

On the Riemann-Hilbert Problem without Index

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Dedicated to 85 years of Academician C. Andreian Cazacu

Abstract - It is proved the existence of solutions for the Riemann-Hilbert problem in the fairly general settings of arbitrary Jordan domains, measurable coefficients and measurable boundary dates. The theorem is formulated in terms of harmonic measure and principal asymptotic values. It is also given the corresponding reinforced criterion for domains with arbitrary rectifiable boundaries stated in terms of the natural parameter and nontangential limits.

Key words and phrases : Riemann-Hilbert problem, Jordan domains, harmonic measure, principal asymptotic values, nontangential limits.

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1. Introduction

Boundary value problems for analytic functions are due to the well-known Riemann dissertation (1851) containing a general setting of a problem on finding analytic functions with a connection between its real and imaginary parts on the boundary. However, it has contained no concrete boundary value problems.

The first concrete problem of such a type has been proposed by Hilbert (1904) and called by the Hilbert problem or the Riemann-Hilbert problem. That consists in finding an analytic function f in a domain bounded by a rectifiable Jordan curve C with the linear boundary condition

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \{ \overline{\lambda(\zeta)} \cdot f(z) \} = \varphi(\zeta) \quad \forall \zeta \in C \quad (1.1)$$

where it was assumed by him that the functions λ and φ are continuously differentiable with respect to the natural parameter s on C and, moreover, $|\lambda| \neq 0$ everywhere on C . Hence without loss of generality one can assume that $|\lambda| \equiv 1$ on C .

The first way for solving this problem based on the theory of singular integral equations was given by Hilbert (1904), see [8]. This attempt was not quite successful because of the theory of singular integral equations has

been not yet enough developed at that time. However, just that way became the main approach in this research direction with important contributions of Georgian and Russian mathematicians and mechanicians, see e.g. [3], [13] and [19]. In particular, the existence of solutions to this problem was in that way proved for Hölder continuous λ and φ . But subsequent weakening conditions on λ and φ led to strengthening conditions on the contour C , say to the Lyapunov curves or the Radon condition of bounded rotation or even to smooth curves.

However, Hilbert (1905) has proposed the second way for solving his problem in setting to (1.1) above based on the reduction it to solving the corresponding two Dirichlet problems, see e.g. [9]. The goal of this paper is to show that this approach is more simple and leads to perfectly general results. That requests to apply some fundamental concepts and facts related to the Dirichlet problem.

2. The case of the unit circle

The following brilliant result of Frederick Gehring is key for our goals, see [5].

Proposition 2.1. *Let $\varphi(\vartheta)$ be real, measurable, almost everywhere finite and have the period 2π . Then there exists a function $u(z)$, harmonic in $|z| < 1$, such that $u(z) \rightarrow \varphi(\vartheta)$ for a.e. ϑ as $z \rightarrow e^{i\vartheta}$ along any nontangential path.*

Since the Gehring proof is very short and nice and has a common interest, we give it for completeness here.

Proof. By a theorem of Lusin, see e.g. Theorem VII(2.3) in [18], p. 217, we can find a continuous function $\Phi(\vartheta)$ such that $\Phi'(\vartheta) = \varphi(\vartheta)$ for a.e. ϑ . Let

$$U(re^{i\vartheta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta-t)+r^2} \Phi(t) dt$$

for $r < 1$. Next, by the well-known result due to Fatou, see e.g. 3.441 in [20], p. 53, $\frac{\partial}{\partial \vartheta} U(z) \rightarrow \Phi'(\vartheta)$ as $z \rightarrow e^{i\vartheta}$ along any nontangential path whenever $\Phi'(\vartheta)$ exists. Thus, the conclusion follows for the function given by $u(z) = \frac{\partial}{\partial \vartheta} U(z)$. \square

Remark 2.1. Recall also the preceding result of W. Kaplan on the existence of a harmonic function $u(z)$ with the radial limits $\varphi(\vartheta)$ a.e., see [10].

It is known that every harmonic function $u(z)$ in $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ has a conjugate function $v(z)$ such that $f(z) = u(z) + iv(z)$ is an analytic function in \mathbb{D} . Hence we have the following consequence of Proposition 2.1.

Corollary 2.1. *Under the conditions of Proposition 2.1, there exists an analytic function f in \mathbb{D} such that $\operatorname{Re} f(z) \rightarrow \varphi(\vartheta)$ for a.e. ϑ as $z \rightarrow e^{i\vartheta}$ along any nontangential path.*

Note that the boundary values of the conjugate function v cannot be prescribed arbitrarily and simultaneously with the boundary values of u because v is uniquely determined by u up to an additive constant.

Denote by h^p , $p \in (0, \infty)$, the class of all harmonic functions u in \mathbb{D} with

$$\sup_{r \in (0,1)} \left\{ \int_0^{2\pi} |u(re^{i\vartheta})|^p d\vartheta \right\}^{\frac{1}{p}} < \infty .$$

It is clear that $h^p \subseteq h^{p'}$ for all $p > p'$ and, in particular, $h^p \subseteq h^1$ for all $p > 1$. It is important that every function in the class h^1 has a.e. nontangential boundary limits, see e.g. Corollary IX.2.2 in [7].

Note also that $v \in h^p$ whenever $u \in h^p$ for all $p > 1$ by the M. Riesz theorem, see [17]. Generally speaking, this fact is not trivial but it follows immediately for $p = 2$ from the Parseval equality. The latter will be sufficient for our goals.

Theorem 2.1. *Let $\lambda : \partial\mathbb{D} \rightarrow \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial\mathbb{D} \rightarrow \mathbb{R}$ be measurable functions. Then there exist analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that along any nontangential path*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \{ \overline{\lambda(\zeta)} \cdot f(z) \} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial\mathbb{D} . \quad (2.1)$$

Proof. First, consider the function $\alpha(\zeta) = \arg \lambda(\zeta)$ where $\arg \omega$ is the principal value of the argument of $\omega \in \mathbb{C}$ with $|\omega| = 1$, i.e., the unique number $\alpha \in (-\pi, \pi]$ such that $\omega = e^{i\alpha}$. Note that the function $\arg \omega$ is continuous on $\partial\mathbb{D} \setminus \{-1\}$ and the sets $\lambda^{-1}(\partial\mathbb{D} \setminus \{-1\})$ and $\lambda^{-1}(-1)$ are measurable because the function $\lambda(\zeta)$ is measurable. Thus, the function $\alpha(\zeta)$ is measurable on $\partial\mathbb{D}$. Furthermore, $\alpha \in L^\infty(\partial\mathbb{D})$ because $|\alpha(\zeta)| \leq \pi$ for all $\zeta \in \partial\mathbb{D}$. Hence

$$g(z) : = \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \alpha(\zeta) \frac{z + \zeta}{z - \zeta} \frac{d\zeta}{\zeta} , \quad z \in \mathbb{D} , \quad (2.2)$$

is an analytic function in \mathbb{D} with $u(z) = \operatorname{Re} g(z) \rightarrow \alpha(\zeta)$ as $z \rightarrow \zeta$ along any nontangential path in \mathbb{D} for a.e. $\zeta \in \partial\mathbb{D}$, see, e.g., Corollary IX.1.1 in [7] and Theorem I.E.1 in [11]. Denote $\mathcal{A}(z) = \exp\{ig(z)\}$ that is an analytic function.

Since $\alpha \in L^\infty(\partial\mathbb{D})$, we have that $u \in h^p$ for all $p \geq 1$, see, e.g., Theorem IX.2.3 in [7], and then $v = \operatorname{Im} g \in h^p$ for all $p \geq 1$ by the theorem of M.

Riesz. Hence there exists a function $\beta : \partial\mathbb{D} \rightarrow \mathbb{R}$, $\beta \in L^p$, for all $p \geq 1$ such that $v(z) \rightarrow \beta(\zeta)$ as $z \rightarrow \zeta$ for a.e. $\zeta \in \partial\mathbb{D}$ along any nontangential path, see e.g. Theorem IX.2.3 and Corollary IX.2.2 in [7]. Thus, by Corollary 2.1 there exists an analytic function $\mathcal{B} : \mathbb{D} \rightarrow \mathbb{C}$ such that $\operatorname{Re} \mathcal{B}(z) = B(\zeta) := \varphi(\zeta) \cdot \exp\{\beta(\zeta)\}$ as $z \rightarrow \zeta$ along any nontangential path for a.e. $\zeta \in \partial\mathbb{D}$. Finally, elementary calculations show that one of the desired analytic functions in (2.1) is $f = \mathcal{A} \cdot \mathcal{B}$. \square

Remark 2.2. As it follows from the formula (2.2), the first analytic function \mathcal{A} in the proof is calculated in the explicit form. The function $\beta : \partial\mathbb{D} \rightarrow \mathbb{R}$ in the proof can also explicitly be calculated by the following formula, see, e.g., Theorem I.E.4.1 in [11], for a.e. $\zeta \in \partial\mathbb{D}$

$$\beta(\zeta) := \lim_{\varepsilon \rightarrow +0} \frac{1}{\pi} \int_{\varepsilon}^{\pi} \frac{\alpha(\zeta e^{-it}) - \alpha(\zeta e^{it})}{2 \operatorname{tg} \frac{t}{2}} dt. \quad (2.3)$$

The second analytic function \mathcal{B} in the proof is equal to $\frac{\partial}{\partial \vartheta} G(z)$, $z = re^{i\vartheta}$, with

$$G(z) := \frac{1}{2\pi i} \int_{\partial\mathbb{D}} \Phi(\zeta) \frac{z + \zeta}{z - \zeta} \frac{d\zeta}{\zeta}, \quad z \in \mathbb{D}, \quad (2.4)$$

where $\Phi : \partial\mathbb{D} \rightarrow \mathbb{R}$ is a continuous function such that $\frac{\partial}{\partial \vartheta} \Phi(\zeta) = B(\zeta)$, $\zeta = e^{i\vartheta}$, for a.e. $\vartheta \in [0, 2\pi]$, see the nontrivial construction of Theorem VII(2.3) in [18].

3. The case of a rectifiable Jordan curve

Theorem 3.1. *Let D be a Jordan domain in \mathbb{C} with a rectifiable boundary and let $\lambda : \partial D \rightarrow \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \rightarrow \mathbb{R}$ be measurable functions with respect to the natural parameter on ∂D . Then there exist analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that along any nontangential path*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \{ \overline{\lambda(\zeta)} \cdot f(z) \} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \quad (3.1)$$

with respect to the natural parameter on ∂D .

Proof. This case is reduced to the case of the unit disk \mathbb{D} in the following way. First, by the Riemann theorem, see e.g. Theorem II.2.1 in [7], there exists a conformal mapping ω of any Jordan domain D onto \mathbb{D} . By the Caratheodory (1912) theorem ω can be extended to a homeomorphisms of \overline{D} onto $\overline{\mathbb{D}}$ and, if ∂D is rectifiable, then by the theorem of F. and M. Riesz (1916) length $\omega^{-1}(E) = 0$ whenever $E \subset \partial\mathbb{D}$ with $|E| = 0$, see e.g. Theorem II.C.1 and Theorems II.D.2 in [11]. Conversely, by the Lavrentiev (1936)

theorem $|\omega(\mathcal{E})| = 0$ whenever $\mathcal{E} \subset \partial D$ and $\text{length } \mathcal{E} = 0$, see [12], see also the point III.1.5 in [16].

Hence ω and ω^{-1} transform measurable sets into measurable sets. Indeed, every measurable set is the union of a sigma-compact set and a set of measure zero, see e.g. Theorem III(6.6) in [18], and continuous mappings transform compact sets into compact sets. Thus, a function $\varphi : \partial D \rightarrow \mathbb{R}$ is measurable with respect to the natural parameter on ∂D if and only if the function $\Phi = \varphi \circ \omega^{-1} : \partial \mathbb{D} \rightarrow \mathbb{R}$ is measurable with respect to the linear measure on $\partial \mathbb{D}$.

By the Lindelöf (1917) theorem, see e.g. Theorem II.C.2 in [11], if ∂D has a tangent at a point ζ , then $\arg [\omega(\zeta) - \omega(z)] - \arg [\zeta - z] \rightarrow \text{const}$ as $z \rightarrow \zeta$. In other words, the conformal images of sectors in D with a vertex at ζ is asymptotically the same as sectors in \mathbb{D} with a vertex at $w = \omega(\zeta)$. Thus, nontangential paths in D are transformed under ω into nontangential paths in \mathbb{D} . Finally, a rectifiable Jordan curve has a tangent a.e. with respect to the natural parameter and, thus, the statement of Theorem 3.1 follows from Theorem 2.1. \square

In particular, choosing $\lambda \equiv 1$ in (3.1), we obtain the following statement.

Proposition 3.1. *Let D be a domain in \mathbb{C} bounded by a rectifiable Jordan curve and $\varphi : \partial D \rightarrow \mathbb{R}$ be measurable. Then there exists an analytic function $f : D \rightarrow \mathbb{C}$ such that*

$$\lim_{z \rightarrow \zeta} \text{Re } f(z) = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \quad (3.2)$$

with respect to the natural parameter on ∂D along any nontangential path.

Corollary 3.1. *Under the conditions of Proposition 3.1, there exists a harmonic function u in D such that $u(z) \rightarrow \varphi(\zeta)$ for a.e. $\zeta \in \partial D$ as $z \rightarrow \zeta$ along any nontangential path.*

4. The case of an arbitrary Jordan curve

The concepts of a harmonic measure introduced by R. Nevanlinna in [14] and a principal asymptotic value based on one nice result of F. Bagemihl [1] make possible with a great simplicity and generality to formulate the existence theorems for the Dirichlet and Riemann-Hilbert problems.

First of all, given a measurable set $E \subseteq \partial \mathbb{D}$ and a point $z \in \mathbb{D}$, a *harmonic measure* of E at z relative to \mathbb{D} is the value at z of the harmonic function u in \mathbb{D} with the boundary values 1 a.e. on E and 0 a.e. on $\partial \mathbb{D} \setminus E$, see Proposition 2.1. In particular, by the mean value theorem for harmonic functions, the

harmonic measure of E at 0 relative to \mathbb{D} is equal to $|E|/2\pi$. In general, the geometric sense of the harmonic measure of E at z_0 relative to \mathbb{D} is the angular measure of view of E from the point z_0 in radians divided by 2π .

Since the harmonic measure zero is invariant under conformal mappings between Jordan domains, given a Jordan domain D , a set $\mathcal{E} \subseteq \partial D$ will be called measurable with respect to harmonic measures in D if $E = \omega(\mathcal{E})$ is measurable with respect to the linear measure on $\partial\mathbb{D}$ where ω is a conformal mapping of D onto the unit disk \mathbb{D} , cf. the proof of Theorem 3.1. Correspondingly, the harmonic measure of $\mathcal{E} \subseteq \partial D$ at $z_0 \in D$ relative to D is the harmonic measure of $\omega_0(\mathcal{E})$ at 0 relative to \mathbb{D} where ω_0 is a conformal mapping of D onto \mathbb{D} with the normalization $\omega_0(z_0) = 0$, i.e., the quantity $|\omega_0(\mathcal{E})|/2\pi$.

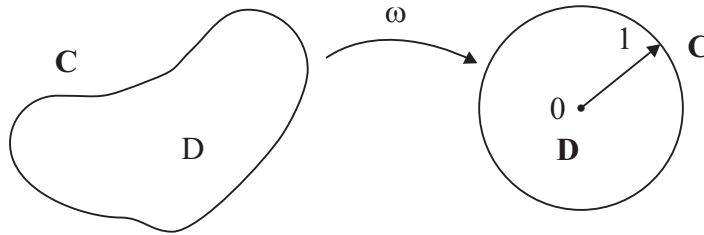


Figure 1. The Riemann mapping

Next, a Jordan curve generally speaking has no tangents. Hence we need a replacement for the notion of a nontangential limit. In this connection, recall Theorem 2 in [1], see also Theorem III.1.8 in [15], stating that, for any function $\Omega : \mathbb{D} \rightarrow \mathbb{C}$, for all pairs of arcs γ_1 and γ_2 in \mathbb{D} terminating at $\zeta \in \partial\mathbb{D}$, except a countable set of $\zeta \in \partial\mathbb{D}$,

$$C(\Omega, \gamma_1) \cap C(\Omega, \gamma_2) \neq \emptyset \quad (4.1)$$

where $C(\Omega, \gamma)$ denotes the *cluster set of Ω at ζ along γ* , i.e.,

$$C(\Omega, \gamma) = \{w \in \overline{\mathbb{C}} : \Omega(z_n) \rightarrow w, z_n \rightarrow \zeta, z_n \in \gamma\}.$$

Immediately by the theorems of Riemann and Caratheodory, this result is extended to an arbitrary Jordan domain D in \mathbb{C} . Given a function $\Omega : D \rightarrow \overline{\mathbb{C}}$ and $\zeta \in \partial D$, denote by $P(\Omega, \zeta)$ the intersection of all cluster sets $C(\Omega, \gamma)$ for arcs γ in D terminating at ζ . Later on, we call the points of the set $P(\Omega, \zeta)$ *principal asymptotic values* of Ω at ζ . Note that, if Ω has a limit along at least one arc in D terminating at a point $\zeta \in \partial D$ with the property (4.1), then the principal asymptotic value is unique.

Thus, by the Bagemihl theorem, we obtain the following result directly from Theorem 2.1.

Theorem 4.1. *Let D be a Jordan domain in \mathbb{C} and let $\lambda : \partial D \rightarrow \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial D \rightarrow \mathbb{R}$ be measurable functions with respect to harmonic measures in D . Then there exist analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \{ \overline{\lambda(\zeta)} \cdot f(z) \} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \quad (4.2)$$

with respect to harmonic measures in D in the sense of the unique principal asymptotic value.

In particular, choosing $\lambda \equiv 1$ in (4.2), we obtain the following consequence.

Proposition 4.1. *Let D be a Jordan domain and $\varphi : \partial D \rightarrow \mathbb{R}$ be measurable with respect to harmonic measures in D . Then there exists an analytic function $f : D \rightarrow \mathbb{C}$ such that*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} f(z) = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D \quad (4.3)$$

with respect to harmonic measures in D in the sense of the unique principal asymptotic value.

Corollary 4.1. *Under the conditions of Proposition 4.1, there exists a harmonic function u in D such that in the same sense*

$$\lim_{z \rightarrow \zeta} u(z) = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial D. \quad (4.4)$$

Remark 4.1. In view of the theorems of Riemann and Caratheodory, this approach makes possible also to formulate the corresponding theorems for arbitrary simply connected domains D in \mathbb{C} having at least 2 boundary points. The only difference is that the functions λ and φ should be given as functions of prime ends of D but not of points of ∂D and harmonic measures of sets of prime ends are given through the natural one-to-one correspondence between the prime ends of D and the boundary points of \mathbb{D} under Riemann mappings $\omega : D \rightarrow \mathbb{D}$, see e.g. [2].

5. On the dimension of spaces of solutions

By the Lindelöf maximum principle, see e.g. Lemma 1.1 in [4], it follows the uniqueness theorem for the Dirichlet problem in the class of **bounded** harmonic functions on the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. In general there is no uniqueness theorem in the Dirichlet problem for the Laplace equation. Furthermore, it can be proved the following result.

Theorem 5.1. *The space of all harmonic functions in \mathbb{D} with nontangential limit 0 at a.e. point of $\partial \mathbb{D}$ has the infinite dimension.*

Proof. Indeed, let $\Phi : [0, 2\pi] \rightarrow \mathbb{R}$ be integrable and differentiable a.e. with $\Phi'(t) = 0$. Then the function

$$U(z) := \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\vartheta-t)+r^2} \Phi(t) dt, \quad z = re^{i\vartheta}, \quad r < 1,$$

is harmonic on \mathbb{D} with $U(z) \rightarrow \Phi(\Theta)$ as $z \rightarrow e^{i\Theta}$, see e.g. Theorem 1.3 in [4] or Theorem IX.1.1 in [7], and $\frac{\partial}{\partial \vartheta} U(z) \rightarrow \Phi'(\Theta)$ as $z \rightarrow e^{i\Theta}$ along any nontangential path whenever $\Phi'(\Theta)$ exists, see e.g. 3.441 in [20], p. 53, or Theorem IX.1.2 in [7]. Thus, the harmonic function $u(z) = \frac{\partial}{\partial \vartheta} U(z)$ has nontangential limit 0 at a.e. point of $\partial\mathbb{D}$.

Let us give a subspace of such functions u with an infinite basis. Namely, let $\varphi : [0, 1] \rightarrow [0, 1]$ be the Cantor function, see e.g. 8.15 in [6], and let $\varphi_n : [0, 2\pi] \rightarrow [0, 1]$ be equal to $\varphi((t - a_{n-1})/(a_n - a_{n-1}))$ on $[a_{n-1}, a_n]$ where $a_0 = 0$ and $a_n = 2\pi(2^{-1} + \dots + 2^{-n})$, $n = 1, 2, \dots$ and 0 outside of $[a_{n-1}, a_n]$. Denote by U_n and u_n the harmonic functions corresponding to φ_n as in the first item.

By the construction the supports of the functions φ_n are mutually disjoint and, thus, the series $\sum_{n=1}^{\infty} \gamma_n \varphi_n$ is well defined for every sequence $\gamma_n \in \mathbb{R}$, $n = 1, 2, \dots$. If in addition we restrict ourselves to the sequences $\gamma = \{\gamma_n\}$ in the space l with the norm $\|\gamma\| = \sum_{n=1}^{\infty} |\gamma_n|$, then the series is a suitable function Φ for the first item.

Denote by U and u the harmonic functions corresponding to the function Φ as in the first item and by \mathcal{H}_0 the class of all such u . Note that u_n , $n = 1, 2, \dots$, form a basis in the space \mathcal{H}_0 with the locally uniform convergence in \mathbb{D} which is metrizable.

Firstly, $\sum_{n=1}^{\infty} \gamma_n u_n \neq 0$ if $\gamma \neq 0$. Really, let us assume that $\gamma_n \neq 0$ for some $n = 1, 2, \dots$. Then $u \neq 0$ because the limits $\lim_{z \rightarrow \zeta} U(z)$ exist for all $\zeta = e^{i\vartheta}$ with $\vartheta \in (a_{n-1}, a_n)$ and can be arbitrarily close to 0 as well as to γ_n .

Secondly, $u_m^* = \sum_{n=1}^m \gamma_n u_n \rightarrow u$ locally uniformly in \mathbb{D} as $m \rightarrow \infty$. Indeed, elementary calculations give the following estimate of the remainder term

$$|u(z) - u_m^*(z)| \leq \frac{2r(1+r)}{(1-r)^3} \cdot \sum_{n=m+1}^{\infty} |\gamma_n| \rightarrow 0 \quad \text{as } m \rightarrow \infty$$

in every disk $\mathbb{D}(r) = \{z \in \mathbb{C} : |z| \leq r\}$, $r < 1$. □

Corollary 5.1. *Given a measurable function $\varphi : \partial\mathbb{D} \rightarrow \mathbb{R}$, the space of all harmonic functions $u : \mathbb{D} \rightarrow \mathbb{R}$ with the limits $\lim_{z \rightarrow \zeta} u(z) = \varphi(\zeta)$ for a.e. $\zeta \in \partial\mathbb{D}$ along nontangential paths has the infinite dimension.*

Indeed, the existence at least one such harmonic function u follows from the known Gehring theorem in [5], see Proposition 2.1 above. Combining this fact with Theorem 5.1, we obtain the conclusion of Corollary 5.1.

Remark 5.1. Note that the harmonic functions u found in Theorem 5.1 themselves cannot be represented in the form of the Poisson integral with any integrable function $\Phi : [0, 2\pi] \rightarrow \mathbb{R}$ because this integral would have nontangential limits Φ a.e., see e.g. Corollary IX.9.1 in [7]. Thus, u do not belong also to the classes h_p for any $p > 1$, see e.g. Theorem IX.2.3 in [7].

The statements on the infinite dimension of the space of solutions can be extended to the Riemann-Hilbert problem because the latter was reduced by us above to the corresponding two Dirichlet problems.

Theorem 5.2. *Let $\lambda : \partial\mathbb{D} \rightarrow \mathbb{C}$, $|\lambda(\zeta)| \equiv 1$, and $\varphi : \partial\mathbb{D} \rightarrow \mathbb{R}$ be measurable functions. Then the space of all analytic functions $f : \mathbb{D} \rightarrow \mathbb{C}$ such that along any nontangential path*

$$\lim_{z \rightarrow \zeta} \operatorname{Re} \{ \overline{\lambda(\zeta)} \cdot f(z) \} = \varphi(\zeta) \quad \text{for a.e. } \zeta \in \partial\mathbb{D} \quad (5.1)$$

has the infinite dimension.

Proof. Let $u : \mathbb{D} \rightarrow \mathbb{R}$ be a harmonic function with nontangential limit 0 at a.e. point of $\partial\mathbb{D}$ from Theorem 5.1. Then there is the unique harmonic function $v : \mathbb{D} \rightarrow \mathbb{R}$ with $v(0) = 0$ such that $\mathcal{C} = u + iv$ is an analytic function. Thus, setting in the proof of Theorem 2.1 $g = \mathcal{A}(\mathcal{B} + \mathcal{C})$ instead of $f = \mathcal{A} \cdot \mathcal{B}$, we obtain by Theorem 5.1 the space of solutions of the Riemann-Hilbert problem (5.1) for analytic functions of the infinite dimension. \square

Remark 5.2. The dimension of the spaces of solutions in Theorems 3.1 and 4.1 is also infinite because the case of the Jordan domains is reduced as in their proofs to the case of the unit disk in Theorem 5.2.

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