

Relations between meromorphic solutions and their derivatives of differential equations and small functions

BENHARRAT BELAÏDI AND HABIB HABIB

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Abstract - In this paper, we investigate relations between solutions, their derivatives of the differential equation

$$f^{(k)} + B_{k-1}e^{b_{k-1}z}f^{(k-1)} + \dots + B_1e^{b_1z}f' + (A_1e^{a_1z} + A_2e^{a_2z})f = 0,$$

and functions of small growth, where $A_j(z) (\neq 0)$ ($j = 1, 2$), $B_l(z) (\neq 0)$ ($l = 1, \dots, k-1$) are meromorphic functions of finite order and b_l ($l = 1, \dots, k-1$), a_j ($j = 1, 2$) are complex constants. We prove that every meromorphic solution $f \neq 0$ to above differential equation whose poles are of uniformly bounded multiplicities and its first and second derivative have infinitely many fixed points. Our results extend the previous results due to Chen and Shon, Peng and Chen.

Key words and phrases : linear differential equations, meromorphic solutions, order of growth, fixed points.

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1. Introduction and statement of results

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions (see [10], [15]). In addition, we will use notations $\sigma(f)$, $\sigma_2(f)$ to denote respectively the order and the hyper-order of growth of a meromorphic function $f(z)$, $\lambda(f)$, $\bar{\lambda}(f)$, to denote respectively the exponents of convergence of the zero-sequence and the sequence of distinct zeros of $f(z)$. See ([2], [10], [13], [15]) for notations and definitions.

To give estimates of fixed points, we define:

Definition 1.1. (see [2], [13], [15]) *Let f be a meromorphic function and let z_1, z_2, \dots , ($|z_j| = r_j$, $0 < r_1 \leq r_2 \leq \dots$) be the sequence of the distinct*

fixed points of f . The exponent of convergence of the sequence of distinct fixed points of $f(z)$ is defined by

$$\bar{\tau}(f) = \inf \left\{ \tau > 0 : \sum_{j=1}^{+\infty} |z_j|^{-\tau} < +\infty \right\}.$$

Clearly,

$$\bar{\tau}(f) = \limsup_{r \rightarrow +\infty} \frac{\log \bar{N} \left(r, \frac{1}{f-z} \right)}{\log r},$$

where $\bar{N} \left(r, \frac{1}{f-z} \right)$ is the integrated counting function of distinct fixed points of $f(z)$ in $\{z : |z| \leq r\}$.

Consider the second order linear differential equation

$$f'' + A_1(z) e^{P(z)} f' + A_0(z) e^{Q(z)} f = 0, \quad (1.1)$$

where $P(z), Q(z)$ are nonconstant polynomials, $A_1(z), A_0(z) (\neq 0)$ are entire functions such that $\sigma(A_1) < \deg P(z)$, $\sigma(A_0) < \deg Q(z)$. Gundersen showed in [8, p. 419] that if $\deg P(z) \neq \deg Q(z)$, then every nonconstant solution of (1.1) is of infinite order. If $\deg P(z) = \deg Q(z)$, then (1.1) may have nonconstant solutions of finite order. For instance $f(z) = e^z + 1$ satisfies $f'' + e^z f' - e^z f = 0$.

In [3], Chen and Shon have investigated the case when $\deg P(z) = \deg Q(z)$ and further proved the following results.

Theorem A (see [3]) *Let $A_j(z) (\neq 0)$ ($j = 0, 1$) be meromorphic functions with $\sigma(A_j) < 1$ ($j = 0, 1$), a, b be nonzero complex numbers such that $\arg a \neq \arg b$ or $a = cb$ ($0 < c < 1$). Then every meromorphic solution $f(z) \neq 0$ that satisfies the equation*

$$f'' + A_1(z) e^{az} f' + A_0(z) e^{bz} f = 0 \quad (1.2)$$

has infinite order.

In the same paper, Chen and Shon investigated the fixed points of solutions, their 1st and 2nd derivatives, and the differential polynomials and obtained the following result.

Theorem B (see [3]) *Let $A_j(z)$ ($j = 0, 1$), a, b, c satisfy the additional hypotheses of Theorem A. Let d_0, d_1, d_2 be complex constants that are not all equal to zero. If $f(z) \neq 0$ is any meromorphic solution of equation (1.2), then:*

(i) f, f', f'' all have infinitely many fixed points and satisfy

$$\bar{\lambda}(f - z) = \bar{\lambda}(f' - z) = \bar{\lambda}(f'' - z) = \infty,$$

(ii) the differential polynomial

$$g(z) = d_2 f'' + d_1 f' + d_0 f$$

has infinitely many fixed points and satisfies $\bar{\lambda}(g - z) = \infty$.

Recently in [12], Peng and Chen have investigated the order and hyper-order of solutions of some second order linear differential equations and proved the following result.

Theorem C (see [12]) *Let $A_j(z)$ ($\neq 0$) ($j = 1, 2$) be entire functions with $\sigma(A_j) < 1$, a_1, a_2 be distinct nonzero complex numbers (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or $a_1 < -1$, then every solution f ($\neq 0$) of the equation*

$$f'' + e^{-z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0$$

has infinite order and $\sigma_2(f) = 1$.

In [9], the authors have investigated the order and hyper-order of solutions of higher order linear differential equations with entire coefficients and further proved the following result.

Theorem D (see [9]) *Let $A_j(z)$ ($\neq 0$) ($j = 1, 2$), $B_l(z)$ ($\neq 0$) ($l = 1, \dots, k-1$), D_m ($m = 0, \dots, k-1$) be entire functions satisfying the condition $\max\{\sigma(A_j), \sigma(B_l), \sigma(D_m)\} < 1$, b_l ($l = 1, \dots, k-1$) be complex constants such that (i) $\arg b_l = \arg a_1$ and $b_l = c_l a_1$ ($0 < c_l < 1$) ($l \in I_1$) and (ii) b_l is a real constant such that $b_l \leq 0$ ($l \in I_2$), where $I_1 \neq \emptyset$, $I_2 \neq \emptyset$, $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, 2, \dots, k-1\}$, and a_1, a_2 are complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or a_1 is a real number such that $a_1 < \frac{b}{1-c}$ where $c = \max\{c_l : l \in I_1\}$ and $b = \min\{b_l : l \in I_2\}$, then every solution f ($\neq 0$) of the equation*

$$\begin{aligned} f^{(k)} + \left(D_{k-1} + B_{k-1} e^{b_{k-1} z}\right) f^{(k-1)} + \dots + \left(D_1 + B_1 e^{b_1 z}\right) f' \\ + (D_0 + A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0 \end{aligned}$$

satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$.

The main purpose of this paper is to extend and improve the results of Theorem C to higher order linear differential equations. In fact we will prove the following results.

Theorem 1.1 *Let $A_j(z)$ ($\neq 0$) ($j = 1, 2$), $B_l(z)$ ($\neq 0$) ($l = 1, \dots, k-1$) be meromorphic functions with*

$$\max \{ \sigma(A_j) \ (j = 1, 2), \sigma(B_l) \ (l = 1, \dots, k-1) \} < 1,$$

b_l ($l = 1, \dots, k-1$) be complex constants such that (i) $b_l = c_l a_1$ ($0 < c_l < 1$) ($l \in I_1$) and (ii) b_l is a real constant such that $b_l < 0$ ($l \in I_2$), where $I_1 \neq \emptyset$, $I_2 \neq \emptyset$, $I_1 \cap I_2 = \emptyset$, $I_1 \cup I_2 = \{1, 2, \dots, k-1\}$, and a_1, a_2 are complex numbers such that $a_1 a_2 \neq 0$, $a_1 \neq a_2$ (suppose that $|a_1| \leq |a_2|$). If $\arg a_1 \neq \pi$ or a_1 is a real number such that $a_1 < \frac{b}{1-c}$ where $c = \max \{c_l, l \in I_1\}$ and $b = \min \{b_l, l \in I_2\}$, then every meromorphic solution f ($\neq 0$) whose poles are of uniformly bounded multiplicities of the equation

$$f^{(k)} + B_{k-1} e^{b_{k-1}z} f^{(k-1)} + \dots + B_1 e^{b_1 z} f' + (A_1 e^{a_1 z} + A_2 e^{a_2 z}) f = 0 \quad (1.3)$$

satisfies $\sigma(f) = +\infty$ and $\sigma_2(f) = 1$.

Theorem 1.2 *Let $A_j(z)$ ($j = 1, 2$), $B_l(z)$ ($l = 1, \dots, k-1$), a_1, a_2, b_l ($l = 1, \dots, k-1$) satisfy the additional hypotheses of Theorem 1.1. If φ ($\neq 0$) is a meromorphic function with order $\sigma(\varphi) < 1$, then every meromorphic solution f ($\neq 0$) whose poles are of uniformly bounded multiplicities of equation (1.3) satisfies*

$$\bar{\lambda}(f - \varphi) = \bar{\lambda}(f' - \varphi) = \bar{\lambda}(f'' - \varphi) = \infty.$$

Remark 1.1 In order to prove Theorem 1.2, we establish two Lemmas 2.12-2.13 from linear algebra, and use them to prove that the equations (4.7) and (4.21) are non-homogeneous equations.

Setting $\varphi(z) = z$ in Theorem 1.2, we obtain the following corollary.

Corollary 1.1 *Let $A_j(z)$ ($j = 1, 2$), $B_l(z)$ ($l = 1, \dots, k-1$), a_1, a_2, b_l ($l = 1, \dots, k-1$) satisfy the additional hypotheses of Theorem 1.1. If f ($\neq 0$) is any meromorphic solution whose poles are of uniformly bounded multiplicities of equation (1.3), then f, f', f'' all have infinitely many fixed points and satisfy*

$$\bar{\tau}(f) = \bar{\tau}(f') = \bar{\tau}(f'') = \infty.$$

2. Preliminary lemmas

We define the linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_0^{+\infty} \chi_E(t) dt$ and the logarithmic measure of a set $F \subset (1, +\infty)$ by $lm(F) = \int_1^{+\infty} \frac{\chi_F(t)}{t} dt$, where χ_H is the characteristic function of a set H .

Lemma 2.1 (see [7]) *Let f be a transcendental meromorphic function with $\sigma(f) = \sigma < +\infty$. Let $\varepsilon > 0$ be a given constant, and let k, j be integers satisfying $k > j \geq 0$. Then, there exists a set $E_1 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ with linear measure zero, such that, if $\psi \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E_1$, then there is a constant $R_0 = R_0(\psi) > 1$, such that for all z satisfying $\arg z = \psi$ and $|z| \geq R_0$, we have*

$$\left| \frac{f^{(k)}(z)}{f^{(j)}(z)} \right| \leq |z|^{(k-j)(\sigma-1+\varepsilon)}.$$

Lemma 2.2 (see [3], [11]) *Consider $g(z) = A(z)e^{az}$, where $A(z) \not\equiv 0$ is a meromorphic function with order $\sigma(A) = \alpha < 1$, a is a complex constant, $a = |a|e^{i\varphi}$ ($\varphi \in [0, 2\pi)$). Set $E_2 = \{\theta \in [0, 2\pi) : \cos(\varphi + \theta) = 0\}$, then E_2 is a finite set. Then for any given ε ($0 < \varepsilon < 1 - \alpha$) there is a set $E_3 \subset [0, 2\pi)$ that has linear measure zero such that if $z = re^{i\theta}$, $\theta \in [0, 2\pi) \setminus (E_2 \cup E_3)$, then we have when r is sufficiently large:*

(i) *If $\cos(\varphi + \theta) > 0$, then*

$$\exp\{(1 - \varepsilon)\delta(az, \theta)r\} \leq |g(z)| \leq \exp\{(1 + \varepsilon)\delta(az, \theta)r\}.$$

(ii) *If $\cos(\varphi + \theta) < 0$, then*

$$\exp\{(1 + \varepsilon)\delta(az, \theta)r\} \leq |g(z)| \leq \exp\{(1 - \varepsilon)\delta(az, \theta)r\},$$

where $\delta(az, \theta) = |a|\cos(\varphi + \theta)$.

Lemma 2.3 (see [12]) *Suppose that $n \geq 1$ is a natural number. Let $P_j(z) = a_{jn}z^n + \dots$ ($j = 1, 2$) be nonconstant polynomials, where a_{jq} ($q = 1, \dots, n$) are complex numbers and $a_{1n}a_{2n} \neq 0$. Set $z = re^{i\theta}$, $a_{jn} = |a_{jn}|e^{i\theta_j}$, $\theta_j \in [-\frac{\pi}{2}, \frac{3\pi}{2})$, $\delta(P_j, \theta) = |a_{jn}|\cos(\theta_j + n\theta)$, then there is a set $E_4 \subset [-\frac{\pi}{2n}, \frac{3\pi}{2n})$ that has linear measure zero. If $\theta_1 \neq \theta_2$, then there exists a ray $\arg z = \theta$, $\theta \in (-\frac{\pi}{2n}, \frac{\pi}{2n}) \setminus (E_4 \cup E_5)$, such that*

$$\delta(P_1, \theta) > 0, \delta(P_2, \theta) < 0$$

or

$$\delta(P_1, \theta) < 0, \delta(P_2, \theta) > 0,$$

where $E_5 = \{\theta \in [-\frac{\pi}{2n}, \frac{3\pi}{2n}) : \delta(P_j, \theta) = 0\}$ is a finite set, which has linear measure zero.

Remark 2.1 (see [12]) We can obtain, in Lemma 2.3, if $\theta \in (-\frac{\pi}{2n}, \frac{\pi}{2n}) \setminus (E_4 \cup E_5)$ is replaced by $\theta \in (\frac{\pi}{2n}, \frac{3\pi}{2n}) \setminus (E_4 \cup E_5)$, the same result.

Lemma 2.4 (see [3]) *Let $f(z)$ be a transcendental meromorphic function of order $\sigma(f) = \alpha < +\infty$. Then for any given $\varepsilon > 0$, there is a set $E_6 \subset [-\frac{\pi}{2}, \frac{3\pi}{2})$ that has linear measure zero such that if $\theta \in [-\frac{\pi}{2}, \frac{3\pi}{2}) \setminus E_6$, then*

there is a constant $R_1 = R_1(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_1$, we have

$$\exp\{-r^{\alpha+\varepsilon}\} \leq |f(z)| \leq \exp\{r^{\alpha+\varepsilon}\}.$$

Using mathematical induction, we can easily prove the following lemma.

Lemma 2.5 *Let $f(z) = g(z)/d(z)$, where $g(z)$ is a transcendental entire function, and let $d(z)$ be the canonical product (or polynomial) formed with the non-zero poles of $f(z)$. Then we have*

$$f^{(n)} = \frac{1}{d} \left[g^{(n)} + D_{n,n-1}g^{(n-1)} + D_{n,n-2}g^{(n-2)} + \cdots + D_{n,1}g' + D_{n,0}g \right]$$

and

$$\frac{f^{(n)}}{f} = \frac{g^{(n)}}{g} + D_{n,n-1}\frac{g^{(n-1)}}{g} + D_{n,n-2}\frac{g^{(n-2)}}{g} + \cdots + D_{n,1}\frac{g'}{g} + D_{n,0},$$

where $D_{n,j}$ are defined as a sum of finite numbers of terms of the type

$$\sum_{(j_1 \cdots j_n)} C_{jj_1 \cdots j_n} \left(\frac{d'}{d} \right)^{j_1} \cdots \left(\frac{d^{(n)}}{d} \right)^{j_n},$$

$C_{jj_1 \cdots j_n}$ are constants, and $j + j_1 + 2j_2 + \cdots + nj_n = n$.

Lemma 2.6 (see [1]) *Let $A_0, A_1, \dots, A_{k-1}, F \not\equiv 0$ be finite order meromorphic functions. If $f(z)$ is an infinite order meromorphic solution of the equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \cdots + A_1f' + A_0f = F,$$

then f satisfies $\bar{\lambda}(f) = \lambda(f) = \sigma(f) = \infty$.

The following lemma, due to Gross (see [6]), is important in the factorization and uniqueness theory of meromorphic functions, playing an important role in this paper as well.

Lemma 2.7 (see [6], [15]) *Suppose that $f_1(z), f_2(z), \dots, f_n(z)$ ($n \geq 2$) are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

- (i) $\sum_{j=1}^n f_j(z) e^{g_j(z)} \equiv 0$;
- (ii) $g_j(z) - g_k(z)$ are not constants for $1 \leq j < k \leq n$;
- (iii) For $1 \leq j \leq n$, $1 \leq h < k \leq n$, $T(r, f_j) = o\{T(r, e^{g_h(z) - g_k(z)})\}$ ($r \rightarrow \infty, r \notin E_7$), where E_7 is a set with finite linear measure.

Then $f_j(z) \equiv 0$ ($j = 1, \dots, n$).

Lemma 2.8 (see [14]) *Suppose that $f_1(z), f_2(z), \dots, f_n(z)$ ($n \geq 2$) are meromorphic functions and $g_1(z), g_2(z), \dots, g_n(z)$ are entire functions satisfying the following conditions:*

(i) $f_{n+1}(z) := \sum_{j=1}^n f_j(z) e^{g_j(z)}$;

(ii) *If $1 \leq j \leq n+1, 1 \leq k \leq n$, the order of f_j is less than the order of $e^{g_k(z)}$. If $n \geq 2, 1 \leq j \leq n+1, 1 \leq h < k \leq n$, and the order of f_j is less than the order of $e^{g_h - g_k}$.*

Then $f_j(z) \equiv 0$ ($j = 1, 2, \dots, n+1$).

Lemma 2.9 (see [7]) *Let $f(z)$ be a transcendental meromorphic function, and let $\alpha > 1$ be a given constant. Then there exist a set $E_8 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on α and i, j ($0 \leq i < j \leq k$), such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, we have*

$$\left| \frac{f^{(j)}(z)}{f^{(i)}(z)} \right| \leq B \left\{ \frac{T(\alpha r, f)}{r} (\log^\alpha r) \log T(\alpha r, f) \right\}^{j-i}.$$

Lemma 2.10 (see [8]) *Let $\varphi : [0, +\infty) \rightarrow \mathbb{R}$ and $\psi : [0, +\infty) \rightarrow \mathbb{R}$ be monotone non-decreasing functions such that $\varphi(r) \leq \psi(r)$ for all $r \notin E_9 \cup [0, 1]$, where $E_9 \subset (1, +\infty)$ is a set of finite logarithmic measure. Let $\gamma > 1$ be a given constant. Then there exists an $r_1 = r_1(\gamma) > 0$ such that $\varphi(r) \leq \psi(\gamma r)$ for all $r > r_1$.*

Lemma 2.11 (see [4]) *Let $k \geq 2$ and A_0, A_1, \dots, A_{k-1} be meromorphic functions. Let $\sigma = \max\{\sigma(A_j), j = 0, \dots, k-1\}$ and assume that all poles of f are of uniformly bounded multiplicity. Then every transcendental meromorphic solution f of the differential equation*

$$f^{(k)} + A_{k-1}f^{(k-1)} + \dots + A_1f' + A_0f = 0$$

satisfies $\sigma_2(f) \leq \sigma$.

Lemma 2.12 *Let a_1, a_2 be two complex numbers such that $a_1 a_2 \neq 0$ and $a_1 \neq a_2$. Let d_i ($i = 1, \dots, p$), d'_j ($j = 1, \dots, q$) be real constants, where $d_i < 0$ and $d'_j < 0$. Let $n \geq 2$ be an integer and $\alpha, \beta, \alpha', \beta', \gamma_i$ ($i = 1, \dots, p$), γ'_j ($j = 1, \dots, q$) be real numbers such that $\alpha \geq 0, \beta \geq 0, \alpha' \geq 0, \beta' \geq 0, \gamma_i \geq 0, \gamma'_j \geq 0, (\alpha, \beta) \neq (0, 0), (\alpha', \beta') \neq (0, 0), 0 < \alpha + \beta + \gamma_1 + \dots + \gamma_p \leq n; 0 < \alpha' + \beta' + \gamma'_1 + \dots + \gamma'_q \leq n$ and $\max\{\alpha, \beta, \alpha', \beta'\} < n$. If $\arg a_1 \neq \pi$ or $a_1 < d$, where $d = \min\{d_i \ (i = 1, \dots, p), d'_j \ (j = 1, \dots, q)\}$ and $na_1 = \alpha a_1 + \beta a_2 + \gamma_1 d_1 + \dots + \gamma_p d_p$, then $na_2 \neq \alpha' a_1 + \beta' a_2 + \gamma'_1 d'_1 + \dots + \gamma'_q d'_q$.*

Proof. Suppose that $na_2 = \alpha'a_1 + \beta'a_2 + \gamma'_1d'_1 + \cdots + \gamma'_qd'_q$, then we have the system

$$\begin{cases} na_1 = \alpha a_1 + \beta a_2 + \gamma_1 d_1 + \cdots + \gamma_p d_p, \\ na_2 = \alpha' a_1 + \beta' a_2 + \gamma'_1 d'_1 + \cdots + \gamma'_q d'_q. \end{cases} \quad (2.1)$$

Since $d \leq d_i$ ($i = 1, \dots, p$) and $d \leq d'_j$ ($j = 1, \dots, q$), then there exist constants c_i ($0 < c_i \leq 1$) and c'_j ($0 < c'_j \leq 1$) such that $d_i = c_i d$ and $d'_j = c'_j d$. The system (2.1) becomes

$$\begin{cases} (n - \alpha) a_1 - \beta a_2 = \gamma d, \\ -\alpha' a_1 + (n - \beta') a_2 = \gamma' d, \end{cases} \quad (2.2)$$

where $\gamma = \gamma_1 c_1 + \cdots + \gamma_p c_p$ and $\gamma' = \gamma'_1 c'_1 + \cdots + \gamma'_q c'_q$. Set $\delta = \alpha + \beta + \gamma$ and $\delta' = \alpha' + \beta' + \gamma'$. We can see that

$$\delta = \alpha + \beta + \gamma_1 c_1 + \cdots + \gamma_p c_p \leq \alpha + \beta + \gamma_1 + \cdots + \gamma_p \leq n$$

and

$$\delta' = \alpha' + \beta' + \gamma'_1 c'_1 + \cdots + \gamma'_q c'_q \leq \alpha' + \beta' + \gamma'_1 + \cdots + \gamma'_q \leq n.$$

The determinant Δ of the system (2.2) is defined as follows

$$\Delta = \begin{vmatrix} n - \alpha & -\beta \\ -\alpha' & n - \beta' \end{vmatrix} = (n - \alpha)(n - \beta') - \alpha'\beta. \quad (2.3)$$

Case 1: If $\alpha'\beta = 0$, then $\Delta = (n - \alpha)(n - \beta') > 0$.

Case 2: If $\alpha'\beta \neq 0$, i.e., $\alpha' \neq 0$ and $\beta \neq 0$. By $\delta \leq n$ and $\delta' \leq n$, we have $n - \alpha \geq \beta + \gamma$ and $n - \beta' \geq \alpha' + \gamma'$. Thus $\Delta \geq S$, where $S = (\beta + \gamma)(\alpha' + \gamma') - \alpha'\beta$.

Subcase 2.1: If $\gamma \neq 0$ and $\gamma' \neq 0$ or $\gamma = 0$ and $\gamma' \neq 0$ or $\gamma \neq 0$ and $\gamma' = 0$, then $S > 0$. Hence $\Delta > 0$.

Subcase 2.2: $\gamma = \gamma' = 0$. Hence $\delta = \alpha + \beta$ and $\delta' = \alpha' + \beta'$.

i) If $\delta < n$ and $\delta' < n$, then $n - \alpha > \beta$ and $n - \beta' > \alpha'$. By this and (2.3), we get $\Delta = (n - \alpha)(n - \beta') - \alpha'\beta > 0$.

ii) If $\delta = n$ and $\delta' < n$, then $n - \alpha = \beta$ and $n - \beta' > \alpha'$. By this and (2.3), we get $\Delta = \beta(n - \beta') - \alpha'\beta > \beta\alpha' - \alpha'\beta = 0$. Thus $\Delta > 0$.

iii) If $\delta < n$ and $\delta' = n$, then $n - \alpha > \beta$ and $n - \beta' = \alpha'$. By this and (2.3), we get $\Delta = (n - \alpha)\alpha' - \alpha'\beta > \beta\alpha' - \alpha'\beta = 0$. Thus $\Delta > 0$.

iiii) If $\delta = n$ and $\delta' = n$, then $n - \alpha = \beta$ and $n - \beta' = \alpha'$. By this and (2.3), we get $\Delta = 0$.

a) For all cases when $\Delta > 0$, by (2.2), we can get $a_1 = Ld$, where

$$L = \frac{\gamma(n - \beta') + \beta\gamma'}{\Delta}.$$

We know that $\gamma(n - \beta') + \beta\gamma' \geq 0$. By using $\gamma = \delta - \alpha - \beta$ and $\gamma' = \delta' - \alpha' - \beta'$, we can obtain

$$\begin{aligned} \gamma(n - \beta') + \beta\gamma' - \Delta &= (n - \beta')\delta - n\beta + \beta\delta' + n\beta' - n^2 \\ &\leq n(n - \beta') - n\beta + n\beta + n\beta' - n^2 = 0. \end{aligned}$$

Thus $0 \leq \gamma(n - \beta') + \beta\gamma' \leq \Delta$, hence $0 \leq L \leq 1$.

(i) When $L = 0$, we have $a_1 = 0$, which is a contradiction.

(ii) When $0 < L \leq 1$, we have $a_1 = Ld$. If $\arg a_1 \neq \pi$ or $a_1 < d$, then $a_1 \neq cd$ ($0 < c \leq 1$). Therefore $a_1 = Ld$ is a contradiction.

b) For the Subcase 2.2 (iii) we have $\Delta = 0$, then by (2.2), we get $a_1 = a_2$, which is a contradiction. \square

Lemma 2.13 *Let a_1 be complex number such that $a_1 \neq 0$. Let further d_i ($i = 1, \dots, p$) be real constants such that $d_i < 0$. Let $n \geq 2$ be an integer and α, γ_i ($i = 1, \dots, p$) be real numbers such that $0 < \alpha < n$, $\gamma_i \geq 0$ and $0 < \alpha + \gamma_1 + \dots + \gamma_p \leq n$. If $\arg a_1 \neq \pi$ or $a_1 < d$ where $d = \min\{d_i : i = 1, \dots, p\}$, then $na_1 \neq \alpha a_1 + \gamma_1 d_1 + \dots + \gamma_p d_p$.*

Proof. Suppose that $na_1 = \alpha a_1 + \gamma_1 d_1 + \dots + \gamma_p d_p$, then we have

$$a_1 = \frac{1}{n - \alpha} (\gamma_1 d_1 + \dots + \gamma_p d_p).$$

Since $d \leq d_i$ ($i = 1, \dots, p$), then there exist constants c_i ($0 < c_i \leq 1$) ($i = 1, \dots, p$) such that $d_i = c_i d$. By this, we get $a_1 = L'd$ where

$$L' = \frac{\gamma_1 c_1 + \dots + \gamma_p c_p}{n - \alpha}.$$

We know that $0 \leq \gamma_1 c_1 + \dots + \gamma_p c_p \leq \gamma_1 + \dots + \gamma_p \leq n - \alpha$, hence we get $0 \leq L' \leq 1$.

(i) When $L' = 0$, we have $a_1 = 0$, which is a contradiction.

(ii) When $0 < L' \leq 1$, we have $a_1 = L'd$. If $\arg a_1 \neq \pi$ or $a_1 < d$, then $a_1 \neq cd$ ($0 < c \leq 1$). Therefore $a_1 = L'd$ is a contradiction. \square

3. Proof of Theorem 1.1

First step. We prove that $\sigma(f) = +\infty$. First of all we prove that equation (1.3) can not have a meromorphic solution $f \not\equiv 0$ with $\sigma(f) < 1$. Assume there exists a meromorphic solution $f \not\equiv 0$ with $\sigma(f) < 1$. By the conditions of Theorem 1.1 we can see that $a_1 \neq a_2$, b_l ($l = 1, \dots, k-1$). Hence, we can rewrite (1.3) in the following form

$$A_1 f e^{a_1 z} + A_2 f e^{a_2 z} + B_{k-1} f^{(k-1)} e^{b_{k-1} z} + \dots + B_1 f' e^{b_1 z} = -f^{(k)}. \quad (3.1)$$

By Lemma 2.7 and Lemma 2.8, we get $A_1 \equiv 0$, which is a contradiction. Therefore $\sigma(f) \geq 1$.

Now, assume that $f \not\equiv 0$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of equation (1.3) with $1 \leq \sigma(f) = \sigma < +\infty$. From equation (1.3), we know that the poles of $f(z)$ can occur only at the poles of A_j ($j = 1, 2$) and B_l ($l = 1, \dots, k-1$). Note that the multiplicities of poles of f are uniformly bounded, and thus by [5], we have

$$\begin{aligned} N(r, f) &\leq M_1 \bar{N}(r, f) \leq M_1 \left(\sum_{j=1}^2 \bar{N}(r, A_j) + \sum_{l=1}^{k-1} \bar{N}(r, B_l) \right) \\ &\leq M \max \{N(r, A_j) \ (j = 1, 2), N(r, B_l) \ (l = 1, \dots, k-1)\}, \end{aligned}$$

where M_1 and M are some suitable positive constants. This gives $\lambda\left(\frac{1}{f}\right) \leq \alpha = \max \{\sigma(A_j) \ (j = 1, 2), \sigma(B_l) \ (l = 1, \dots, k-1)\} < 1$. Let $f = g/d$, d be the canonical product formed with the nonzero poles of $f(z)$, with $\sigma(d) = \lambda(d) = \lambda\left(\frac{1}{f}\right) = \beta \leq \alpha < 1$, g be an entire function and $1 \leq \sigma(g) = \sigma(f) = \sigma < \infty$. Substituting $f = g/d$ into (1.3), by Lemma 2.5 we can get

$$\begin{aligned} &\frac{g^{(k)}}{g} + \left[B_{k-1} e^{b_{k-1}z} + D_{k,k-1} \right] \frac{g^{(k-1)}}{g} \\ &+ \left[B_{k-2} e^{b_{k-2}z} + D_{k,k-2} + B_{k-1} e^{b_{k-1}z} D_{k-1,k-2} \right] \frac{g^{(k-2)}}{g} \\ &+ \left[B_{k-3} e^{b_{k-3}z} + D_{k,k-3} + \sum_{i=k-2}^{k-1} B_i e^{b_i z} D_{i,k-3} \right] \frac{g^{(k-3)}}{g} \\ &+ \dots + \left[B_s e^{b_s z} + D_{k,s} + \sum_{i=s+1}^{k-1} B_i e^{b_i z} D_{i,s} \right] \frac{g^{(s)}}{g} \\ &+ \dots + \left[B_2 e^{b_2 z} + D_{k,2} + \sum_{i=3}^{k-1} B_i e^{b_i z} D_{i,2} \right] \frac{g''}{g} \\ &+ \left[B_1 e^{b_1 z} + D_{k,1} + \sum_{i=2}^{k-1} B_i e^{b_i z} D_{i,1} \right] \frac{g'}{g} \\ &+ \sum_{i=1}^{k-1} B_i e^{b_i z} D_{i,0} + D_{k,0} + A_1 e^{a_1 z} + A_2 e^{a_2 z} = 0. \end{aligned} \quad (3.2)$$

By Lemma 2.4, for any given ε ($0 < \varepsilon < 1 - \alpha$), there is a set $E_6 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ that has linear measure zero such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_6$, then there is

a constant $R_1 = R_1(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| \geq R_1$, we have

$$|B_l(z)| \leq \exp\{r^{\alpha+\varepsilon}\} \quad (l = 1, \dots, k-1). \quad (3.3)$$

By Lemma 2.1, for any given $\varepsilon \left(0 < \varepsilon < \min\left\{\frac{|a_2|-|a_1|}{|a_2|+|a_1|}, 1-\alpha, \frac{1-c}{2(1+c)}\right\}\right)$, there exists a set $E_1 \subset \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$ of linear measure zero, such that if $\theta \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right) \setminus E_1$, then there is a constant $R_0 = R_0(\theta) > 1$, such that for all z satisfying $\arg z = \theta$ and $|z| = r \geq R_0$, we have

$$\left|\frac{g^{(j)}(z)}{g(z)}\right| \leq r^{k(\sigma-1+\varepsilon)}, \quad j = 1, \dots, k, \quad (3.4)$$

$$\left|\frac{d^{(j)}(z)}{d(z)}\right| \leq r^{k(\beta-1+\varepsilon)}, \quad j = 1, \dots, k \quad (3.5)$$

and

$$\begin{aligned} |D_{k,j}| &= \left| \sum_{(j_1 \dots j_k)} C_{jj_1 \dots j_k} \left(\frac{d'}{d}\right)^{j_1} \left(\frac{d''}{d}\right)^{j_2} \dots \left(\frac{d^{(k)}}{d}\right)^{j_k} \right| \\ &\leq \sum_{(j_1 \dots j_k)} |C_{jj_1 \dots j_k}| \left|\frac{d'}{d}\right|^{j_1} \left|\frac{d''}{d}\right|^{j_2} \dots \left|\frac{d^{(k)}}{d}\right|^{j_k} \\ &\leq \sum_{(j_1 \dots j_k)} |C_{jj_1 \dots j_k}| r^{j_1(\beta-1+\varepsilon)} r^{2j_2(\beta-1+\varepsilon)} \dots r^{kj_k(\beta-1+\varepsilon)} \\ &= \sum_{(j_1 \dots j_k)} |C_{jj_1 \dots j_k}| r^{(j_1+2j_2+\dots+kj_k)(\beta-1+\varepsilon)}. \end{aligned} \quad (3.6)$$

By $j_1 + \dots + kj_k = k - j \leq k$ and (3.6), we have

$$|D_{k,j}| \leq M r^{k(\beta-1+\varepsilon)}, \quad (3.7)$$

where $M > 0$ is a some constant. Let $z = r e^{i\theta}$, $a_1 = |a_1| e^{i\theta_1}$, $a_2 = |a_2| e^{i\theta_2}$, $\theta_1, \theta_2 \in \left[-\frac{\pi}{2}, \frac{3\pi}{2}\right)$. We know that $\delta(b_l z, \theta) = \delta(c_l a_1 z, \theta) = c_l \delta(a_1 z, \theta)$ ($l \in I_1$).

Case 1: $\arg a_1 \neq \pi$, which is $\theta_1 \neq \pi$.

(i) Assume that $\theta_1 \neq \theta_2$. By Lemma 2.2 and Lemma 2.3, for the above ε , there is a ray $\arg z = \theta$ such that $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ (where E_4 and E_5 are defined as in Lemma 2.3, $E_1 \cup E_4 \cup E_5 \cup E_6$ is of linear measure zero), and satisfying

$$\delta(a_1 z, \theta) > 0, \quad \delta(a_2 z, \theta) < 0$$

or

$$\delta(a_1 z, \theta) < 0, \delta(a_2 z, \theta) > 0.$$

a) When $\delta(a_1 z, \theta) > 0, \delta(a_2 z, \theta) < 0$, for sufficiently large r , we get by Lemma 2.2

$$|A_1 e^{a_1 z}| \geq \exp\{(1 - \varepsilon) \delta(a_1 z, \theta) r\}, \quad (3.8)$$

$$|A_2 e^{a_2 z}| \leq \exp\{(1 - \varepsilon) \delta(a_2 z, \theta) r\} < 1. \quad (3.9)$$

By (3.8) and (3.9) we have

$$\begin{aligned} |A_1 e^{a_1 z} + A_2 e^{a_2 z}| &\geq |A_1 e^{a_1 z}| - |A_2 e^{a_2 z}| \geq \exp\{(1 - \varepsilon) \delta(a_1 z, \theta) r\} - 1 \\ &\geq (1 - o(1)) \exp\{(1 - \varepsilon) \delta(a_1 z, \theta) r\}. \end{aligned} \quad (3.10)$$

By (3.2), we get

$$\begin{aligned} |A_1 e^{a_1 z} + A_2 e^{a_2 z}| &\leq \left| \frac{g^{(k)}}{g} \right| + \left[|B_{k-1} e^{b_{k-1} z}| + |D_{k,k-1}| \right] \left| \frac{g^{(k-1)}}{g} \right| \\ &+ \left[|B_{k-2} e^{b_{k-2} z}| + |D_{k,k-2}| + |B_{k-1} e^{b_{k-1} z}| |D_{k-1,k-2}| \right] \left| \frac{g^{(k-2)}}{g} \right| \\ &+ \left[|B_{k-3} e^{b_{k-3} z}| + |D_{k,k-3}| + \sum_{i=k-2}^{k-1} |B_i e^{b_i z}| |D_{i,k-3}| \right] \left| \frac{g^{(k-3)}}{g} \right| \\ &+ \cdots + \left[|B_s e^{b_s z}| + |D_{k,s}| + \sum_{i=s+1}^{k-1} |B_i e^{b_i z}| |D_{i,s}| \right] \left| \frac{g^{(s)}}{g} \right| \\ &+ \cdots + \left[|B_2 e^{b_2 z}| + |D_{k,2}| + \sum_{i=3}^{k-1} |B_i e^{b_i z}| |D_{i,2}| \right] \left| \frac{g''}{g} \right| \\ &+ \left[|B_1 e^{b_1 z}| + |D_{k,1}| + \sum_{i=2}^{k-1} |B_i e^{b_i z}| |D_{i,1}| \right] \left| \frac{g'}{g} \right| + \sum_{i=1}^{k-1} |B_i e^{b_i z}| |D_{i,0}| + |D_{k,0}|. \end{aligned} \quad (3.11)$$

For $l \in I_1$, we have

$$\left| B_l(z) e^{b_l z} \right| \leq \exp\{(1 + \varepsilon) c_l \delta(a_1 z, \theta) r\} \leq \exp\{(1 + \varepsilon) c \delta(a_1 z, \theta) r\}. \quad (3.12)$$

For $l \in I_2$, we have

$$\left| B_l(z) e^{b_l z} \right| = |B_l(z)| \left| e^{b_l z} \right| \leq \exp\{r^{\alpha+\varepsilon}\} \exp\{b_l r \cos \theta\} \leq \exp\{r^{\alpha+\varepsilon}\}, \quad (3.13)$$

because $b_l < 0$ and $\cos \theta > 0$. Substituting (3.4), (3.7), (3.10), (3.12) and (3.13) into (3.11), we obtain

$$\begin{aligned} & (1 - o(1)) \exp \{ (1 - \varepsilon) \delta(a_1 z, \theta) r \} \\ & \leq M_1 r^{M_2} \exp \{ r^{\alpha + \varepsilon} \} \exp \{ (1 + \varepsilon) c \delta(a_1 z, \theta) r \}, \end{aligned} \quad (3.14)$$

where $M_1 > 0$ and $M_2 > 0$ are some constants. From (3.14) and $0 < \varepsilon < \frac{1-c}{2(1+c)}$, we get

$$(1 - o(1)) \exp \left\{ \frac{1-c}{2} \delta(a_1 z, \theta) r \right\} \leq M_1 r^{M_2} \exp \{ r^{\alpha + \varepsilon} \}. \quad (3.15)$$

By $\delta(a_1 z, \theta) > 0$ and $\alpha + \varepsilon < 1$ we know that (3.15) is a contradiction.

b) When $\delta(a_1 z, \theta) < 0$, $\delta(a_2 z, \theta) > 0$, for sufficiently large r , we get by Lemma 2.2

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geq (1 - o(1)) \exp \{ (1 - \varepsilon) \delta(a_2 z, \theta) r \}. \quad (3.16)$$

For $l \in I_1$, we have

$$\left| B_l(z) e^{b_l z} \right| \leq \exp \{ (1 - \varepsilon) c_l \delta(a_1 z, \theta) r \} < 1. \quad (3.17)$$

Substituting (3.4), (3.7), (3.13), (3.16) and (3.17) into (3.11), we obtain

$$(1 - o(1)) \exp \{ (1 - \varepsilon) \delta(a_2 z, \theta) r \} \leq M_1 r^{M_2} \exp \{ r^{\alpha + \varepsilon} \}. \quad (3.18)$$

By $\delta(a_2 z, \theta) > 0$ and $\alpha + \varepsilon < 1$ we know that (3.18) is a contradiction.

(ii) Assume that $\theta_1 = \theta_2$. By Lemma 2.3, for the above ε , there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ and $\delta(a_1 z, \theta) > 0$. Since $|a_1| \leq |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, it follows that $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta) > 0$. For sufficiently large r , we have by Lemma 2.2

$$|A_1 e^{a_1 z}| \leq \exp \{ (1 + \varepsilon) \delta(a_1 z, \theta) r \}, \quad (3.19)$$

$$|A_2 e^{a_2 z}| \geq \exp \{ (1 - \varepsilon) \delta(a_2 z, \theta) r \}. \quad (3.20)$$

By (3.19) and (3.20) we get

$$\begin{aligned} & |A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geq |A_2 e^{a_2 z}| - |A_1 e^{a_1 z}| \\ & \geq \exp \{ (1 - \varepsilon) \delta(a_2 z, \theta) r \} - \exp \{ (1 + \varepsilon) \delta(a_1 z, \theta) r \} \\ & = \exp \{ (1 + \varepsilon) \delta(a_1 z, \theta) r \} [\exp \{ \eta r \} - 1], \end{aligned} \quad (3.21)$$

where

$$\eta = (1 - \varepsilon) \delta(a_2 z, \theta) - (1 + \varepsilon) \delta(a_1 z, \theta).$$

Since $0 < \varepsilon < \frac{|a_2| - |a_1|}{|a_2| + |a_1|}$, it follows that

$$\begin{aligned} \eta &= (1 - \varepsilon) |a_2| \cos(\theta_2 + \theta) - (1 + \varepsilon) |a_1| \cos(\theta_1 + \theta) \\ &= (1 - \varepsilon) |a_2| \cos(\theta_1 + \theta) - (1 + \varepsilon) |a_1| \cos(\theta_1 + \theta) \\ &= [(1 - \varepsilon) |a_2| - (1 + \varepsilon) |a_1|] \cos(\theta_1 + \theta) \\ &= [|a_2| - |a_1| - \varepsilon(|a_2| + |a_1|)] \cos(\theta_1 + \theta) > 0. \end{aligned}$$

Then, by $\eta > 0$ we get from (3.21) that

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geq (1 - o(1)) \exp\{[(1 + \varepsilon) \delta(a_1 z, \theta) + \eta] r\}. \quad (3.22)$$

Substituting (3.4), (3.7), (3.12), (3.13) and (3.22) into (3.11), we obtain

$$\begin{aligned} &(1 - o(1)) \exp\{[(1 + \varepsilon) \delta(a_1 z, \theta) + \eta] r\} \\ &\leq M_1 r^{M_2} \exp\{r^{\alpha + \varepsilon}\} \exp\{(1 + \varepsilon) c \delta(a_1 z, \theta) r\}. \end{aligned} \quad (3.23)$$

By (3.23), we have

$$(1 - o(1)) \exp\{[(1 + \varepsilon) (1 - c) \delta(a_1 z, \theta) + \eta] r\} \leq M_1 r^{M_2} \exp\{r^{\alpha + \varepsilon}\}. \quad (3.24)$$

By $\delta(a_1 z, \theta) > 0$, $\eta > 0$ and $\alpha + \varepsilon < 1$ we know that (3.24) is a contradiction.

Case 2: $a_1 < \frac{b}{1-c}$, which is $\theta_1 = \pi$.

(i) Assume that $\theta_1 \neq \theta_2$, then $\theta_2 \neq \pi$. By Lemma 2.3, for the above ε , there is a ray $\arg z = \theta$ such that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$ and $\delta(a_2 z, \theta) > 0$. Because $\cos \theta > 0$, we have $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos \theta < 0$. For sufficiently large r , we obtain by Lemma 2.2

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \geq (1 - o(1)) \exp\{(1 - \varepsilon) \delta(a_2 z, \theta) r\}. \quad (3.25)$$

Using the same reasoning as in Case 1(i), we can get a contradiction.

(ii) Assume that $\theta_1 = \theta_2$, then $\theta_1 = \theta_2 = \pi$.

By Lemma 2.3, for the above ε , there is a ray $\arg z = \theta$ such that $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, then $\cos \theta < 0$, $\delta(a_1 z, \theta) = |a_1| \cos(\theta_1 + \theta) = -|a_1| \cos \theta > 0$, $\delta(a_2 z, \theta) = |a_2| \cos(\theta_2 + \theta) = -|a_2| \cos \theta > 0$. Since $|a_1| \leq |a_2|$, $a_1 \neq a_2$ and $\theta_1 = \theta_2$, it follows that $|a_1| < |a_2|$, thus $\delta(a_2 z, \theta) > \delta(a_1 z, \theta)$, for sufficiently large r , we get (3.19), (3.20) and (3.22) hold. For $l \in I_2$, we have

$$\begin{aligned} |B_l(z) e^{b_l z}| &= |B_l(z)| |e^{b_l z}| \leq \exp\{r^{\alpha + \varepsilon}\} \exp\{b_l r \cos \theta\} \\ &\leq \exp\{r^{\alpha + \varepsilon}\} \exp\{b r \cos \theta\} \end{aligned} \quad (3.26)$$

because $b_l < 0$, $b = \min \{b_l : l \in I_2\}$ and $\cos \theta < 0$. Substituting (3.4), (3.7), (3.12), (3.22) and (3.26) into (3.11), we obtain

$$\begin{aligned} & (1 - o(1)) \exp \{[(1 + \varepsilon) \delta(a_1 z, \theta) + \eta] r\} \\ & \leq M_1 r^{M_2} \exp \{r^{\alpha + \varepsilon}\} \exp \{(1 + \varepsilon) c \delta(a_1 z, \theta) r\} \exp \{br \cos \theta\}. \end{aligned} \quad (3.27)$$

From (3.27), we have

$$(1 - o(1)) \exp \{\gamma r\} \leq M_1 r^{M_2} \exp \{r^{\alpha + \varepsilon}\}, \quad (3.28)$$

where $\gamma = (1 + \varepsilon)(1 - c) \delta(a_1 z, \theta) + \eta - b \cos \theta$. Since $\eta > 0$, $\cos \theta < 0$, $\delta(a_1 z, \theta) = -|a_1| \cos \theta$, $a_1 < \frac{b}{1-c}$ and $b < 0$, then it follows that

$$\begin{aligned} \gamma &= -(1 + \varepsilon)(1 - c)|a_1| \cos \theta - b \cos \theta + \eta \\ &= -[(1 + \varepsilon)(1 - c)|a_1| + b] \cos \theta + \eta \\ &> -\left[(1 + \varepsilon)(1 - c) \frac{|b|}{1 - c} + b\right] \cos \theta + \eta \\ &= -[-(1 + \varepsilon)b + b] \cos \theta + \eta = \eta + b\varepsilon \cos \theta > 0. \end{aligned}$$

By $\alpha + \varepsilon < 1$ and $\gamma > 0$, we know that (3.28) is a contradiction. Concluding the above proof, we obtain $\sigma(f) = \sigma(g) = +\infty$.

Second step. We prove that $\sigma_2(f) = 1$. By

$$\max \left\{ \sigma(A_1 e^{a_1 z} + A_2 e^{a_2 z}), \sigma(B_l e^{b_l z}) \ (l = 1, \dots, k-1) \right\} = 1$$

and Lemma 2.11, we obtain $\sigma_2(f) \leq 1$. By Lemma 2.9, we know that there exists a set $E_8 \subset (1, +\infty)$ with finite logarithmic measure and a constant $B > 0$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_8$, we get

$$\left| \frac{f^{(j)}(z)}{f(z)} \right| \leq B [T(2r, f)]^{j+1} \quad (j = 1, \dots, k). \quad (3.29)$$

By (1.3), we have

$$|A_1 e^{a_1 z} + A_2 e^{a_2 z}| \leq \left| \frac{f^{(k)}}{f} \right| + \left| B_{k-1} e^{b_{k-1} z} \right| \left| \frac{f^{(k-1)}}{f} \right| + \dots + \left| B_1 e^{b_1 z} \right| \left| \frac{f'}{f} \right|. \quad (3.30)$$

Case 1: $\arg a_1 \neq \pi$

(i) ($\theta_1 \neq \theta_2$) In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, satisfying

$$\delta(a_1 z, \theta) > 0, \delta(a_2 z, \theta) < 0 \text{ or } \delta(a_1 z, \theta) < 0, \delta(a_2 z, \theta) > 0.$$

a) When $\delta(a_1z, \theta) > 0$, $\delta(a_2z, \theta) < 0$, for sufficiently large r , we get (3.10) holds. Substituting (3.10), (3.12), (3.13) and (3.29) into (3.30), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0, 1] \cup E_8$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$

$$\begin{aligned} & (1 - o(1)) \exp \{ (1 - \varepsilon) \delta(a_1z, \theta) r \} \\ & \leq M \exp \{ r^{\alpha+\varepsilon} \} \exp \{ (1 + \varepsilon) c \delta(a_1z, \theta) r \} [T(2r, f)]^{k+1}, \end{aligned} \quad (3.31)$$

where $M > 0$ is a some constant. From (3.31) and $0 < \varepsilon < \frac{1-c}{2(1+c)}$, we get

$$(1 - o(1)) \exp \left\{ \frac{1-c}{2} \delta(a_1z, \theta) r \right\} \leq M \exp \{ r^{\alpha+\varepsilon} \} [T(2r, f)]^{k+1}. \quad (3.32)$$

Since $\delta(a_1z, \theta) > 0$, $\alpha + \varepsilon < 1$, then by using Lemma 2.10 and (3.32), we obtain $\sigma_2(f) \geq 1$, hence $\sigma_2(f) = 1$.

b) When $\delta(a_1z, \theta) < 0$, $\delta(a_2z, \theta) > 0$, using a proof similar to the above, we can also get $\sigma_2(f) = 1$.

(ii) ($\theta_1 = \theta_2$) In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, satisfying $\delta(a_2z, \theta) > \delta(a_1z, \theta) > 0$ and for sufficiently large r , we get (3.22) holds. Substituting (3.12), (3.13), (3.22) and (3.29) into (3.30), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0, 1] \cup E_8$, $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$

$$\begin{aligned} & (1 - o(1)) \exp \{ [(1 + \varepsilon) \delta(a_1z, \theta) + \eta] r \} \\ & \leq M \exp \{ r^{\alpha+\varepsilon} \} \exp \{ (1 + \varepsilon) c \delta(a_1z, \theta) r \} [T(2r, f)]^{k+1}. \end{aligned} \quad (3.33)$$

By (3.33), we have

$$(1 - o(1)) \exp \{ [(1 + \varepsilon) (1 - c) \delta(a_1z, \theta) + \eta] r \} \leq M \exp \{ r^{\alpha+\varepsilon} \} [T(2r, f)]^{k+1}. \quad (3.34)$$

Since $\delta(a_1z, \theta) > 0$, $\eta > 0$ and $\alpha + \varepsilon < 1$, then by using Lemma 2.10 and (3.34), we obtain $\sigma_2(f) \geq 1$, hence $\sigma_2(f) = 1$.

Case 2: $a_1 < \frac{b}{1-c}$.

(i) ($\theta_1 \neq \theta_2$) In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, satisfying $\delta(a_2z, \theta) > 0$ and $\delta(a_1z, \theta) < 0$ and for sufficiently large r , we get (3.25) holds. Using the same reasoning as in second step (**Case 1** (i)), we can get $\sigma_2(f) = 1$.

(ii) ($\theta_1 = \theta_2$) In first step, we have proved that there is a ray $\arg z = \theta$ where $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$, satisfying $\delta(a_2z, \theta) > \delta(a_1z, \theta) > 0$ and for sufficiently large r , we get (3.22) holds. Substituting (3.12), (3.22), (3.26) and (3.29) into (3.30), we obtain for all $z = re^{i\theta}$ satisfying $|z| = r \notin [0, 1] \cup E_8$, $\theta \in (\frac{\pi}{2}, \frac{3\pi}{2}) \setminus (E_1 \cup E_4 \cup E_5 \cup E_6)$

$$\begin{aligned} & (1 - o(1)) \exp \{ [(1 + \varepsilon) \delta(a_1z, \theta) + \eta] r \} \\ & \leq M \exp \{ r^{\alpha+\varepsilon} \} \exp \{ (1 + \varepsilon) c \delta(a_1z, \theta) r \} \exp \{ br \cos \theta \} [T(2r, f)]^{k+1}. \end{aligned} \quad (3.35)$$

From (3.35), we have

$$(1 - o(1)) \exp \{ \gamma r \} \leq M \exp \{ r^{\alpha + \varepsilon} \} [T(2r, f)]^{k+1}, \quad (3.36)$$

where $\gamma = (1 + \varepsilon)(1 - c)\delta(a_1 z, \theta) + \eta - b \cos \theta$. Since $\gamma > 0$, $\alpha + \varepsilon < 1$, then by using Lemma 2.10 and (3.36), we obtain $\sigma_2(f) \geq 1$, hence $\sigma_2(f) = 1$. Concluding the above proof, we obtain that every meromorphic solution $f (\neq 0)$ whose poles are of uniformly bounded multiplicities of (1.3) satisfies $\sigma_2(f) = 1$. The proof of Theorem 1.1 is complete.

4. Proof of Theorem 1.2

Set $R_0(z) = A_1 e^{a_1 z} + A_2 e^{a_2 z}$ and $R_j(z) = B_j e^{b_j z}$ ($j = 1, \dots, k-1$). Assume $f (\neq 0)$ is a meromorphic solution whose poles are of uniformly bounded multiplicities of equation (1.3), then $\sigma(f) = +\infty$ by Theorem 1.1. Set $g_0(z) = f(z) - \varphi(z)$. We have $g_0(z)$ is a meromorphic function and $\sigma(g_0) = \sigma(f) = \infty$. Substituting $f = g_0 + \varphi$ into (1.3), we have

$$\begin{aligned} & g_0^{(k)} + R_{k-1} g_0^{(k-1)} + \dots + R_2 g_0'' + R_1 g_0' + R_0 g_0 \\ &= - \left[\varphi^{(k)} + R_{k-1} \varphi^{(k-1)} + \dots + R_2 \varphi'' + R_1 \varphi' + R_0 \varphi \right]. \end{aligned} \quad (4.1)$$

We can rewrite (4.1) in the following form

$$g_0^{(k)} + h_{0,k-1} g_0^{(k-1)} + \dots + h_{0,2} g_0'' + h_{0,1} g_0' + h_{0,0} g_0 = h_0, \quad (4.2)$$

where

$$h_0 = - \left[\varphi^{(k)} + R_{k-1} \varphi^{(k-1)} + \dots + R_2 \varphi'' + R_1 \varphi' + R_0 \varphi \right].$$

We prove that $h_0 \neq 0$. In fact, if $h_0 \equiv 0$, then

$$\varphi^{(k)} + R_{k-1} \varphi^{(k-1)} + \dots + R_2 \varphi'' + R_1 \varphi' + R_0 \varphi = 0.$$

Hence, $\varphi \neq 0$ is a solution of equation (1.3) with $\sigma(\varphi) = +\infty$ by Theorem 1.1, it is a contradiction. Hence, $h_0 \neq 0$ is proved. By Lemma 2.6 and (4.2) we know that $\bar{\lambda}(g_0) = \bar{\lambda}(f - \varphi) = \sigma(g_0) = \sigma(f) = \infty$.

Now we prove that $\bar{\lambda}(f' - \varphi) = \infty$. Set $g_1(z) = f'(z) - \varphi(z)$, then $g_1(z)$ is a meromorphic function and $\sigma(g_1) = \sigma(f') = \sigma(f) = \infty$. Differentiating both sides of equation (1.3), we have

$$\begin{aligned} & f^{(k+1)} + R_{k-1} f^{(k)} + (R'_{k-1} + R_{k-2}) f^{(k-1)} + (R'_{k-2} + R_{k-3}) f^{(k-2)} \\ &+ \dots + (R'_3 + R_2) f''' + (R'_2 + R_1) f'' + (R'_1 + R_0) f' + R'_0 f = 0. \end{aligned} \quad (4.3)$$

By (1.3), we have

$$f = -\frac{1}{R_0} \left[f^{(k)} + R_{k-1}f^{(k-1)} + \dots + R_2f'' + R_1f' \right]. \quad (4.4)$$

Substituting (4.4) into (4.3), we have

$$\begin{aligned} & f^{(k+1)} + \left(R_{k-1} - \frac{R'_0}{R_0} \right) f^{(k)} + \left(R'_{k-1} + R_{k-2} - R_{k-1} \frac{R'_0}{R_0} \right) f^{(k-1)} \\ & + \left(R'_{k-2} + R_{k-3} - R_{k-2} \frac{R'_0}{R_0} \right) f^{(k-2)} + \dots + \left(R'_3 + R_2 - R_3 \frac{R'_0}{R_0} \right) f''' \\ & + \left(R'_2 + R_1 - R_2 \frac{R'_0}{R_0} \right) f'' + \left(R'_1 + R_0 - R_1 \frac{R'_0}{R_0} \right) f' = 0. \end{aligned} \quad (4.5)$$

We can denote equation (4.5) by the following form

$$f^{(k+1)} + h_{1,k-1}f^{(k)} + h_{1,k-2}f^{(k-1)} + \dots + h_{1,2}f''' + h_{1,1}f'' + h_{1,0}f' = 0, \quad (4.6)$$

where

$$\begin{aligned} h_{1,i} &= R'_{i+1} + R_i - R_{i+1} \frac{R'_0}{R_0} \quad (i = 0, 1, \dots, k-2), \\ h_{1,k-1} &= R_{k-1} - \frac{R'_0}{R_0}. \end{aligned}$$

Substituting $f^{(j+1)} = g_1^{(j)} + \varphi^{(j)}$ ($j = 0, \dots, k$) into (4.6), we get

$$g_1^{(k)} + h_{1,k-1}g_1^{(k-1)} + h_{1,k-2}g_1^{(k-2)} + \dots + h_{1,2}g_1'' + h_{1,1}g_1' + h_{1,0}g_1 = h_1, \quad (4.7)$$

where

$$h_1 = - \left[\varphi^{(k)} + h_{1,k-1}\varphi^{(k-1)} + h_{1,k-2}\varphi^{(k-2)} + \dots + h_{1,2}\varphi'' + h_{1,1}\varphi' + h_{1,0}\varphi \right].$$

We can get,

$$h_{1,i}(z) = \frac{N_i(z)}{R_0(z)} \quad (i = 0, 1, \dots, k-1), \quad (4.8)$$

where

$$N_0 = R'_1R_0 + R_0^2 - R_1R'_0, \quad (4.9)$$

$$N_i = R'_{i+1}R_0 + R_iR_0 - R_{i+1}R'_0 \quad (i = 1, 2, \dots, k-2), \quad (4.10)$$

$$N_{k-1} = R_{k-1}R_0 - R'_0. \quad (4.11)$$

Now we prove that $h_1 \not\equiv 0$. In fact, if $h_1 \equiv 0$, then $\frac{h_1}{\varphi} \equiv 0$. Hence, by (4.8) we get

$$\frac{\varphi^{(k)}}{\varphi}R_0 + \frac{\varphi^{(k-1)}}{\varphi}N_{k-1} + \frac{\varphi^{(k-2)}}{\varphi}N_{k-2} + \dots + \frac{\varphi''}{\varphi}N_2 + \frac{\varphi'}{\varphi}N_1 + N_0 = 0. \quad (4.12)$$

Obviously, $\frac{\varphi^{(j)}}{\varphi}$ ($j = 1, \dots, k$) are meromorphic functions with $\sigma\left(\frac{\varphi^{(j)}}{\varphi}\right) < 1$. By (4.9) – (4.11) we can rewrite (4.12) in the form

$$\begin{aligned} f_{1,0}e^{a_1z} + f_{2,0}e^{a_2z} + \sum_{i=1}^{k-1} f_{1,i}e^{(a_1+b_i)z} + \sum_{i=1}^{k-1} f_{2,i}e^{(a_2+b_i)z} \\ + 2A_1A_2e^{(a_1+a_2)z} + A_1^2e^{2a_1z} + A_2^2e^{2a_2z} = 0, \end{aligned} \quad (4.13)$$

where $f_{1,i}, f_{2,i}$ ($i = 0, 1, \dots, k-1$) are meromorphic functions of order less than 1. Set $I = \{2a_1, 2a_2, a_1+a_2, a_1, a_2, a_1+b_i, a_2+b_i \ (i = 1, \dots, k-1)\}$. It is clear that $2a_1 \neq a_1, a_1+a_2, 2a_2$ and by Lemma 2.13 we have $2a_1 \neq a_1+b_i$ ($i = 1, \dots, k-1$).

(i) If $2a_1 \neq a_2, a_2+b_i$ ($i = 1, \dots, k-1$), then we can rewrite (4.13) in the form

$$A_1^2e^{2a_1z} + \sum_{\beta \in \Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_1 \subseteq I \setminus \{2a_1\}$ and α_β ($\beta \in \Gamma_1$) are meromorphic functions of order less than 1. By Lemma 2.7 and Lemma 2.8, we get $A_1 \equiv 0$, which is a contradiction.

(ii) If $2a_1 = \gamma$ such that $\gamma \in \{a_2, a_2+b_i \ (i = 1, \dots, k-1)\}$, then by Lemma 2.12 we have $2a_2 \neq \beta$ for all $\beta \in I \setminus \{2a_2\}$. Hence, we can rewrite (4.13) in the form

$$A_2^2e^{2a_2z} + \sum_{\beta \in \Gamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_2 \subseteq I \setminus \{2a_2\}$ and α_β ($\beta \in \Gamma_2$) are meromorphic functions of order less than 1. By Lemma 2.7 and Lemma 2.8, we get $A_2 \equiv 0$, it is a contradiction. Hence, $h_1 \neq 0$ is proved. By Lemma 2.6 and (4.7) we know that $\bar{\lambda}(g_1) = \bar{\lambda}(f' - \varphi) = \sigma(g_1) = \sigma(f) = \infty$.

Now we prove that $\bar{\lambda}(f'' - \varphi) = \infty$. Set $g_2(z) = f''(z) - \varphi(z)$, then $g_2(z)$ is a meromorphic function and $\sigma(g_2) = \sigma(f'') = \sigma(f) = \infty$. Differentiating both sides of equation (1.3), we have

$$\begin{aligned} f^{(k+2)} + R_{k-1}f^{(k+1)} + (2R'_{k-1} + R_{k-2})f^{(k)} + (R''_{k-1} + 2R'_{k-2} + R_{k-3})f^{(k-1)} \\ + (R''_{k-2} + 2R'_{k-3} + R_{k-4})f^{(k-2)} + \dots + (R''_3 + 2R'_2 + R_1)f''' \\ + (R''_2 + 2R'_1 + R_0)f'' + (R''_1 + 2R'_0)f' + R''_0f = 0. \end{aligned} \quad (4.14)$$

By (4.4) and (4.14), we have

$$\begin{aligned} f^{(k+2)} + R_{k-1}f^{(k+1)} + \left(2R'_{k-1} + R_{k-2} - \frac{R''_0}{R_0}\right)f^{(k)} \\ + \left(R''_{k-1} + 2R'_{k-2} + R_{k-3} - R_{k-1}\frac{R''_0}{R_0}\right)f^{(k-1)} \end{aligned}$$

$$\begin{aligned}
& + \cdots + \left(R_4'' + 2R_3' + R_2 - R_4 \frac{R_0''}{R_0} \right) f^{(4)} + \left(R_3'' + 2R_2' + R_1 - R_3 \frac{R_0''}{R_0} \right) f''' \\
& + \left(R_2'' + 2R_1' + R_0 - R_2 \frac{R_0''}{R_0} \right) f'' + \left(R_1'' + 2R_0' - R_1 \frac{R_0''}{R_0} \right) f' = 0. \quad (4.15)
\end{aligned}$$

Now we prove that $R_1' + R_0 - R_1 \frac{R_0'}{R_0} \neq 0$. Suppose that $R_1' + R_0 - R_1 \frac{R_0'}{R_0} \equiv 0$, then we have

$$f_1 e^{(a_1+b_1)z} + f_2 e^{(a_2+b_1)z} + 2A_1 A_2 e^{(a_1+a_2)z} + A_1^2 e^{2a_1 z} + A_2^2 e^{2a_2 z} = 0, \quad (4.16)$$

where f_j ($j = 1, 2$) are meromorphic functions of order less than 1. By using the same reasoning as above, we can get a contradiction. Hence, $R_1' + R_0 - R_1 \frac{R_0'}{R_0} \neq 0$ is proved. Set

$$\psi(z) = R_1' R_0 + R_0^2 - R_1 R_0' \text{ and } \phi(z) = R_1'' R_0 + 2R_0' R_0 - R_1 R_0''. \quad (4.17)$$

By (4.5) and (4.17), we get

$$\begin{aligned}
f' = \frac{-R_0}{\psi(z)} \left\{ f^{(k+1)} + \left(R_{k-1} - \frac{R_0'}{R_0} \right) f^{(k)} + \left(R_{k-1}' + R_{k-2} - R_{k-1} \frac{R_0'}{R_0} \right) f^{(k-1)} \right. \\
\left. + \left(R_{k-2}' + R_{k-3} - R_{k-2} \frac{R_0'}{R_0} \right) f^{(k-2)} + \cdots + \left(R_2' + R_1 - R_2 \frac{R_0'}{R_0} \right) f' \right\}. \quad (4.18)
\end{aligned}$$

Substituting (4.17) and (4.18) into (4.15), we obtain

$$\begin{aligned}
& f^{(k+2)} + \left[R_{k-1} - \frac{\phi}{\psi} \right] f^{(k+1)} + \left[2R_{k-1}' + R_{k-2} - \frac{R_0''}{R_0} - \frac{\phi}{\psi} \left(R_{k-1} - \frac{R_0'}{R_0} \right) \right] f^{(k)} \\
& + \left[R_{k-1}'' + 2R_{k-2}' + R_{k-3} - R_{k-1} \frac{R_0''}{R_0} - \frac{\phi}{\psi} \left(R_{k-1}' + R_{k-2} - R_{k-1} \frac{R_0'}{R_0} \right) \right] f^{(k-1)} \\
& + \cdots + \left[R_3'' + 2R_2' + R_1 - R_3 \frac{R_0''}{R_0} - \frac{\phi}{\psi} \left(R_3' + R_2 - R_3 \frac{R_0'}{R_0} \right) \right] f''' \\
& + \left[R_2'' + 2R_1' + R_0 - R_2 \frac{R_0''}{R_0} - \frac{\phi}{\psi} \left(R_2' + R_1 - R_2 \frac{R_0'}{R_0} \right) \right] f'' = 0. \quad (4.19)
\end{aligned}$$

We can denote equation (4.19) by the following form

$$f^{(k+2)} + h_{2,k-1} f^{(k+1)} + h_{2,k-2} f^{(k)} + \cdots + h_{2,2} f^{(4)} + h_{2,1} f''' + h_{2,0} f'' = 0, \quad (4.20)$$

where

$$\begin{aligned}
h_{2,i} &= R_{i+2}'' + 2R_{i+1}' + R_i - R_{i+2} \frac{R_0''}{R_0} \\
&\quad - \frac{\phi(z)}{\psi(z)} \left(R_{i+2}' + R_{i+1} - R_{i+2} \frac{R_0'}{R_0} \right) \quad (i = 0, 1, \dots, k-3),
\end{aligned}$$

$$h_{2,k-2} = 2R'_{k-1} + R_{k-2} - \frac{R''_0}{R_0} - \frac{\phi(z)}{\psi(z)} \left(R_{k-1} - \frac{R'_0}{R_0} \right),$$

$$h_{2,k-1} = R_{k-1} - \frac{\phi(z)}{\psi(z)}.$$

Substituting $f^{(j+2)} = g_2^{(j)} + \varphi^{(j)}$ ($j = 0, \dots, k$) into (4.20) we get

$$g_2^{(k)} + h_{2,k-1}g_2^{(k-1)} + h_{2,k-2}g_2^{(k-2)} + \dots + h_{2,1}g_2' + h_{2,0}g_2 = h_2, \quad (4.21)$$

where

$$h_2 = - \left[\varphi^{(k)} + h_{2,k-1}\varphi^{(k-1)} + h_{2,k-2}\varphi^{(k-2)} + \dots + h_{2,2}\varphi'' + h_{2,1}\varphi' + h_{2,0}\varphi \right].$$

We can get

$$h_{2,i} = \frac{L_i(z)}{\psi(z)} \quad (i = 0, 1, \dots, k-1), \quad (4.22)$$

where

$$L_0(z) = R''_2R'_1R_0 + R''_2R_0^2 - R''_2R_1R'_0 + 2R_1^2R_0 + 3R'_1R_0^2 - 2R'_1R_1R'_0 + R_0^3$$

$$- 3R_1R'_0R_0 - R_2R'_1R''_0 - R_2R''_0R_0 - R'_2R'_1R_0 - 2R'_2R'_0R_0 + R'_2R_1R''_0$$

$$- R_1^2R_1R_0 + R_1^2R''_0 + R_2R''_1R'_0 + 2R_2R_0^2, \quad (4.23)$$

$$L_i = R''_{i+2}R'_1R_0 + R''_{i+2}R_0^2 - R''_{i+2}R_1R'_0 + 2R'_{i+1}R'_1R_0 + 2R'_{i+1}R_0^2 - 2R'_{i+1}R_1R'_0$$

$$+ R_iR'_1R_0 + R_iR_0^2 - R_iR_1R'_0 - R_{i+2}R'_1R''_0 - R_{i+2}R''_0R_0 - R'_{i+2}R''_1R_0$$

$$- 2R'_{i+2}R'_0R_0 + R'_{i+2}R_1R''_0 - R_{i+1}R'_1R_0 - 2R_{i+1}R'_0R_0 + R_{i+1}R_1R''_0$$

$$+ R_{i+2}R''_1R'_0 + 2R_{i+2}R_0^2 \quad (i = 1, 2, \dots, k-3), \quad (4.24)$$

$$L_{k-2} = 2R'_{k-1}R'_1R_0 + 2R'_{k-1}R_0^2 - 2R'_{k-1}R_1R'_0 + R_{k-2}R'_1R_0 + R_{k-2}R_0^2$$

$$- R_{k-2}R_1R'_0 - R'_1R''_0 - R''_0R_0 - R_{k-1}R'_1R_0 - 2R_{k-1}R'_0R_0$$

$$+ R_{k-1}R_1R''_0 + R''_1R'_0 + 2R_0^2, \quad (4.25)$$

$$L_{k-1} = R_{k-1}R'_1R_0 + R_{k-1}R_0^2 - R_{k-1}R_1R'_0 - R''_1R_0 - 2R'_0R_0 + R_1R''_0. \quad (4.26)$$

Therefore

$$\frac{-h_2}{\varphi} = \frac{1}{\psi} \left[\frac{\varphi^{(k)}}{\varphi} \psi + \frac{\varphi^{(k-1)}}{\varphi} L_{k-1} + \dots + \frac{\varphi''}{\varphi} L_2 + \frac{\varphi'}{\varphi} L_1 + L_0 \right]. \quad (4.27)$$

Now we prove that $h_2 \neq 0$. In fact, if $h_2 \equiv 0$, then $\frac{-h_2}{\varphi} \equiv 0$. Hence, by (4.27) we have

$$\frac{\varphi^{(k)}}{\varphi} \psi + \frac{\varphi^{(k-1)}}{\varphi} L_{k-1} + \dots + \frac{\varphi''}{\varphi} L_2 + \frac{\varphi'}{\varphi} L_1 + L_0 = 0, \quad (4.28)$$

Obviously, $\frac{\varphi^{(j)}}{\varphi}$ ($j = 1, \dots, k$) are meromorphic functions with $\sigma\left(\frac{\varphi^{(j)}}{\varphi}\right) < 1$. By (4.17) and (4.23)-(4.26), we can rewrite (4.28) in the form

$$\begin{aligned} & f_{1,0}e^{2a_1z} + f_{2,0}e^{2a_2z} + f_{3,0}e^{(a_1+a_2)z} + \sum_{i=1}^{k-1} f_{1,i}e^{(2a_1+b_i)z} + \sum_{i=1}^{k-1} f_{2,i}e^{(2a_2+b_i)z} \\ & + \sum_{i=1}^{k-1} f_{3,i}e^{(a_1+a_2+b_i)z} + l_{1,0}e^{(a_1+b_1)z} + l_{2,0}e^{(a_2+b_1)z} \\ & + \sum_{i=1}^{k-1} l_{1,i}e^{(a_1+b_1+b_i)z} + \sum_{i=1}^{k-1} l_{2,i}e^{(a_2+b_1+b_i)z} \\ & + A_1^3e^{3a_1z} + A_2^3e^{3a_2z} + 3A_1^2A_2e^{(2a_1+a_2)z} + 3A_1A_2^2e^{(a_1+2a_2)z} = 0, \end{aligned} \quad (4.29)$$

where $f_{1,i}, f_{2,i}, f_{3,i}, l_{1,i}, l_{2,i}$ ($i = 0, 1, \dots, k-1$) are meromorphic functions of order less than 1. Set $J = \{3a_1, 3a_2, 2a_1 + a_2, a_1 + 2a_2, 2a_1, 2a_2, a_1 + a_2, a_1 + b_1, a_2 + b_1, 2a_1 + b_i, 2a_2 + b_i, a_1 + a_2 + b_i, a_1 + b_1 + b_i, a_2 + b_1 + b_i$ ($i = 1, \dots, k-1$)}. It is clear that $3a_1 \neq 2a_1, 2a_1 + a_2, a_1 + 2a_2, 3a_2$ and by Lemma 2.13 we have $3a_1 \neq a_1 + b_1, 2a_1 + b_i, a_1 + b_1 + b_i$ ($i = 1, \dots, k-1$). (i) If $3a_1 \neq 2a_2, a_1 + a_2, a_2 + b_1, 2a_2 + b_i, a_1 + a_2 + b_i, a_2 + b_1 + b_i$ ($i = 1, \dots, k-1$), then we can rewrite (4.29) in the form

$$A_1^3e^{3a_1z} + \sum_{\beta \in \Gamma_1} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_1 \subseteq J \setminus \{3a_1\}$ and α_β ($\beta \in \Gamma_1$) are meromorphic functions of order less than 1. By Lemma 2.7 and Lemma 2.8, we get $A_1 \equiv 0$, which is a contradiction.

(ii) If $3a_1 = \gamma$ such that $\gamma \in \{2a_2, a_1 + a_2, a_2 + b_1, 2a_2 + b_i, a_1 + a_2 + b_i, a_2 + b_1 + b_i$ ($i = 1, \dots, k-1$)}, then by Lemma 2.12 we have $3a_2 \neq \beta$ for all $\beta \in J \setminus \{3a_2\}$. Hence, we can rewrite (4.29) in the form

$$A_2^3e^{3a_2z} + \sum_{\beta \in \Gamma_2} \alpha_\beta e^{\beta z} = 0,$$

where $\Gamma_2 \subseteq J \setminus \{3a_2\}$ and α_β ($\beta \in \Gamma_2$) are meromorphic functions of order less than 1. By Lemma 2.7 and Lemma 2.8, we get $A_2 \equiv 0$, it is a contradiction. Hence, $h_2 \neq 0$ is proved. By Lemma 2.6 and (4.21), we have $\bar{\lambda}(g_2) = \bar{\lambda}(f'' - \varphi) = \sigma(g_2) = \sigma(f) = \infty$. The proof of Theorem 1.2 is complete.

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Benharrat Belaïdi

Department of Mathematics, Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem-(Algeria)
E-mail: belaidi@univ-mosta.dz

Habib Habib

Department of Mathematics, Laboratory of Pure and Applied Mathematics
University of Mostaganem (UMAB)
B. P. 227 Mostaganem-(Algeria)
E-mail: habibhabib2927@yahoo.fr