Annals of the University of Bucharest (mathematical series)
5 (LXIII) (2014), 157–168

Geometric and ergodic aspects in conformal dynamics on invariant fractals

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To Prof. Cabiria Andreian Cazacu on 85th birthday

Abstract - We survey recently discovered aspects in the geometric theory of higher dimensional dynamical systems on basic fractal sets. For most of these systems, hyperbolicity and the various invariant measures play a central role. We present several results which underline the rich connections between geometric theory of currents, ergodic theory, and thermodynamical formalism. Also we explain the interplay between the geometry of folded fractal sets and the ergodic properties of equilibrium measures supported on them.

Key words and phrases : hyperbolic dynamical systems on basic sets, dimension theory, fractals, equilibrium measures, global unstable sets, positive currents.

Mathematics Subject Classification (2010) **:** 37D20, 37A35, 37C40, 28D20, 60B05.

1. Introduction

Dynamics and smooth ergodic theory have had a remarkable growth in the last 30 years and, besides their own rich theory and structure, these fields employ methods and ideas from several areas such as differentiable dynamics and fractal theory, geometric analysis, measure theory, nonlinear analysis, statistical physics, complex geometry, stochastics. We give in this paper a short survey of some recent results, pertaining to applications of thermodynamic formalism, in particular on projective spaces. Methods of thermodynamic formalism and smooth ergodic theory have started to be applied relatively recently in conformal dynamics in higher dimensions. In one complex variable, they have been applied by several authors in the past, starting with the pioneering work of Ruelle [26]. The main idea is to study the metric properties of the complicated fractal sets invariated by dynamical systems with the help of concepts such as: topological pressure, various types of entropy, equilibrium measures, decay of correlations, folded fractals theory, stochastic methods, mixing, etc.

Let us consider a holomorphic map $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$ which is non-degenerate; then f is given by three homogeneous polynomials $[P_0 : P_1 : P_2]$

in the complex variables z_0, z_1, z_2, each of these polynomials having the same degree $d \geq 2$, which is called the *degree* (or the *algebraic degree*) of f (cf. [9]). Consider a *basic set* Λ for the map $f : \mathbb{P}^2\mathbb{C} \to \mathbb{P}^2\mathbb{C}$, i.e a compact f-invariant fractal set Λ such that f is topologically transitive on Λ and there exists a neighbourhood U of Λ with $\Lambda = \bigcap\limits_{n\in\mathbb{Z}} f^n(U)$. The general notion of basic set originated from the sets of the spectral decomposition theorem (cf. [3]). Given a basic set Λ, one can construct the *natural extension* (or *inverse limit*), $\hat{\Lambda} := \{(x, x_{-1}, x_{-2}, \dots)\},\ f(x_{-i}) = x_{-i+1},\ x_{-i} \in \Lambda,\ i \geq 1\}$. The natural extension is a compact metric space with the canonical metric (see [24], [9], etc.) We now obtain a shift homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$ on the natural extension, which is defined by $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, \dots),\ \hat{x} \in \hat{\Lambda}$.

An important notion in the sequel is that of *hyperbolicity for endomorphisms*; this notion is different from that of expanding maps in the one dimensional case, or from that of hyperbolicity for diffeomorphisms. For endomorphisms, hyperbolicity is defined as a continuous invariant splitting of the tangent bundle $T_{\hat{\Lambda}} := \{(\hat{x}, v), \hat{x} \in \hat{\Lambda}, v \in T_{x_0}\mathbb{P}^2\}$ over $\hat{\Lambda}$, into stable E^s_x and unstable $E^u_{\hat{x}}$ directions, for all $\hat{x} \in \hat{\Lambda}$ (see Ruelle [24]). While for diffeomorphisms, both stable and unstable directions are uniquely determined by their base point, for non-invertible maps the unstable directions depend on *whole prehistories* $\hat{x} \in \hat{\Lambda}$ (i.e past trajectories); this follows naturally from the non-invertibility of f, and the fact that $Df_x(E^s_x) \subset E^s_{f(x)}$ and $Df_x(E^u_{\hat{x}}) \subset E^u_{\hat{f}(\hat{x})}$. If f is hyperbolic on Λ then we have local stable manifolds $W^s_r(x)$ and local unstable manifolds $W^u_r(\hat{x})$ where $\hat{x} \in \hat{\Lambda}$. An additional difficulty is that Λ is not necessarily totally invariant. Hence for $x \in \Lambda$ we could have some f-preimages of x in Λ and others outside Λ. Also the number of f-preimages of x that remain in Λ can vary with x. Thus the case of smooth endomorphisms is subtle, and is very different from the case of diffeomorphisms. Hyperbolic non-invertible maps on invariant fractal sets have been studied also in [9], [28], [15], [19], etc. A *saddle basic set* is a basic set Λ on which f is hyperbolic and has both stable and unstable directions, i.e the dimensions of the stable/unstable tangent subspaces are nonzero. Define also the *stable dimension* at a point x, as

$$\delta^s(x) := HD(W^s_r(x) \cap \Lambda),\ x \in \Lambda$$

Let us remark that there exist classes of endomorphisms which are hyperbolic on saddle sets Λ, but which have a very strong non-invertible character, while at the same time, are also far from being constant-to-1 on Λ; such examples were constructed in [12]. These skew product endomorphisms also present new phenomena w.r.t the stable dimension; by using a type of Newhouse phenomenon for intersections of Cantor sets in fibers, we proved in [12] that there exist uncountably many points in Λ having one 1-preimage in Λ, and also uncountably many points having two 1-preimages in Λ.

Assume now that μ is a probability measure on a compact metric space X; the *lower pointwise dimension* and the *upper pointwise dimension* of μ at the point $x \in X$ are defined respectively by:

$$\underline{\delta}_\mu(x) := \liminf_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}, \text{ and } \bar{\delta}_\mu(x) := \limsup_{\rho \to 0} \frac{\log \mu(B(x,\rho))}{\log \rho}$$

If they are equal, the common value $\delta_\mu(x)$ is the *pointwise dimension of* μ at $x \in X$ (see [23]).

For measures with one contracting direction on average (i.e one negative Lyapunov exponent) and one expanding direction (one positive Lyapunov exponent), Young proved the following formula

Theorem 1.1. (Young, [30]) *Consider a hyperbolic f-invariant measure μ (i.e μ has only non-zero Lyapunov exponents), where f is a smooth diffeomorphism of a surface M. Then, μ-a.e we have*

$$\delta_\mu = h_\mu\left(\frac{1}{\chi_u(\mu)} - \frac{1}{\chi_s(\mu)}\right),$$

where $\chi_s(\mu), \chi_u(\mu)$ are the negative, respectively positive Lyapunov exponents of μ.

Mańe showed in [11] that $HD(\mu) = \frac{h_\mu}{\chi(\mu)}$ for any ergodic probability measure μ, f-invariant with respect to a rational endomorphism f in one variable, and such that μ has positive Lyapunov exponents. However the situation for higher dimensional endomorphisms and their invariant measures is different (see also [8], [16], [12], [13]). Nevertheless, the study of the measure of maximal entropy of the *restriction* $f|_\Lambda$ to a saddle basic set Λ of a non-invertible map, is different from above.

In [7], Fornaess and Mihailescu studied a much larger class of measures, namely equilibrium measures μ_ϕ for certain Hölder potentials ϕ on Λ. In this way, the structure of Λ is more intimately investigated. Also in [7], we gave optimal estimates for the pointwise dimension of these equilibrium measures, and also for the Hausdorff dimension of μ_ϕ. This answers a question from [8], in a more general setting. Also, we proved that for certain basic saddle sets Λ, the measure of maximal entropy on Λ can be described geometrically as a wedge product of two positive closed currents.

Another direction in which we present results, is that of the relationships between two apparently distant notions, i.e 1-sided Bernoullicity for certain invariant probabilities and stable dimension. In [13] we proved a *geometric flattening phenomenon* associated to the stable dimension. Namely we showed that if the stable dimension is zero a some point in the basic set Λ, then the fractal Λ must be contained in a submanifold (or a union of finitely many submanifolds), and f is expanding on Λ. Moreover in the same paper

[13] we classified the possible dynamical behaviours and established when is the system (Λ, f, μ) 1-sided or 2-sided Bernoulli for certain equilibrium measures, in the case of holomorphic perturbations of a product map.

In [15] we studied the stable dimension in the case of partially conformal hyperbolic maps, and its connections to a notion of inverse pressure introduced in [20]. The estimates on the Hausdorff dimension of the stable section through the fractal basic set Λ, as well as the estimates on the Hausdorff dimension of the global unstable set $W^u(\hat{\Lambda})$ complete the results obtained by Mihailescu in [16]; where we showed that, if a holomorphic endomorphism $f : \mathbb{P}^2 \to \mathbb{P}^2$ is s-hyperbolic, then the set K^-, equal to the complement of the set U^- of points z so that every y in a neighbourhood of z has its preimages $f^{-n}(y)$ converging to suppμ_G, has in fact empty interior (see also [18]). This represents a fundamental difference from the case of Hénon diffeomorphisms hyperbolic on their respective Julia sets, for which Bedford and Smillie showed in [2], that the set K^- of the points with bounded inverse iterates, may contain repelling basins of periodic repelling points.

Moreover in [19], Mihailescu and Urbański also found estimates for the stable dimension, by using continuous maps $\omega(\cdot)$ which bound the preimage counting function $d(\cdot)$ from above.

Also in [14], E. Mihailescu studied the asymptotic distributions of those consecutive preimages belonging to a saddle invariant set, showing that they approach a certain equilibrium measure.

2. Geometric and ergodic properties of measures and currents.

Let us consider a non-degenerate holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ of degree $d \geq 2$, so $f = [P_0 : P_1 : P_2]$, where P_0, P_1, P_2 are homogeneous polynomials in z_0, z_1, z_2 each of them having degree d. Then the topological entropy of f is equal to $\log d^2$; also there exists a Green function G on $\mathbb{C}^3 \setminus \{0\}$ and a positive closed current T on $\mathbb{P}^2$ (see [9], etc.), which satisfy $G(F(z)) = d \cdot G(z)$, and $\pi_2^* T = dd^c G$. Moreover the Green measure μ_G is defined as the wedge product $T \wedge T$. In [4] it was shown that in fact μ_G is the measure of maximal entropy for f on $\mathbb{P}^2$, so the measure-theoretic entropy of μ_G is $\log d^2$. It was shown by Briend that all Lyapunov exponents of μ_G are positive.

However if we consider instead, the restriction of f to a saddle set Λ, then the properties of the measure of maximal entropy of the *restriction* $f|_\Lambda$ are completely different.

Recall now that in general, given a hyperbolic endomorphism f on a basic set Λ, and a Hölder continuous potential ϕ on Λ, there exists a unique f-invariant probability measure μ_ϕ on Λ such that the supremum in the Variational Principle is attained ([3]). So μ_ϕ is the unique f-invariant measure for which we have $P(\phi) = h_{\mu_\phi} + \int_\Lambda \phi d\mu_\phi$. This follows from the hyperbolicity of the endomorphism f on Λ and from the Hölder continuity of ϕ, similarly to the case of diffeomorphisms. In particular, if $\phi \equiv 0$ on Λ, then μ_0 is just the unique measure of maximal entropy for the restriction $f|_\Lambda$. We showed in [17] that in general, for the case of a hyperbolic smooth map on a basic set of saddle type Λ, the equilibrium measure μ_ϕ satisfies the following estimates on Bowen balls:

$$\frac{1}{C}e^{S_n\phi(x)-nP(\phi)} \leq \mu_\phi(B_n(x,\varepsilon)) \leq Ce^{S_n\phi(x)-nP(\phi)}, \ \forall n > 0, \tag{2.1}$$

where $S_n\phi(x) := \phi(x) + \ldots + \phi(f^{n-1}(x))$, and $C > 0$ is some constant independent of x, n.

We give now some constructions of currents and measures. In [9] Fornaess and Sibony studied *s-hyperbolic* Axiom A holomorphic maps on $\mathbb{P}^2$ and *minimal* saddle basic sets, i.e minimal for the ordering between saddle basic sets defined by $\Lambda_i \succ \Lambda_j$ iff $W^u(\hat\Lambda_i) \cap W^s(\Lambda_j) \neq \emptyset$. A related notion introduced in [6] is that of a *terminal* set; here f is not assumed to have Axiom A and the condition refers strictly to the saddle set Λ, which is called *terminal* if for any $\hat x \in \hat\Lambda$, the iterates f^n, n restricted to $W^u_{loc}(\hat x) \setminus \Lambda$ form a normal family. If f is Axiom A and Λ is minimal, then for any $\hat x \in \hat\Lambda$ the global unstable set $W^u(\hat x)$ does not intersect any global stable set of any other basic set, thus $W^u(\hat\Lambda) \setminus \Lambda$ is contained in the union of basins of attraction of attracting cycles. Hence in this case, minimal sets are also terminal. Examples of terminal/minimal sets can be obtained by perturbations of products or skew products, or as maps constructed by Ueda's method.

Now, given a minimal set Λ for a s-hyperbolic map $f : \mathbb{P}^2 \to \mathbb{P}^2$, one can construct an f-invariant measure on Λ, as a wedge product of certain positive closed currents:

Theorem 2.1. (Fornaess and Sibony, [9]) *If f is a holomorphic non-degenerate map on $\mathbb{P}^2$, which is s-hyperbolic, then for any minimal set Λ there exists a positive closed current σ constructed by using forward iterates of unstable disks, namely if D is a local unstable disk centered on Λ, then*

$$\frac{f^n_\star([D])}{d^n} \to \sigma \cdot \int D \wedge T$$

This implies that there exists an f-invariant measure ν on Λ, defined as $\nu = \sigma \wedge T$.

We will need also the notion of *transversal measures* $\hat\mu^s_x$; these are similar to the transversal measures obtained in the diffeomorphism case by Ruelle and Sullivan [27] (see also Sinai [29]), but are more difficult to construct on the natural extension $\hat\Lambda$. One obtains a system of transversal measures $\hat\mu^s_x$ on $\hat W^s_{loc}(x)$, where by $\hat W^s_{loc}(x)$ and $\hat W^u_{loc}(\hat x)$ are denoted the lifts to $\hat\Lambda$, of the local stable intersection $W^s_{loc}(x) \cap \Lambda$, respectively of the local unstable intersection $W^u_{loc}(\hat{x}) \cap \Lambda$, i.e $\hat{W}^s_{loc}(x) := \pi^{-1}(W^s_{loc}(x) \cap \Lambda)$, and $\hat{W}^u_{loc}(\hat{x}) := \pi^{-1}(W^u_{loc}(\hat{x}) \cap \Lambda)$, $\hat{x} \in \hat{\Lambda}$. Then the family of measures $\hat{\mu}^s_x$ satisfies the following properties:

i) if $\chi^s_{x,y} : \hat{W}^s_r(x) \to \hat{W}^s_r(y)$ is the holonomy map given by $\chi^s_{x,y}(\hat{\xi}) = \hat{W}^u_r(\hat{\xi}) \cap \hat{W}^s_r(y)$, then $\hat{\mu}^s_x(A) = \hat{\mu}^s_y(\chi^s_{x,y}(A))$ for any borelian set A.

ii) $\hat{f}_\star \hat{\mu}^s_x = e^{h_{top}(f|_\Lambda)} \hat{\mu}^s_{f(x)}|_{\hat{f}(\hat{W}^s_r(x))}$

iii) supp $\hat{\mu}^s_x = \hat{W}^s(x)$.

In fact from [27] and [29] applied to our case on $\hat{\Lambda}$, it follows that there exist also unstable transversal measures, denoted by $\hat{\mu}^u_{\hat{x}}$ on $\hat{W}^u_r(\hat{x}), \hat{x} \in \hat{\Lambda}$ with similar properties. And that the measure of maximal entropy on $\hat{\Lambda}$ denoted by $\hat{\mu}_0$, can be written as the product of transversal stable measures $\hat{\mu}^s_y$ with transversal unstable measures $\hat{\mu}_{\hat{x}}$ i.e that:

$$\hat{\mu}_0(\phi) = \int_{\hat{W}^s_r(x)} (\int_{\hat{W}^u_r(\hat{y})} \phi \; d\hat{\mu}^u_{\hat{y}}) \; d\hat{\mu}^s_x(\hat{y}),$$

for any function ϕ defined on a neighbourhood of $\hat{x} \in \hat{\Lambda}$.

In [6] Diller and Jonsson introduced a positive current σ^u by using transversal measures (see also the diffeomorphism case in [27], [29]); namely in a neighbourhood of $x \in \Lambda$, $< \sigma^u, \chi >= \int_{\hat{W}^s_{loc}} (\int_{W^u_{loc}(\hat{y})} \chi) d\hat{\mu}^s_x(\hat{y})$, where $\hat{\mu}^s_x$ are transversal measures on $\hat{W}^s_{loc}(x) := \pi^{-1}(W^s_{loc}(x)$. If the saddle set Λ is terminal, then they defined an invariant probability measure on Λ, namely $\nu_i = \sigma^u \wedge T$. We use the notation ν_i in order to emphasize the way the current σ^u was constructed, by using the inverse limit. By using Spectral Decomposition Theorem ([3]), there is no loss of generality in assuming in the sequel that f itself is mixing on Λ. In the next Theorem we obtained a geometric description of the measure of maximal entropy of the restriction $f|_\Lambda$ as a product of two positive closed currents.

Theorem 2.2. (Fornaess and Mihailescu, [7]) *a) Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ holomorphic map of degree d and Λ be a terminal mixing saddle set. Then ν_i is equal to the measure of maximal entropy μ_0 on Λ.*

b) Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be an Axiom A holomorphic map of degree d, c-hyperbolic and mixing on a minimal saddle set Λ. Then $\nu_i = \nu = \mu_0$, where μ_0 is the measure of maximal entropy on Λ.

Now we shall give several results about the pointwise dimension for equilibrium measures for holomorphic maps. For a saddle basic set Λ, a point $z \in \Lambda$, integers n, k and $\varepsilon > 0$, define the set:

$$B(n, k, z, \varepsilon) := f^n(B_{n+k}(z, \varepsilon))$$

The question of establishing the pointwise dimension for invariant measures supported on fractal sets is an important one. For hyperbolic measures invariated by diffeomorphisms, it was solved by Barreira, Pesin and Schmeling in [1]. However for endomorphisms there appear many different phenomena and one cannot hope in general for a formula, due to the fact that the preimage counting function $d(x) := Card\{y \in \Lambda, f(y) = x\}, x \in \Lambda$ associated to f and Λ, may be non-constant. In the case when $d(\cdot)$ is constant, we studied this problem in [7].

Theorem 2.3. (Fornaess and Mihailescu, [7]) *Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ be a holomorphic map of degree d and Λ be a basic set, such that f is c-hyperbolic on Λ and the preimage counting function $d(\cdot)$ is constant and equal to d' on Λ. Consider a Hölder continuous function ϕ on Λ, such that $\phi(x) + \log d' < P(\phi), \forall x \in \Lambda$. Then we obtain the following comparison, which does not depend on n, k, z:*

$$\mu_\phi(B(n, k, z, \varepsilon)) \approx \frac{e^{S_{n+k}\phi(z)}}{(d')^k}$$

It follows that the pointwise dimension of μ_ϕ exists μ_ϕ-a.e and denote it by δ_{μ_ϕ}. Then μ_ϕ-a.e,

$$\delta_{\mu_\phi} = HD(\mu_\phi) = h_{\mu_\phi}(\frac{1}{\chi_u(\mu_\phi)} - \frac{1}{\chi_s(\mu_\phi)}) + \log d' \cdot \frac{1}{\chi_s(\mu_\phi)}$$

This Theorem was proved by studying the measure of the iterates of Bowen balls and carefully comparing the measures coming from various preimages, and then adjusting the stable and unstable sides of the iterates conformally, so that we obtain "roughly round" balls. These iterates can then be used in order to obtain the pointwise dimension of μ_ϕ.

If the preimage counting function $d(\cdot)$ is not constant on Λ, we can still obtain bounds for the measure of iterates of Bowen balls, and estimates for the lower pointwise dimension:

Corollary 2.1. (see [7]) *In the setting of Theorem 2.3 assume the preimage counting function satisfies $d(x) \le d'$ for μ_ϕ-a.e $x \in \Lambda$ and that $\phi(x) + \log d' < P(\phi)$ for all $x \in \Lambda$. Then for μ_ϕ-a.e $x \in \Lambda$,*

$$\underline{\delta}_{\mu_\phi}(x) \ge h_{\mu_\phi}(\frac{1}{\chi_u(\mu_\phi)} - \frac{1}{\chi_s(\mu_\phi)}) + \log d' \cdot \frac{1}{\chi_s(\mu_\phi)}$$

From Theorem 2.3 it follows that the measure ν_i is equal to the measure of maximal entropy μ_0, and to ν, so now by employing also Theorem 2.3, we found its pointwise dimension in:

Corollary 2.2. (Fornaess and Mihailescu, [7])

a) Let Λ be a mixing terminal saddle set for a holomorphic map $f : \mathbb{P}^2 \to \mathbb{P}^2$ of degree d, s.t Λ does not intersect the critical set C_f of f. If each *point in Λ has at most d' f-preimages in Λ and if $d' < d$, then for μ_ϕ-a.e z,* $\underline{\delta}_{\nu_i}(z) \ge \log d \cdot (\frac{1}{\chi_u(\nu_i)} - \frac{1}{\chi_s(\nu_i)}) + \log d' \cdot \frac{1}{\chi_s(\nu_i)}$.

b) If Λ is a mixing terminal saddle set for a holomorphic map f on $\mathbb{P}^2$ of degree d, if $C_f \cap \Lambda = \emptyset$ and if the preimage counting function is constant equal to d' on Λ for $d' \le d$, then we have:

$$\delta_{\nu_i} = HD(\nu_i) = \log d \cdot (\frac{1}{\int \log |Df_u| d\nu_i} - \frac{1}{\int \log |Df_s| d\nu_i}) + \log d' \cdot \frac{1}{\int \log |Df_s| d\nu_i}$$

c) Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ a holomorphic Axiom A map of degree d, which is c-hyperbolic on a connected minimal saddle set Λ. If d' denotes the constant number of f-preimages in Λ of a point, then:

$$\delta_\nu = HD(\nu) = \log d \cdot (\frac{1}{\chi_u(\nu)} - \frac{1}{\chi_s(\nu)}) + \log d' \cdot \frac{1}{\chi_s(\nu)}$$

In general the map f is not necessarily constant-to-1 on Λ. However when it happens that $f|_\Lambda$ is expanding and constant-to-1 as above, then we proved that the measure-preserving system (Λ, f, μ_0) is 1-sided Bernoulli, namely it is isomorphic as a Lebesgue space to a shift space of type $(\Sigma_m^+, \sigma_m, \nu_{\bar{p}})$, for some integer $m \ge 2$ and some probability vector $\bar{p}$. Notice that, while for 2-sided Bernoulli shifts, their isomorphism class is determined by measure-theoretic entropy only (Ornstein), for 1-sided Bernoulli shifts this is not true anymore ([22], [5]) and the problem of isomorphism class for endomorphisms is more subtle. In fact one has the following result:

Theorem 2.4. (Parry and Walters, [22]) *There exist non-isomorphic exact endomorphisms S, T of a Lebesgue space $(X, \mathcal{B}, \mu)$ so that $S^2 = T^2$ (hence S, T have the same entropy w.r.t μ), $S^{-n}\mathcal{B} = T^{-n}\mathcal{B}, n \ge 0$ and s.t the Jacobians of S and T w.r.t μ are equal.*

However for expanding maps on folded fractals we proved 1-sided Bernoullicity in some cases:

Theorem 2.5. (Mihailescu, [13]) *Assume that Λ is a hyperbolic basic set for a smooth endomorphism f, such that $f|_\Lambda$ is d-to-1, $t_d = 0$ and $f|_\Lambda$ is expanding. Then (Λ, f, μ_0) is 1-sided Bernoulli, where μ_0 is the unique measure of maximal entropy.*

In general when the stable dimension is zero, we showed in [13] that there are strong geometric consequences (geometric rigidity) for the fractal.

Theorem 2.6. (Mihailescu, [13]) *Let $f : M \to M$ be a smooth endomorphism which is hyperbolic on a basic set Λ with $C_f \cap \Lambda = \emptyset$ and such that f is conformal on local stable manifolds. Assume that d is the maximum possible value of $d(\cdot)$ on Λ, and that there exists a point $x \in \Lambda$ where $\delta^s(x) = t_d = 0$.*

Then it follows that $d(\cdot) \equiv d$ on Λ and there exist a finite number of unstable manifolds whose union contains Λ. In particular if Λ is connected, then there exists an invariant unstable manifold containing Λ, and f is expanding on Λ.

Thus, from a local condition such as stable dimension being zero at a point, we obtain the global restriction that Λ is contained in a finite union of unstable manifolds.

Jointly with Stratmann, see [21] we obtained the following upper estimate for the stable dimension, using the pressure function.

Theorem 2.7. (Mihailescu and Stratmann, [21]) *Consider a $\mathcal{C}^2$-endomorphism f c-hyperbolic on a basic set Λ of f, s.t there exists a continuous function $\omega : \Lambda \to \mathbb{R}$ with $\Delta(x) \geq \omega(x)$, for all $x \in \Lambda$. It then follows that $\delta^s(x) \leq t_\omega$, where t_ω is the unique zero of $t \mapsto P(t\Phi^s - \log \omega)$.*

As we proved, saddle basic sets are local repellers (see [15]), respectively attractors, if and only if certain conditions on stable/unstable dimensions hold. However the proofs for these two cases are different, due to the non-invertibility.

Theorem 2.8. (Mihailescu, [13]) *a) Let Λ be a basic set for a hyperbolic endomorphism f such that $C_f \cap \Lambda = \emptyset$ and f is conformal on local stable manifolds. Then Λ is a local repeller if and only if there exists $x \in \Lambda$ with $\delta^s(x) = d_s$.*

b) Let Λ a hyperbolic basic set for a smooth endomorphism $f : M \to M$ defined on a Riemannian manifold. Assume that f is conformal on local unstable manifolds (which are supposed to have real dimension d_u). Then Λ is an attractor for f if and only if there exists $\hat{x} \in \hat{\Lambda}$ with $\delta^u(\hat{x}) = d_u$.

We proved in [13] a Classification Theorem for a class of perturbations:

Theorem 2.9. (Mihailescu, [13]) *For some small $|c|, c \in \mathbb{C} \setminus \{0\}$, let us consider the polynomial map $f(z, w) = (z^2 + c, w^2), (z, w) \in \mathbb{C}^2$. Let also a polynomial f_ε which is a smooth perturbation of f and let Λ_ε be the corresponding basic set of f_ε close to the set $\Lambda := \{p_c\} \times S^1$ (where p_c is the fixed attracting point of $z \to z^2 + c$). Then we may have exactly one of the following possibilities:*

a) There exists a point $x \in \Lambda_\varepsilon$ where $\delta^s(x) = 0$. Then there exists a manifold W such that $\Lambda_\varepsilon \subset W$, $f_\varepsilon|_{\Lambda_\varepsilon}$ is expanding and $f_\varepsilon|_{\Lambda_\varepsilon}$ is 2-to-1. In this case the stable dimension is 0 at any point from Λ_ε, and the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_{0,\varepsilon})$ is 1-sided Bernoulli (where $\mu_{0,\varepsilon}$ is the unique measure of maximal entropy for $f_\varepsilon|_{\Lambda_\varepsilon}$).

b) There exists a point $x \in \Lambda_\varepsilon$ with $0 < \delta^s(x) < 2$. Then the stable dimension is positive at any point of Λ_ε, and the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_{s,\varepsilon})$ cannot be 1-sided Bernoulli, where $\mu_{s,\varepsilon}$ is the equilibrium measure of the potential $\delta^s(x)\Phi^s_\varepsilon$. We have two subcases:

b1) $f_\varepsilon|_{\Lambda_\varepsilon}$ is a homeomorphism, and in this case the measure preserving system $(\Lambda_\varepsilon, f_\varepsilon, \mu_\phi)$ is 2-sided Bernoulli for any Holder continuous potential ϕ, where μ_ϕ is the equilibrium measure of ϕ.

b2) there exist both points with only one f_ε-preimage in Λ_ε, as well as points with two f_ε-preimages in Λ_ε; the set of points with one f_ε-preimage in Λ_ε has non-empty interior.

In the case when the stable dimension is minimal, we obtained the following global result, about preimages in Λ.

Theorem 2.10. (Mihailescu and Urbański, [19]) *Assume that f is c-hyperbolic on a basic set Λ and that the preimage counting function $d(\cdot)$ reaches a maximum value of d on Λ. If there exists a point $x \in \Lambda$ such that $\delta^s(x) = t_d$, where t_d is the unique zero of the pressure function $t \to P(t\Phi^s - \log d)$, then $d(y) = d,\ y \in \Lambda$. Hence the stable dimension at every point of Λ must be equal to t_d.*

Now, another question is under what conditions the stable dimension can be maximal. This question is related to the Hausdorff dimension of the global unstable set, and to the dimension of K^- (similar to the set of points with bounded backwards iterates for Hénon maps). In [15] we found a bound for the stable upper box dimension, by the zero $t^s(\varepsilon)$ of the ε-inverse pressure P^-_ε of $\log|Df_s|$; we showed that unless Λ is a local repeller, the stable upper box dimension cannot be maximal.

Theorem 2.11. (Mihailescu, [15]) *Consider a non-degenerate holomorphic $f : \mathbb{P}^2 \to \mathbb{P}^2$ c-hyperbolic on a basic saddle set Λ. Then for any $x \in \Lambda$, we have $\delta^s(x) \le t^s(\varepsilon) < 2$, for some $\varepsilon > 0$.*

By using the above result, we showed in [15] that in certain conditions, the Hausdorff dimension of the global unstable set $W^u(\hat\Lambda)$ is less than maximal. The result improves qualitatively the theorem of [16] namely that for an s-hyperbolic map f, the interior of the set K^- is empty.

Theorem 2.12. (Mihailescu, [15]) *a) Let $f : \mathbb{P}^2 \to \mathbb{P}^2$ to be a holomorphic map which is c-hyperbolic on a basic set of saddle type Λ, and assume that the following derivatives condition is satisfied:*

$$\sup_{\hat\xi \in \hat\Lambda} |Df_u(\hat\xi)| \cdot |Df_s(\xi)| < 1 \tag{2.2}$$

Then $HD(W^u(\hat\Lambda)) < 4$.

b) Consider holomorphic perturbations g, of the holomorphic map given by $f(z,w) = (z^2 + c, w^2)$ for small $|c|$. Then $HD(W^u(\hat{\Lambda}_g)) < 4$, for the respective basic set Λ_g of g, which is close to the set $\{p_0(c)\} \times S^1$, where $p_0(c)$ is the fixed attracting point of $z \to z^2 + c$.

Condition (2.2) can be verified also in the case of saddle sets for certain skew products with overlaps in fibers. Then from [17], the unstable dimension δ_g^u is constant on $\hat{\Lambda}_g$, and varies real-analytically when the parameters of the perturbation g of a map f vary holomorphically.

Acknowledgements

Paper supported by grant PCE-2011-3-0269.

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