of the porous Mises solid, the most pronounced difference in the response is between the axisymmetric states (i.e. between $R(\gamma = \frac{\pi}{6})$ and $R(\gamma = -\frac{\pi}{6})$, for a porous Tresca no general conclusions can be drawn because the specific expression of $R(\gamma)$ depends both on the level of porosity and the level of D_m .

References

[1] J.L. Alves, B. Revil-Baudard and O. Cazacu, Importance of the coupling between the sign of the mean stress and the third invariant on the rate of void growth and collapse in porous solids with a von Mises matrix, *Model. Simul. Mater. Sc.*, **22**, *2* (2014), 025005, http://dx.doi.org/10.1088/0965-0393/22/2/025005.

[2] O. Cazacu, B. Plunkett and F. Barlat, Orthotropic yield criterion for hexagonal closed packed metals, *Int. J. Plasticity*, **22** (2006), 1171-1194, http://dx.doi.org/10.1016/j.ijplas.2005.06.001.

[3] O. Cazacu and J.B. Stewart, Plastic potentials for porous aggregates with the matrix exhibiting tension-compression asymmetry, *J. Mech. Phys. Solids*, **57** (2009), 325-341, http://dx.doi.org/10.1016/j.jmps.2008.10.010.

[4] O. Cazacu, I.R. Ionescu and J.W. Yoon, Orthotropic strain-rate potential for the description of anisotropy in tension and compression of metals, *Int. J. Plasticity*, **26** (2010), 887-904, http://dx.doi.org/10.1016/j.ijplas.2009.11.005.

[5] O. Cazacu, B. Revil-Baudard, R.A. Lebensohn and M. Garajeu, New analytic criterion describing the combined effect of pressure and third invariant on yielding of porous solids with von Mises matrix, *J. Appl. Mech.*, **80**, *6* (2013), 64501-1-64501-5, http://dx.doi.org/10.1115/1.4024074.

[6] O. Cazacu, B. Revil-Baudard, N. Chandola and D. Kondo, New analytical criterion for porous solids with Tresca matrix under axisymmetric loadings, *Int. J. Solids Struct.*, **51** (2014), 861-874, http://dx.doi.org/10.1016/j.ijsolstr.2013.11.010.

[7] O. Cazacu, N. Chandola, J. L. Alves and B. Revil-Baudard, Importance of the consideration of the specificities of local plastic deformation on the response of porous solids with Tresca matrix, *Eur. J. Mech. A Solids*, **47** (2014), 194-205, http://dx.doi.org/10.1016/j.euromechsol.2014.04.004.

[8] D. Drucker, Relation of experiments to mathematical theories of plasticity, *J. Appl. Mech.*, **16** (1949), 349-357.

[9] M. Gologanu, *Etude de quelques problemes de rupture ductile des metaux*, Ph.D. Thesis, Univ. Pierre and Marie Curie, Paris, France, 1997.

[10] A. Gurson, *Plastic flow and fracture behavior of ductile materials incorporating void nucleation, growth, and interaction*, PhD Thesis, Brown University, Rhode Island, 1975.

[11] A.L. Gurson, Continuum theory of ductile rupture by void nucleation and growth. Part I: Yield criteria and flow rules for porous ductile media, *J. Eng. Mater.-T. ASME, Series H*, **99** (1977), 2-15, http://dx.doi.org/10.1115/1.3443401.

[12] R. Hill, A theory of yielding and plastic flow of anisotropic metals, *Proc. R. Soc. Lond. Ser. A*, **193** (1948), 281-297, http://dx.doi.org/10.1098/rspa.1948.0045.

[13] R. Hill, The essential structure of constitutive laws for metal composites and polycrystals, *J. Mech. Phys. Solids*, **15** (1967), 79-95, http://dx.doi.org/10.1016/0022-5096(67)90018-x.

[14] R. Hill, Constitutive dual potentials in classical plasticity, *J. Mech. Phys. Solids*, **35** (1987), 23-33, http://dx.doi.org/10.1016/0022-5096(87)90025-1.

[15] R.A. Lebensohn and O. Cazacu, Effect of single-crystal plastic deformation mechanisms on the dilatational plastic response of porous polycrystals, *Int. J. Solids Struct.*, **49** (2012), 3838-3852, http://dx.doi.org/10.1016/j.ijsolstr.2012.08.019 .

[16] J. Lubliner, *Plasticity theory*, Dover Publications Inc., Mineola, New York, 2008.

[17] J. Mandel, *Plasticite classique et viscoplasticite*, Int. Centre Mech Sci., Courses and lectures, 97, Udine 1971, Springer, Wien, New York, 1972.

[18] B. Revil-Baudard and O. Cazacu, Role of the plastic flow of the matrix on yielding and void evolution of porous solids: comparison between the theoretical response of porous solids with Tresca and von Mises matrices, *Mech. Res. Commun.*, **56** (2014), 69-75, http://dx.doi.org/10.1016/j.mechrescom.2013.11.008.

[19] B. Revil-Baudard and O. Cazacu, New three-dimensional strain-rate potentials for isotropic porous metals: role of the plastic flow of the matrix, *Int. J. Plasticity*, **60** (2014), 101-117, http://dx.doi.org/10.1016/j.ijplas.2014.04.003.

[20] A.B. Richelsen and V. Tvergaard, Dilatant plasticity or upper bound estimates for porous ductile solids, *Acta Metall. Mater.*, **42** (1994), 2561-2577, http://dx.doi.org/10.1016/0956-7151(94)90198-8.

[21] J.R. Rice and D.M. Tracey, On the ductile enlargement of voids in triaxial stress fields, *J. Mech. Phys. Solids*, **17** (1969), 201-217, http://dx.doi.org/10.1016/0022-5096(69)90033-7.

[22] J. Salencon, *Yield design and limit analysis*, Eds. Presses de l'ENPC, Paris, 1983.

[23] D.R.S. Talbot and J.R. Willis, Variational principles for inhomogeneous non-linear media, *IMA J. Appl. Math.*, **35**, *1* (1985), 39-54, http://dx.doi.org/10.1093/imamat/35.1.39.

[24] V. Tvergaard, Influence of voids on shear band instabilities under plane strain conditions, *Int. J. Fracture*, **17** (1981), 389-407, http://dx.doi.org/10.1007/bf00036191.

[25] P. Van Houtte and A. Van Bael, Convex plastic potentials of fourth and sixth rank for anisotropic materials, *Int. J. Plasticity*, **20** (2004), 1505-1524, http://dx.doi.org/10.1016/j.ijplas.2003.11.005.

[26] J.H. Yoon, O. Cazacu and J.W. Yoon, Strain rate potential based elastic/plastic anisotropic model for metals displaying tension-compression asymmetry, *Comput. Methods Appl. Mech. Engrg.*, **200** (2011), 1993-2004, http://dx.doi.org/10.1016/j.cma.2011.03.003.

[27] J.W. Yoon, Y. Lou, J.H. Yoon and M.V. Glazov, Asymmetric yield functions based on stress invariant for pressure-sensitive materials, *Int. J. Plasticity*, **56** (2014), 184-202, http://dx.doi.org/10.1016/j.ijplas.2013.11.008.

[28] H. Ziegler, *An introduction to thermodynamics*, North-Holland, Amsterdam, 1977.

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