Annals of the University of Bucharest (mathematical series)
6 (LXIV) (2015), 19–34

Lattice preradicals versus module preradicals

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Communicated by Constantin Năstăsescu

Dedicated to Professors Nicolae Dinculeanu and Solomon Marcus
in honour of their 90th birthdays

Abstract - This paper investigates the connections between lattice preradicals and module preradicals. We show that to any lattice preradical one associates in a canonical way a module preradical, but not conversely. However, to any module preradical, or more generally, to any preradical on a locally small Abelian category, we may associate a weaker form of a lattice preradical by introducing and investigating a class of subcategories, not necessarily full, of the category $\mathcal{LM}$ of all linear modular lattices, we call linearly closed.

Key words and phrases : Modular lattice, linear modular lattice, lattice preradical, weakly lattice preradical, module preradical, linearly closed subcategory, Abelian category, hereditary torsion theory.

Mathematics Subject Classification (2010) **:** 06C05, 06C99, 16S90, 18E10.

Introduction

The aim of this paper is to investigate the connections between the lattice preradicals introduced in [4] and the usual module preradicals.

Section 0 collects together some general notation and terminology on lattices, modules, and hereditary torsion theories needed in the sequel.

Section 1 presents some basic definitions and results of [3], [4], and [5] on linear morphisms of lattices and lattice preradicals.

In Section 2 we give the main results of the paper. Firstly, we show that any lattice preradical naturally induces a module preradical, or more generally, a preradical on any locally small Abelian category, but not conversely. Then, we introduce and investigate the concept of a linearly closed subcategory of the category $\mathcal{LM}$ of all linear modular lattices; these are subcategories of $\mathcal{LM}$ that are not necessarily full but enjoy some natural conditions that are in particular satisfied when considering subcategories of locally small Abelian categories or subcategories associated with τ-saturated submodules with respect to a hereditary torsion theory τ on the category Mod-R of all right R-modules over a unital ring R.

Section 3 presents the more general concept of a preradical on a linearly closed subcategory of $\mathcal{LM}$. Then, we show that we can naturally associate to preradicals on locally small Abelian categories and module categories equipped with hereditary torsion theories lattice preradicals on the linearly closed subcategories $\mathcal{SC}_{\mathcal{X}}$ and $\mathcal{SC}_{\mathcal{H}}$ discussed in Examples 2.7 and 2.9, respectively. In the final part of this section we show how the main results of [5] about lattice preradicals on C_{11} lattices also hold for preradicals on linearly closed subcategories that are weakly hereditary.

0. Preliminaries

All lattices considered in this paper are assumed to be *bounded,* i.e., they have a least element denoted by 0 and a greatest element denoted by 1. Throughout this paper, L will always denote such a lattice. We shall denote by $\mathcal{L}$ the class of all (bounded) lattices and by $\mathcal{M}$ the class of all (bounded) modular lattices.

For a lattice L and elements $a \leqslant b$ in L we write

$$b/a := [a,b] = \{\ x \in L \mid a \leqslant x \leqslant b\,\}.$$

An *initial interval* of b/a is any interval c/a for some $c \in b/a$.

For all other undefined notation and terminology on lattices, the reader is referred to [1], [2], [7], and/or [8].

Throughout this paper R will denote an associative ring with non-zero identity element, and Mod-R (respectively, R-Mod) the category of all unital right (respectively, left) R-modules. The notation M_R will be used to designate a unital right R-module M, and $N \leqslant M$ will mean that N is a submodule of M. The lattice of all submodules of a module M will be denoted by $\mathcal{L}(M)$.

A *preradical* on Mod-R is a subfunctor q of the identity functor $1_{\text{Mod-}R}$ of Mod-R. This means that q assigns to each right R-module M a submodule $q(M)$ of M such that each morphism $f : M \longrightarrow N$ in Mod-R induces by restriction a morphism $q(f) : q(M) \longrightarrow q(N)$, i.e., $f(q(M)) \leqslant q(N)$.

In this paper $\tau = (\mathcal{T}, \mathcal{F})$ will denote a fixed hereditary torsion theory on Mod-R and $t_\tau(M)$ the τ-torsion submodule of a right R-module M. It is well-known that the assignment $M \mapsto t_\tau(M)$, $M \in$ Mod-R, defines a left exact (pre)radical on Mod-R. For any M_R we shall denote

$$\text{Sat}_\tau(M) \ := \ \{\ N \mid N \leqslant M \text{ and } M/N \in \mathcal{F}\ \},$$

and for any $N \leqslant M$ we shall denote by $\overline{N}$ the τ-*saturation* of N (in M) defined by $\overline{N}/N = t_\tau(M/N)$. The submodule N is called τ-*saturated* if $N = \overline{N}$. Note that

$$\text{Sat}_\tau(M) = \{\, N \mid N \leqslant M,\ N = \overline{N}\,\},$$

so $\mathrm{Sat}_\tau(M)$ is the set of all τ-*saturated* submodules of M.

It is well-known that for any M_R, $\mathrm{Sat}_\tau(M)$ is an upper continuous modular lattice with respect to the inclusion $\subseteq$ and the operations $\bigvee$ and $\bigwedge$ defined as follows:

$$\bigvee_{i\in I} N_i := \overline{\sum_{i\in I} N_i} \qquad \text{and} \qquad \bigwedge_{i\in I} N_i := \bigcap_{i\in I} N_i,$$

having least element $\tau(M)$ and greatest element M (see [8, Chapter 9, Proposition 4.1]).

The reader is referred to [8] for more about hereditary torsion theories.

1. Linear morphisms of lattices and lattice preradicals

In this section we recall from [3] and [4] the concepts of a *linear morphism* and of a *lattice preradical,* respectively, and list some of their basic properties. We also present from [5] the concept of a *weakly lattice preradical.*

As in [3], a mapping $f : L \longrightarrow L'$ between a lattice L with least element 0 and greatest element 1 and a lattice L' with least element $0'$ and greatest element $1'$ is called a *linear morphism* if there exist $k \in L$, called a *kernel* of f, and $a' \in L'$ such that the following two conditions are satisfied.

- $f(x) = f(x \vee k), \forall\, x \in L.$
- f induces a lattice isomorphism

$$\bar{f} : 1/k \xrightarrow{\sim} a'/0',\ \bar{f}(x) = f(x),\ \forall\, x \in 1/k.$$

If $f : L \longrightarrow L'$ is a linear morphism of lattices, then f is an increasing mapping, commutes with arbitrary joins (i.e., $f\left(\bigvee_{i\in I} x_i\right) = \bigvee_{i\in I} f(x_i)$ for any family $(x_i)_{i\in I}$ of elements of L, provided both joins exist), preserves intervals (i.e., for any $u \leqslant v$ in L, one has $f(v/u) = f(v)/f(u)$), and its kernel k is uniquely determined.

As in [3], the class $\mathcal{M}$ of all (bounded) modular lattices becomes a category, denoted by $\mathcal{LM}$, (for "linear modular") if for any $L, L' \in \mathcal{M}$ one takes as morphisms from L to L' all the linear morphisms from L to L'.

The isomorphisms in the category $\mathcal{LM}$ are exactly the isomorphisms in the full category $\mathcal{M}$ of the category $\mathcal{L}$ of all (bounded) lattices. The monomorphisms (respectively, epimorphisms) in the category $\mathcal{LM}$ are exactly the injective (respectively, surjective) linear morphisms. Moreover, the subobjects of $L \in \mathcal{LM}$ can be viewed as the intervals $a/0$ for any $a \in L$.

As in [5], a non-empty class $\mathcal{C}$ of lattices is said to be *weakly hereditary* if $a/0 \in \mathcal{C}$ for any $L \in \mathcal{C}$ and $a \in L$. According to [6], an *abstract class* of lattices is a subclass $\varnothing \neq \mathcal{C} \subseteq \mathcal{L}$ which is closed under lattice isomorphisms, i.e., if $L, K \in \mathcal{L}$, $K \simeq L$, and $L \in \mathcal{C}$, then $K \in \mathcal{C}$. Thus, a *hereditary class* of lattices as defined in [6] is nothing else than a weakly hereditary class which additionally is an abstract class.

For any non-empty subclass $\mathcal{C}$ of $\mathcal{M}$ we shall denote by $\mathcal{LC}$ the full subcategory of $\mathcal{LM}$ having $\mathcal{C}$ as the class of its objects.

Let $\mathcal{C}$ be a weakly hereditary subclass of $\mathcal{M}$. As in [5], a *weakly lattice preradical* on $\mathcal{C}$ is any functor $r : \mathcal{LC} \longrightarrow \mathcal{LC}$ satisfying the following two conditions.

- $r(L)$ is an initial interval of L for any $L \in \mathcal{LC}$.
- For any morphism $f : L \longrightarrow L'$ in $\mathcal{LC}$, $r(f) : r(L) \longrightarrow r(L')$ is the restriction and corestriction of f to $r(L)$ and $r(L')$, respectively.

The *lattice preradicals* defined in [4] are precisely the weakly lattice preradicals on hereditary classes $\mathcal{C} \subseteq \mathcal{M}$. As in the case of "true" lattice preradicals, for a weakly lattice preradical r on the weakly hereditary class $\mathcal{C} \subseteq \mathcal{M}$, we set $r(a/0) := a^r/0$ for any $a \in L$ and $L \in \mathcal{C}$.

If $a \leqslant b$ in L then $a/0, b/0$ are both in $\mathcal{C}$ because $\mathcal{C}$ is weakly hereditary. The inclusion mapping $\iota : a/0 \hookrightarrow b/0$ is clearly a linear morphism, thus it is a morphism in $\mathcal{LC}$. Applying now r we obtain $r(\iota) : a^r/0 \longrightarrow b^r/0$ as a restriction of ι, and so $a^r \leqslant b^r$.

2. Connections between lattice preradicals and module preradicals

This section contains the main results of the paper. We first show that any lattice preradical naturally induces a module preradical, or more generally a preradical on any locally small Abelian category, but not conversely. Then, we introduce the concept of a linearly closed subcategory of $\mathcal{LM}$ and show that, based on this, the main results of [5] about lattice preradicals on C_{11} lattices also hold for preradicals on linearly closed subcategories that are weakly hereditary; so, they can be at once applied to Grothendieck categories and module categories equipped with hereditary torsion theories.

Proposition 2.1. *For any lattice preradical r on $\mathcal{LM}$, the assignment $M_R \mapsto M^r$ defines a preradical $\underline{r}$ on* Mod-R*.*

Proof. Recall that, when we specialize the notation $a^r/0 := r(a/0)$, $a \in L$, $L \in \mathcal{LM}$, for $L = \mathcal{L}(M_R)$ and $a = M$, we have $M^r/0 = r(\mathcal{L}(M_R))$ in the lattice $\mathcal{L}(M_R) = M/0$.

Clearly $\underline{r}(M) := M^r \leqslant M$. Let $f : M \longrightarrow M'$ be a morphism of right R-modules. Then f induces a mapping

$$\overline{f} : \mathcal{L}(M) \longrightarrow \mathcal{L}(M'),\, \overline{f}(N) = f(N),\ \forall\, N \leqslant M,$$

which is a linear morphism of lattices. Since r is a preradical on $\mathcal{LM}$, we have

$$\overline{f}(M^r/0) = \overline{f}(r(\mathcal{L}(M)) \subseteq r(\mathcal{L}(M')) = M'^r/0,$$

and so, $\overline{f}(M^r) \subseteq M'^r$, that is, $f(\underline{r}(M)) \subseteq \underline{r}(M')$. Thus $\underline{r}$ is a module preradical. $\square$

More generally, we may consider instead of Mod-R any locally small Abelian category. Recall that an Abelian category $\mathcal{A}$ is said to be *locally small* if the class $\mathcal{L}(X)$ of all subobjects of each object X of $\mathcal{A}$ is a set, and in this case, $\mathcal{L}(X)$ is actually a modular lattice. We shall use the standard notation $A \subseteq X$ to designate an element $A \in \mathcal{L}(X)$. As it is well-known, any Grothendieck category is locally small. To extend Proposition 2.1 to a locally small Abelian category $\mathcal{A}$, it suffices to observe that, by [4, Lemma 5.1], for any morphism $f : X \longrightarrow Y$ in $\mathcal{A}$, the induced mapping

$$f_* : \mathcal{L}(X) \longrightarrow \mathcal{L}(Y),\ f_*(A) = f(A),\ \forall\, A \subseteq X,$$

is a linear morphism of lattices.

The next example shows that a module preradical does not necessarily define a lattice preradical.

Example 2.2. For any $M \in \mathbb{Z}$-Mod, denote $\underline{r}(M) = \{\, x \in M \,|\, 2x = 0 \,\}$. Then $\underline{r}$ is a preradical on $\mathbb{Z}$-Mod. We claim that there is no lattice preradical r such that $\underline{r}$ is obtained from r as in Proposition 2.1.

To see this, suppose that such an r exists. Consider the cyclic Abelian groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$. Since their lattices of subgroups $\mathcal{L}(\mathbb{Z}_2)$ and $\mathcal{L}(\mathbb{Z}_3)$ are two-element chains, they are isomorphic, and let $\varphi : \mathcal{L}(\mathbb{Z}_2) \overset{\sim}{\longrightarrow} \mathcal{L}(\mathbb{Z}_3)$ be the (unique) lattice isomorphism. Then $\varphi(r(\mathcal{L}(\mathbb{Z}_2)) \subseteq r(\mathcal{L}(\mathbb{Z}_3))$. But

$$\mathbb{Z}_2^r = \underline{r}(\mathbb{Z}_2) = \mathbb{Z}_2 \ \text{ and } \ \mathbb{Z}_3^r = \underline{r}(\mathbb{Z}_3) = 0,$$

so $r(\mathcal{L}(\mathbb{Z}_2)) = \{\mathbb{Z}_2, 0\}$ and $r(\mathcal{L}(\mathbb{Z}_3)) = \{0\}$, and then

$$\mathbb{Z}_3 = \varphi(\mathbb{Z}_2) \in \varphi(r(\mathcal{L}(\mathbb{Z}_2)) = \{0\},$$

which is a contradiction. □

We are now going to investigate when a module preradical produces a sort of a lattice preradical. Thus, we introduce the concept of a *linearly closed* subcategory of the category $\mathcal{LM}$; these are subcategories of $\mathcal{LM}$ that are not necessarily full but enjoy some natural conditions that are in particular satisfied when considering subcategories of locally small Abelian categories or subcategories associated with τ-saturated submodules with respect to a hereditary torsion theory τ on the category Mod-R.

Definition 2.3. *Let $\mathcal{SC}$ be a subcategory (not necessarily full) of $\mathcal{LM}$ having as class of objects a non-empty subclass $\mathcal{C}$ of $\mathcal{M}$. We say that $\mathcal{SC}$ is* linearly closed *if its class of morphisms* Mor$\,(\mathcal{SC})$ *satisfies the following four properties.*

(1) *If $L \in \mathcal{C}$, $a \in L$, and $a/0 \in \mathcal{C}$, then the inclusion mapping*

$$i : a/0 \hookrightarrow L,\ i(x) = x,\ \forall\, x \in a/0,$$

is in $\mathrm{Mor}\,(\mathcal{SC})$.

(2) *If* $L \in \mathcal{C}$, $a \in L$, *and* $1/a \in \mathcal{C}$, *then the linear morphism*

$$p : L \longrightarrow 1/a,\ p(x) = x \vee a,\ \forall\, x \in L,$$

is in $\mathrm{Mor}\,(\mathcal{SC})$.

(3) *If* $f : L \longrightarrow L'$ *is in* $\mathrm{Mor}\,(\mathcal{SC})$, k *is the kernel of* f, *and* $a' \in L'$ *is such that* $\overline{f} : 1/k \overset{\sim}{\longrightarrow} a'/0'$ *is the induced isomorphism, then*

$$1/k \in \mathcal{C},\ a'/0' \in \mathcal{C},\ \textit{and}\ \overline{f} \in \mathrm{Mor}\,(\mathcal{SC}).$$

(4) *If* $f : L \overset{\sim}{\longrightarrow} L'$ *is in* $\mathrm{Mor}\,(\mathcal{SC})$ *and is an isomorphism in* $\mathcal{LM}$, *then its inverse* f^{-1} *is in* $\mathrm{Mor}\,(\mathcal{SC})$ (*i.e.*, f *is an isomorphism in* $\mathcal{SC}$). □

The next result has a series of consequences that will be essentially used in our forthcoming paper [5] investigating the behavior under lattice preradicals of the condition (C_{11}) in modular lattices.

Proposition 2.4. *Let* $\mathcal{SC}$ *be a linearly closed subcategory of* $\mathcal{LM}$, *let* $f : L \longrightarrow L'$ *be a morphism in* $\mathcal{SC}$ *with kernel* k, *and let* $a, b \in L$ *such that* $a/0$ *and* $1'/f(b)$ *are in* $\mathcal{SC}$. *Then* $a/((b \vee k) \wedge a)$ *and* $f(a \vee b)/f(b)$ *are both in* $\mathcal{SC}$, *and the canonical morphism*

$$\overline{g} : a/((b \vee k) \wedge a) \longrightarrow f(a \vee b)/f(b),\ x \mapsto f(x) \vee f(b),$$

induced by f *is an isomorphism in* $\mathrm{Mor}\,(\mathcal{SC})$.

Proof. By Definition 2.3(1), the inclusion mapping $i : a/0 \hookrightarrow L$ is in $\mathrm{Mor}\,(\mathcal{SC})$, and, by Definition 2.3(2) the projection

$$p : L' \longrightarrow 1'/f(b),\ p(y) = y \vee f(b),\ \forall\, y \in L',$$

is also in $\mathrm{Mor}\,(\mathcal{SC})$. Thus $g := p \circ f \circ i : a/0 \longrightarrow 1'/f(b)$ is in $\mathrm{Mor}\,(\mathcal{SC})$.

The kernel of g is $(b \vee k) \wedge a$. Indeed, for $x \in a/0$, we have

$$g(x) = f(b) \Longleftrightarrow f(x) \vee f(b) = f(b) \Longleftrightarrow f(x \vee b) = f(b) \Longleftrightarrow x \vee b \vee k = b \vee k$$

$$\Longleftrightarrow x \leqslant b \vee k \Longleftrightarrow x \leqslant (b \vee k) \wedge a.$$

Since $g(a) = f(a \vee b)$, it follows that the isomorphism induced by the linear morphism g is

$$\overline{g} : a/((b \vee k) \wedge a) \longrightarrow f(a \vee b)/f(b),\ x \mapsto f(x) \vee f(b).$$

By Definition 2.3(3), we obtain the desired conclusion. □

Corollary 2.5. *The following assertions hold for a linearly closed subcategory* $\mathcal{SC}$ *of* $\mathcal{LM}$*, a morphism* $f : L \longrightarrow L'$ *in* $\mathcal{SC}$ *with kernel* k*, and elements* $a, b \in L$.

(1) *If* $a/0 \in \mathcal{SC}$*, then both intervals* $a/(a \wedge k)$ *and* $f(a)/0'$ *are in* $\mathcal{SC}$*, and the canonical morphism*

$$\alpha : a/(a \wedge k) \longrightarrow f(a)/0'$$

induced by f *is an isomorphism in* $\mathrm{Mor}\,(\mathcal{SC})$.

(2) *If* $1'/f(b) \in \mathcal{SC}$*, then both intervals* $1/(b \vee k)$ *and* $f(1)/f(b)$ *are in* $\mathcal{SC}$*, and the canonical morphism*

$$\beta : 1/(b \vee k) \longrightarrow f(1)/f(b)$$

induced by f *is an isomorphism in* $\mathrm{Mor}\,(\mathcal{SC})$.

Proof. (1) Apply Proposition 2.4 first for $b = 0$, and then for $a = 1$. □

Corollary 2.6. *The following assertions hold for a linearly closed subcategory* $\mathcal{SC}$ *of* $\mathcal{LM}$*,* $L \in \mathcal{C}$*, and* $a, b \in L$.

(1) *If* $a/0 \in \mathcal{C}$ *and* $1/b \in \mathcal{C}$*, then* $a/(a \wedge b) \in \mathcal{C}$*,* $(a \vee b)/b \in \mathcal{C}$*, and the canonical isomorphisms*

$$\varphi : a/(a \wedge b) \xrightarrow{\sim} (a \vee b)/b,\ \varphi(x) = x \vee b,\ \forall\, x \in a/(a \wedge b),$$

$$\psi : (a \vee b)/b \xrightarrow{\sim} a/(a \wedge b),\ \psi(y) = y \wedge a,\ \forall\, y \in (a \vee b)/b,$$

are both in $\mathrm{Mor}\,(\mathcal{SC})$.

(2) *Suppose that* $1 = a \dot{\vee} b$ *(this means that* $1 = a \vee b$ *and* $a \wedge b = 0$*). If* $a/0 \in \mathcal{C}$ *and* $1/b \in \mathcal{C}$*, then the linear morphism*

$$q : L \longrightarrow a/0,\ q(x) := (x \vee b) \wedge a,\ \forall\, x \in L,$$

is in $\mathrm{Mor}\,(\mathcal{SC})$*. Moreover,* q *is a surjective linear morphism with kernel* b.

(3) *If* $0/0 \in \mathcal{C}$*, then, the mapping* $o : L \longrightarrow 0/0,\ o(x) = 0,\ \forall\, x \in L$*, is in* $\mathrm{Mor}\,(\mathcal{SC})$.

(4) *If* $K \in \mathcal{C}, 0/0 \in \mathcal{C}$*, and there exists a morphism from* K *to* L *in* $\mathrm{Mor}\,(\mathcal{SC})$*, then the mapping* $K \longrightarrow L, x \mapsto 0$*, is in* $\mathrm{Mor}\,(\mathcal{SC})$.

Proof. (1) Apply Proposition 2.4 for $L' = L$ and $f = 1_L$. Then $\overline{g} = \varphi$ is in $\mathrm{Mor}\,(\mathcal{SC})$. Since $\psi = \varphi^{-1}$, by Definition 2.3(4), we have $\psi \in \mathrm{Mor}\,(\mathcal{SC})$.

(2) With notation from (1) above, we have $q = \psi \circ p$, where

$$p : L \longrightarrow 1/b,\ p(x) = x \vee b,\ \forall\, x \in L.$$

For the last part of (2), see [4, Example 0.2(3)].

(3) Take $a = 0$ and $b = 1$ in (2).

(4) Compose the inclusion mapping of $0/0$ into L with the previous mapping o and the supposed morphism from K to L. □

We present now two examples where linearly closed subcategories naturally occur: in locally small Abelian categories and in τ-saturated submodules with respect to a hereditary torsion theory τ on the category Mod-R.

Example 2.7. Let $\mathcal{X}$ be a non-empty class of objects of a locally small Abelian category $\mathcal{A}$, in particular a non-empty class of right R-modules. We assume that $\mathcal{X}$ is *hereditary,* i.e., it is closed under subobjects; this means that for every $X \in \mathcal{X}$ and subobject Y of X in $\mathcal{A}$, we have $Y \in \mathcal{X}$.

For any $X' \subseteq X$ in $\mathcal{A}$, we denote by $[X', X]$ the interval in the lattice $\mathcal{L}(X)$, and by

$$\varphi_{X/X'} : [X', X] \xrightarrow{\sim} \mathcal{L}(X/X')$$

the canonical lattice isomorphism $Z \mapsto Z/X'$, which is clearly a linear morphism of lattices.

We shall associate to $\mathcal{X}$ a linearly closed subcategory $\mathcal{SC}_{\mathcal{X}}$ having

$$\mathcal{C}_{\mathcal{X}} := \{\, [X', X] \,|\, X \in \mathcal{X},\ X' \subseteq X\}$$

as class of objects, and as morphisms those mappings that are induced by morphisms $f : X/X' \longrightarrow Y/Y'$ in $\mathcal{A}$, i.e., arise as compositions

$$[X', X] \xrightarrow{\varphi_{X/X'}} \mathcal{L}(X/X') \xrightarrow{f_*} \mathcal{L}(Y/Y') \xrightarrow{\varphi^{-1}_{Y/Y'}} [Y', Y].$$

Recall that for any morphism $f : A \longrightarrow B$ in $\mathcal{A}$ we denoted by f_* the so called *direct image* mapping

$$f_* : \mathcal{L}(A) \longrightarrow \mathcal{L}(B),\ f_*(A') = f(A'),\ \forall\, A' \in \mathcal{L}(A).$$

By [4, Lemma 5.1], any such mapping f_* is a linear morphism of lattices, so, the morphisms in $\mathcal{SC}_{\mathcal{X}}$, as compositions of linear morphisms of lattices, are also linear morphisms of lattices.

Notice that the transition from morphisms in $\mathcal{A}$ to their direct image mappings is functorial, i.e., for any morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ we have $(g \circ f)_* = g_* \circ f_*$ and $(1_A)_* = 1_{\mathcal{L}(A)}$. Therefore, if f is an isomorphism in $\mathcal{A}$, then f_* is a linear lattice isomorphism and $(f_*)^{-1} = (f^{-1})_*$.

Clearly, $\mathcal{SC}_{\mathcal{X}}$ is a subcategory, not necessarily full, of the category $\mathcal{LM}$. We are now going to show that $\mathcal{SC}_{\mathcal{X}}$ is indeed a linearly closed subcategory of $\mathcal{LM}$, i.e., it verifies the properties (1) - (4) of Definition 2.3.

For property (1), let $[X', X] \in \mathcal{C}_{\mathcal{X}}$, and let $Y \in [X', X]$. Because the class $\mathcal{X}$ is hereditary, we have $Y \in \mathcal{X}$. Clearly, the inclusion mapping $[X', Y] \overset{\iota}{\hookrightarrow} [X', X]$ is induced by the inclusion morphism $Y/X' \hookrightarrow X/X'$ in $\mathcal{A}$, so $\iota \in \mathrm{Mor}\,(\mathcal{SC}_{\mathcal{X}})$, as desired.

For property (2), let $[X', X] \in \mathcal{C}_{\mathcal{X}}$, and let $Y \in [X', X]$. Then $Y \in \mathcal{X}$. We have to prove that the mapping

$$\pi : [X', X] \longrightarrow [Y, X],\ \pi(Z) = Y + Z,\ \forall\, Z \in [X', X],$$

is induced by a certain morphism in $\mathcal{A}$, namely by the canonical epimorphism $q : X/X' \longrightarrow X/Y$ in $\mathcal{A}$, i.e.,

$$\pi = \varphi^{-1}_{X/Y} \circ q_* \circ \varphi_{X/X'}.$$

Indeed

$$(\varphi^{-1}_{X/Y} \circ q_* \circ \varphi_{X/X'})(Z) = (\varphi^{-1}_{X/Y} \circ q_*)(Z/X') = \varphi^{-1}_{X/Y}((Y + Z)/Y) = \\ = Y + Z = \pi(Z),\ \forall\, Z \in [X', X].$$

To verify the property (3), let $\alpha : [X', X] \longrightarrow [Y', Y]$ be a morphism in $\mathcal{SC}_{\mathcal{X}}$. This means that α is induced by a morphism $f : X/X' \longrightarrow Y/Y'$ in $\mathcal{A}$, i.e.,

$$\alpha = \varphi^{-1}_{Y/Y'} \circ f_* \circ \varphi_{X/X'}.$$

Set $K := \mathrm{Ker}\,(f)$ and $I := \mathrm{Im}\,(f)$. Since $\mathcal{A}$ is an Abelian category, we have $K = U/X'$ and $I = V/Y'$ for some $X' \subseteq U \subseteq X$ and $Y' \subseteq V \subseteq Y$. Now, observe that $V \in \mathcal{C}$ because the given class $\mathcal{C}$ is hereditary, so $[U, X] \in \mathcal{C}_{\mathcal{X}}$ and $[Y', V] \in \mathcal{C}_{\mathcal{X}}$.

Further, let

$$\overline{f} : (X/X')/(U/X') \overset{\sim}{\longrightarrow} V/Y' \quad \text{and} \quad h : X/U \overset{\sim}{\longrightarrow} (X/X')/(U/X')$$

be the canonical isomorphisms in $\mathcal{A}$, and set $g := \overline{f} \circ h$. Then $g_* = \overline{f}_* \circ h_*$ is an isomorphism in $\mathcal{LM}$. If we set $\overline{\alpha} := \varphi_{V/Y'} \circ g_* \circ \varphi_{X/U}$, then it is easily checked that the obtained isomorphism $\overline{\alpha} : [U, X] \overset{\sim}{\longrightarrow} [Y', V]$ in $\mathcal{LM}$ is a restriction of the given morphism α, i.e., $\overline{\alpha}(Z) = \alpha(Z),\ \forall\, Z \in [U, X]$. To conclude that $\overline{\alpha} \in \mathrm{Mor}\,(\mathcal{SC}_{\mathcal{X}})$, we have to prove that U is the kernel of the given linear mapping α, i.e., $\alpha(W + U) = \alpha(W),\ \forall\, W \in [X', X]$.

Indeed, for any $W \in [X', X]$, we have

$$\alpha(W + U) = (\varphi^{-1}_{Y/Y'} \circ f_* \circ \varphi_{X/X'})(W + U) = (\varphi^{-1}_{Y/Y'} \circ f_*)((W + U)/X') = \\ = \varphi^{-1}_{Y/Y'}(f((W + U)/X')) = \varphi^{-1}_{Y/Y'}(f(W/X') + f(U/X')) = \\ = \varphi^{-1}_{Y/Y'}(f(W/X') + f(K)) = \varphi^{-1}_{Y/Y'}(f(W/X')) = \alpha(W).$$

To prove the property (4), let $\alpha : [X', X] \longrightarrow [Y', Y]$, $\alpha \in \mathrm{Mor}\,(\mathcal{SC}_{\mathcal{X}})$. This means that α is induced by a morphism $f : X/X' \longrightarrow Y/Y'$ in $\mathcal{A}$, i.e.,

$$\alpha = \varphi_{Y/Y'}^{-1} \circ f_* \circ \varphi_{X/X'}.$$

Assume that α is an isomorphism in $\mathcal{LM}$, so a bijective mapping. Then f_* is also a bijective mapping.

Let $K := \mathrm{Ker}\,(f)$. Then, $f_*(K) = f(K) = 0 = f_*(0)$, where 0 is the zero object of $\mathcal{A}$, so $K = 0$ because f_* is an injective mapping. Thus f is a monomorphism. We also have $\alpha(X) = Y$ because α, as a lattice isomorphism, carries the greatest element of $[X', X]$ onto the greatest element of $[Y', Y]$. Then $f(X/X') = Y/Y'$, i.e., f is an epimorphism, so a bimorphism. Thus f is an isomorphism in $\mathcal{A}$. This implies that α^{-1} is induced by f^{-1}, i.e., α^{-1} is an isomorphism in $\mathrm{Mor}\,(\mathcal{SC}_{\mathcal{X}})$, as desired. □

We shall discuss now another circumstance where the linearly closed subcategories naturally occur, namely in lattices of τ-saturated submodules with respect to a hereditary torsion theory τ on the category Mod-R. To do that, we recall the following result.

Lemma 2.8. ([1, Lemma 3.4.4]). *The following statements hold for a module M_R and submodules $P \subseteq N$ of M_R.*

(1) *The mapping $\alpha : \mathrm{Sat}_\tau(N/P) \longrightarrow \mathrm{Sat}_\tau(\overline{N}/\overline{P})$, $X/P \mapsto \overline{X}/\overline{P}$, is a lattice isomorphism.*

(2) $\mathrm{Sat}_\tau(N) \simeq \mathrm{Sat}_\tau(\overline{N})$.

(3) *If $M/N \in \mathcal{T}$, then $\mathrm{Sat}_\tau(M) \simeq \mathrm{Sat}_\tau(N)$.*

(4) *If $N, P \in \mathrm{Sat}_\tau(M)$, then the assignment $X \mapsto X/P$ defines a lattice isomorphism from the interval $[P, N]$ of the lattice $\mathrm{Sat}_\tau(M)$ onto the lattice $\mathrm{Sat}_\tau(N/P)$.* □

Example 2.9. Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on Mod-R, and let $\mathcal{H}$ be a non-empty class of right R-modules which is *τ-hereditary.* Recall from [5] that $\mathcal{H}$ is said to be *τ-hereditary* if for any $M \in \mathcal{H}$ and $N \in \mathrm{Sat}_\tau(M)$ one has $N \in \mathcal{H}$.

For any $M \in \mathcal{H}$ and $M' \in \mathrm{Sat}_\tau(M)$, we denote by $[M', M]$ the interval in the lattice $\mathrm{Sat}_\tau(M)$, and by

$$\psi_{M/M'} : [M', M] \xrightarrow{\sim} \mathrm{Sat}_\tau(M/M'),\ \psi(N) := N/M',\ \forall\, N \in [M', M],$$

the canonical lattice isomorphism in Lemma 2.8(4), which is clearly a linear morphism of lattices.

We shall associate to $\mathcal{H}$ a linearly closed subcategory $\mathcal{SC}_{\mathcal{H}}$ having

$$\mathcal{C}_{\mathcal{H}} := \{\, [M', M] \mid M \in \mathcal{H},\ M' \in \mathrm{Sat}_\tau(M)\}$$

as class of objects and as morphisms those mappings that are induced by morphisms $f : M/M' \longrightarrow P/P'$ in Mod-R, i.e., arise as compositions

$$[M', M] \overset{\psi_{M/M'}}{\longrightarrow} \mathrm{Sat}_{\tau}(M/M') \overset{f_\tau}{\longrightarrow} \mathrm{Sat}_{\tau}(P/P') \overset{\psi^{-1}_{P/P'}}{\longrightarrow} [P', P].$$

where, for any morphism $f : A \longrightarrow B$ in Mod-R, f_τ denotes the mapping

$$f_\tau : \mathrm{Sat}_{\tau}(A) \longrightarrow \mathrm{Sat}_{\tau}(B),\ f_\tau(X) = \overline{f(X)},\ \forall\, X \in \mathrm{Sat}_{\tau}(A).$$

Notice that f_τ is a linear morphism of lattices by [4, Lemma 6.6]. We deduce that the morphisms in $\mathcal{SC}_{\mathcal{H}}$, as compositions of linear morphisms of lattices, are also so.

We are now going to show that $\mathcal{SC}_{\mathcal{H}}$ is indeed a linearly closed subcategory of $\mathcal{LM}$, i.e., it verifies the properties (1) - (4) of Definition 2.3. Essentially, we shall proceed as in Example 2.7 by replacing the lattices $\mathcal{L}(X/X')$ with the lattices $\mathrm{Sat}_{\tau}(M/M')$, and the intervals $[X', X]$ in the lattice $\mathcal{L}(X)$ with the intervals $[M', M]$ in the lattice $\mathrm{Sat}_{\tau}(M)$.

For instance, to check the property (1), let $[M', M] \in \mathcal{C}_{\mathcal{H}}$ and $N \in [M', M]$. Because the class $\mathcal{H}$ is τ-hereditary, we have $N \in \mathcal{H}$. Clearly, the inclusion mapping $\iota : [M', N] \hookrightarrow [M', M]$ is induced by the inclusion morphism $N/M' \hookrightarrow M/M'$ in Mod-R, so $\iota \in \mathrm{Mor}\,(\mathcal{SC}_{\mathcal{H}})$, as desired.

Similarly, to prove the property (2), let $[M', M] \in \mathcal{C}_{\mathcal{H}}$ and $N \in [M', M]$. Then $N \in \mathcal{H}$. We have to prove that the mapping

$$\pi : [M', M] \longrightarrow [N, M],\ \pi(P) = N \vee P,\ \forall\, P \in [M', M],$$

is induced by a certain morphism in Mod-R, namely by the canonical epimorphism $q : M/M' \longrightarrow M/N,\ q(U/M') = (N + U)/N$, in Mod-$R$, i.e.,

$$\pi = \psi^{-1}_{M/N} \circ q_\tau \circ \psi_{M/M'}.$$

Indeed, $q_\tau(P/M') = \overline{q(P/M')} = \overline{(N + P)/N} = (\,\overline{N + P}\,)/N = (N \vee P)/N$, so we have

$$\begin{aligned}(\psi^{-1}_{M/N} \circ q_\tau \circ \psi_{M/M'})(P) = (\psi^{-1}_{M/N} \circ q_\tau)(P/M') = \psi^{-1}_{M/N}((N \vee P)/N) = \\ = N \vee P = \pi(P),\ \forall\, P \in [M', M].\end{aligned}$$

To verify the property (3), let $\alpha : [M', M] \longrightarrow [N', N]$ be a morphism in $\mathcal{SC}_{\mathcal{H}}$. This means that α is induced by a morphism $f : M/M' \longrightarrow N/N'$ in Mod-R, i.e.,

$$\alpha = \psi^{-1}_{N/N'} \circ f_\tau \circ \psi_{M/M'}.$$

Set $K := \mathrm{Ker}\,(f)$ and $I := \mathrm{Im}\,(f)$. We have $K = U/M'$ and $I = V/N'$ for some $M' \leqslant U \leqslant M$ and $N' \leqslant V \leqslant N$.

Further, let

$$\overline{f} : (M/M')/(U/M') \xrightarrow{\sim} V/N' \quad \text{and} \quad h : M/U \xrightarrow{\sim} (M/M')/(U/M')$$

be the canonical module isomorphisms, and set $g := \overline{f} \circ h$. Then $g_\tau = \overline{f}_\tau \circ h_\tau$ is an isomorphism in $\mathcal{LM}$.

Because $\overline{U} \in \operatorname{Sat}_\tau(M)$, $\overline{V} \in \operatorname{Sat}_\tau(N)$, and the class $\mathcal{H}$ is hereditary, we have $[\overline{U}, M]$, $[N', \overline{V}] \in \mathcal{C}_\mathcal{H}$. We are going to prove that there exists a linear lattice isomorphism $\beta : [\overline{U}, M] \xrightarrow{\sim} [N', \overline{V}]$ such that β is the restriction of the given morphism $\alpha \in \operatorname{Mor}(\mathcal{SC}_\mathcal{H})$.

Indeed, the lattice isomorphism $g_\tau : \operatorname{Sat}_\tau(M/U) \xrightarrow{\sim} \operatorname{Sat}_\tau(V/N')$ yields by Lemma 2.8 the following sequence of canonical lattice isomorphisms

$$[\overline{U}, M] \xrightarrow{\sim} \operatorname{Sat}_\tau(M/\overline{U}) \xrightarrow{\sim} \operatorname{Sat}_\tau(\overline{V}/N') \xrightarrow{\sim} [N', \overline{V}].$$

It is straightforward to check that their composition β is exactly the restriction of the given morphism $\alpha : [M', M] \longrightarrow [N', N]$ in $\mathcal{SC}_\mathcal{H}$, i.e., $\alpha(Z) = \beta(Z), \forall Z \in [U, M]$.

To conclude, we have to prove that $\overline{U}$ is the kernel of the given linear mapping α, i.e.,

$$\alpha(W \vee \overline{U}) = \alpha(W), \forall W \in [M', M].$$

First, notice that $f(\overline{K}) \subseteq \overline{f(K)}$ (see the proof of [4, Lemma 6.6]), so $\overline{0} \subseteq \overline{f(\overline{K})} \subseteq \overline{\overline{f(K)}} = \overline{f(K)} = \overline{0}$, and then $\overline{f(K)} = \overline{f(\overline{K})} = \overline{0}$. We have

$$\begin{aligned} \alpha(W \vee \overline{U}) = (\psi_{N/N'}^{-1} \circ f_\tau \circ \psi_{M/M'})(W \vee \overline{U}) = (\psi_{N/N'}^{-1} \circ f_\tau)((W \vee \overline{U})/M') = \\ = \psi_{N/N'}^{-1}(f_\tau((W \vee \overline{U})/M')) = \psi_{N/N'}^{-1}(f_\tau(\psi_{M/M'}(W))) = \\ = (\psi_{N/N'}^{-1} \circ f_\tau \circ \psi_{M/M'})(W) = \alpha(W), \end{aligned}$$

as desired, because

$$\begin{aligned} \overline{f_\tau((W \vee \overline{U})/M')} = \overline{f_\tau((\overline{W + \overline{U}})/M')} = \overline{f_\tau((\overline{W + U})/M')} = \\ = f_\tau(\overline{(W+U)/M'}) = f_\tau(\overline{(W/M') + (U/M')}) = f_\tau((\overline{W/M'}) \vee (\overline{U/M'})) = \\ = f_\tau(W/M') \vee f_\tau(\overline{K}) = f(W/M') \vee \overline{f(\overline{K})} = f(W/M') \vee \overline{0} = f_\tau(\psi_{M/M'}(W). \end{aligned}$$

To prove the property (4), let $\alpha : [M', M] \longrightarrow [N', N]$, $\alpha \in \operatorname{Mor}(\mathcal{SC}_\mathcal{H})$. This means that α is induced by a morphism $f : M/M' \longrightarrow N/N'$ in Mod-R, i.e.,

$$\alpha = \psi_{N/N'}^{-1} \circ f_\tau \circ \psi_{M/M'}.$$

Assume that α is an isomorphism in $\mathcal{LM}$, so a bijective mapping. Then f_τ is also a bijective mapping. Notice that $M/M', N/N' \in \mathcal{F}$ because $M' \in \operatorname{Sat}_\tau(M)$ and $N' \in \operatorname{Sat}_\tau(N)$.

Let $K := \operatorname{Ker}(f)$. Then, $f_\tau(K) = \overline{f(K)} = \overline{0} = 0 = f_\tau(0)$, so $K = 0$ because f_τ is an injective mapping, so f is a monomorphism.

We have also $\alpha(M) = N$ because α, as a lattice isomorphism, carries the greatest element of $[M', M]$ onto the greatest element of $[N', N]$. Then $f(M/M') = N/N'$, i.e., f is an epimorphism, so an isomorphism in Mod-R. This implies that α^{-1} is induced by f^{-1}, which shows that α^{-1} is an isomorphism in $\operatorname{Mor}(\mathcal{SC}_\mathcal{H})$, as desired. □

3. Preradicals on linearly closed subcategories of $\mathcal{LM}$

In this section we define the more general concept of a preradical on a linearly closed subcategory of $\mathcal{LM}$ and show that we can associate to preradicals on locally small Abelian categories and module categories equipped with hereditary torsion theories lattice preradicals on the linearly closed subcategories $\mathcal{SC}_{\mathcal{X}}$ and $\mathcal{SC}_{\mathcal{H}}$ discussed in Examples 2.7 and 2.9, respectively. Finally we show that how the main results of [5] also hold for any preradical on a linearly closed subcategory of $\mathcal{LM}$ which is weakly hereditary.

Proposition 3.1. *The following assertions are equivalent for a a linearly closed subcategory $\mathcal{SC}$ of $\mathcal{LM}$.*

(1) *$\mathcal{C}$ is weakly hereditary.*

(2) *The monomorphisms in the category $\mathcal{SC}$ are injective.*

(3) *For any $L \in \mathcal{C}$, the subobjects of L in the category $\mathcal{SC}$ can be regarded as the initial intervals $a/0$ of $L = 1/0$, $a \in L$.*

Proof. (1)$\Longrightarrow$(2): Let $f : L \longrightarrow L'$ be a monomorphism in $\mathcal{SC}$. If k is the kernel of f, then $K := k/0 \in \mathcal{C}$ since $\mathcal{C}$ is weakly hereditary. By Definition 2.3, the inclusion mapping $\kappa : K \hookrightarrow L$ is in $\mathrm{Mor}\,(\mathcal{SC})$. Also, since $\mathcal{C}$ is weakly hereditary, we have $0/0 \in \mathcal{C}$, and by Corollary 2.6(4) the zero mapping $o : K \longrightarrow L$ is in $\mathrm{Mor}\,(\mathcal{SC})$. We have $f \circ \kappa = f \circ o$, and since f is a monomorphism, we deduce that $\kappa = o$, thus $k = 0$, and consequently, f is injective.

(2)$\Longrightarrow$(3): Let (S, α) be a subobject of L in $\mathcal{SC}$. Then α is a monomorphism, thus injective by (2). By Definition 2.3, its image $a/0 \in \mathcal{C}$, for $a \in L$, and since its kernel is zero, α induces an isomorphism $\overline{\alpha} : S \xrightarrow{\sim} a/0$, which is in $\mathrm{Mor}\,(\mathcal{SC})$. Since the inclusion mapping of $i : a/0 \hookrightarrow L$ is a monomorphism in $\mathrm{Mor}\,(\mathcal{SC})$, it follows that $(a/0, i)$ is a subobject of L in $\mathcal{SC}$ that is isomorphic to (S, α) via $\overline{\alpha}$.

(3)$\Longrightarrow$(1): For $a \in L$ and inclusion mapping $i : a/0 \hookrightarrow L$, $(a/0, i)$ is a subobject of L in $\mathcal{SC}$, hence $a/0 \in \mathcal{C}$. □

Definition 3.2. *Let $\mathcal{SC}$ be a linearly closed subcategory of $\mathcal{LM}$ such that its class of objects $\mathcal{C}$ is weakly hereditary. A* lattice preradical *on $\mathcal{SC}$ is any functor $r : \mathcal{SC} \longrightarrow \mathcal{SC}$ satisfying the following two conditions.*

(1) *$r(L) \leqslant L$, i.e., $r(L)$ is a subobject of L, for any $L \in \mathcal{SC}$.*

(2) *For any morphism $f : L \longrightarrow L'$ in $\mathcal{SC}$, $r(f) : r(L) \longrightarrow r(L')$ is the restriction and corestriction of f to $r(L)$ and $r(L')$, respectively.* □

Let $\mathcal{SC}$ be a linearly closed subcategory of $\mathcal{LM}$ such that its class of objects $\mathcal{C}$ is weakly hereditary, and let $r : \mathcal{SC} \longrightarrow \mathcal{SC}$ be a lattice preradical on $\mathcal{SC}$. By Proposition 3.1, for every $L \in \mathcal{C}$ and $a \in L$, the subobject $r(a/0)$ of L in $\mathcal{SC}$ is necessarily an initial interval of $a/0$. We denote

$$r(a/0) := a^r/0.$$

If $a \leqslant b$ in L then $a/0, b/0$ are in $\mathcal{C}$ because $\mathcal{C}$ is weakly hereditary. The inclusion mapping $i : a/0 \hookrightarrow b/0$ is in $\mathrm{Mor}\,(\mathcal{SC})$ since $\mathcal{SC}$ is linearly closed. Applying r we obtain the morphism $r(i) : a^r/0 \longrightarrow b^r/0$ as a restriction of i, and so $a^r \leqslant b^r$.

Recall that a *preradical* on an Abelian category $\mathcal{A}$ is just a subfunctor of the identity functor $1_{\mathcal{A}}$ of $\mathcal{A}$.

Proposition 3.3. *Let $\mathcal{X}$ be a hereditary class of objects of a locally small Abelian category $\mathcal{A}$, and let r be a preradical on $\mathcal{A}$. Then r canonically yields a preradical ϱ on the linearly closed subcategory*

$$\mathcal{SC}_{\mathcal{X}} := \{\, [X', X] \mid X \in \mathcal{X},\ X' \subseteq X\}$$

of $\mathcal{LM}$ discussed in Example 2.7.

Proof. With notation of Example 2.7, let $[X', X] \in \mathcal{SC}_{\mathcal{X}}$. Then $r(X/X') = Y/X'$ for some $Y \in \mathcal{A}$ with $X' \subseteq Y \subseteq X$. We set $X^r := Y$. Because $\mathcal{X}$ is a hereditary subclass of $\mathcal{A}$, we have $X^r \in \mathcal{X}$, so we can define the following mapping

$$\varrho : \mathcal{SC}_{\mathcal{X}} \longrightarrow \mathcal{SC}_{\mathcal{X}},\ \varrho([X', X]) := [X', X^r],\ \forall\, [X', X] \in \mathcal{SC}_{\mathcal{X}}.$$

By definition, $\varrho([X', X])$ is a subobject of $[X', X]$ for any $[X', X] \in \mathcal{SC}_{\mathcal{X}}$. To conclude that ϱ is a preradical on $\mathcal{SC}_{\mathcal{X}}$, we must show that for any morphism $\alpha : [X', X] \longrightarrow [Y', Y]$ in $\mathcal{SC}_{\mathcal{X}}$, we have

$$\alpha(\varrho([X', X])) \subseteq \varrho([Y', Y]), \text{ i.e., } \alpha([X', X^r]) \subseteq [Y', Y^r],\ \forall\, [X', X] \in \mathcal{SC}_{\mathcal{X}}.$$

Indeed, by the definition of the morphisms in $\mathcal{SC}_{\mathcal{X}}$, α is induced by a morphism $f : X/X' \longrightarrow Y/Y'$ in $\mathcal{A}$, i.e., arises as a composition

$$[X', X] \overset{\varphi_{X/X'}}{\longrightarrow} \mathcal{L}(X/X') \overset{f_*}{\longrightarrow} \mathcal{L}(Y/Y') \overset{\varphi^{-1}_{Y/Y'}}{\longrightarrow} [Y', Y].$$

Now, the morphism f yields a morphism $r(f) : r(X/X') \longrightarrow r(Y/Y')$, i.e., a morphism $r(f) : X^r/X' \longrightarrow Y^r/Y'$, and then $f_*(X^r/X') \subseteq Y^r/Y'$. This shows that

$$\alpha([X', X^r]) = (\varphi^{-1}_{Y/Y'} \circ f_* \circ \varphi_{X/X'})([X', X^r]) \subseteq \varphi^{-1}_{Y/Y'}(Y^r/Y') = [Y', Y^r],$$

as desired. $\square$

Proposition 3.4. *Let $\tau = (\mathcal{T}, \mathcal{F})$ be a hereditary torsion theory on* Mod-R, *let $\mathcal{H}$ be a τ-hereditary class of right R-modules, and let r be preradical on* Mod-R. *Then r canonically yields a preradical ϱ_τ on the linearly closed subcategory*

$$\mathcal{SC}_{\mathcal{H}} := \{\, [M', M] \mid M \in \mathcal{H},\ M' \in \operatorname{Sat}_\tau(M)\}$$

of $\mathcal{LM}$ discussed in Example 2.9.

Proof. With notation of Example 2.9, let $[M', M] \in \mathcal{SC}_{\mathcal{H}}$. Then $r(M/M') = P/M'$ for some $P \in$ Mod-R with $M' \leqslant P \leqslant M$. We set $M^r := \overline{P}$. Because $M \in \mathcal{H}$ and $\mathcal{H}$ is a τ-hereditary subclass of Mod-R, we have $M^r \in \mathcal{H}$, so we can define the following mapping

$$\varrho_\tau : \mathcal{SC}_{\mathcal{H}} \longrightarrow \mathcal{SC}_{\mathcal{H}},\ \varrho_\tau([M', M]) := [M', M^r],\ \forall\, [M', M] \in \mathcal{SC}_{\mathcal{H}}.$$

By definition, $\varrho_\tau([M', M])$ is a subobject of $[M', M]$ for any $[M', M] \in \mathcal{SC}_{\mathcal{H}}$. To conclude that ϱ_τ is a preradical on $\mathcal{SC}_{\mathcal{H}}$, we must show that for any morphism $\alpha : [M', M] \longrightarrow [N', N]$ in $\mathcal{SC}_{\mathcal{H}}$, we have

$$\alpha(\varrho_\tau([M', M]) \subseteq \varrho([N', N]), \text{ i.e., } \alpha([M', M^r]) \subseteq [N', N^r],\ \forall\, [M', M] \in \mathcal{SC}_{\mathcal{H}}.$$

Indeed, by the definition of the morphisms in $\mathcal{SC}_{\mathcal{H}}$, α is induced by a morphism $f : M/M' \longrightarrow N/N'$ in Mod-R, i.e., arises as a composition

$$[M', M] \overset{\psi_{M/M'}}{\longrightarrow} \operatorname{Sat}_\tau(M/M') \overset{f_\tau}{\longrightarrow} \operatorname{Sat}_\tau(N/N') \overset{\psi^{-1}_{N/N'}}{\longrightarrow} [N', N].$$

Now, the morphism f yields a morphism

$$r(f) : P/M' = r(M/M') \longrightarrow r(N/N') = Q/N',$$

i.e., $f(P/M') \subseteq Q/N'$. Then, by [5, Lemma 4.4], $f(\,\overline{P/M'}\,) \subseteq \overline{Q/N'}$, so

$$f_\tau(M^r/M') = \overline{f(\overline{P}/M')} = \overline{f(\,\overline{P/M'}\,)} \subseteq \overline{\overline{Q/N'}} = \overline{Q/N'} = \overline{Q}/N' = N^r/N'.$$

This shows that

$$\alpha([M', M^r]) = (\psi^{-1}_{N/N'} \circ f_\tau \circ \psi_{M/M'})([M', M^r]) \subseteq \psi^{-1}_{N/N'}(N^r/N') = [N', N^r],$$

as desired. □

Remarks 3.5. (1) Observe that Proposition 3.3 (respectively, Proposition 3.4) also holds when the given preradical r on the category $\mathcal{A}$ (respectively, Mod-R) is a preradical only on the given hereditary class $\mathcal{X}$ (respectively, τ-hereditary class $\mathcal{H}$) under the additional condition that $\mathcal{X}$ is a cohereditary (respectively, $\mathcal{H}$ is a τ-cohereditary) class. Recall that a non-empty subclass of $\mathcal{A}$ is said to be *cohereditary* if it is closed under quotient objects, and if $\tau = (\mathcal{T}, \mathcal{F})$ is a hereditary torsion theory on Mod-R, then, a non-empty class $\mathcal{H}$ of right R-modules is said to be τ-*cohereditary* if for any $M \in \mathcal{H}$ and $M' \in \mathrm{Sat}_{\tau}(M)$ one has $M/M' \in \mathcal{H}$.

(2) A thorough examination of the proofs in [5] shows that they are performed using only morphisms as in Definition 2.3 and Corollary 2.6. So, all the results of [5], in particular [5, Theorem 2.4] and its Corollary 2.5 also hold for any lattice preradical on a linearly closed subcategory of $\mathcal{LM}$ which is weakly hereditary. □

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