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Extremal quasiconformal parametric representation of a quasicircle

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Abstract - For a given quasicircle we are looking for a parametric representation with an extremal quasiconformal extension to the whole Riemann sphere.

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1. Introduction

The theory of analytic functions and of conformal mappings became important new aspects and a great enrichment by introducing quasiconformal mappings. The latest great survey of this field was given by Cabiria Andreian Cazacu in her profound Handbook article [3]. Furthermore, beside her own important contributions, she also gave one of the first introduction to the foundation: [1], [2].

Here we will consider the following question. Let a (closed) Jordan curve $\mathfrak{C}$ be given in the complex w-plane. Then we have a great variety of continuous parametric representations of the form

$$w = w(z) \quad \text{with} \quad z = e^{it},\ 0 \le t \le 2\pi,$$

where $w(z)$ is a continuous and schlicht mapping of the unit circle $|z| = 1$ onto $\mathfrak{C}$. Our aim is the question: Is there a distinguished parametric representation, from the point of view of complex analysis? For this reason, we ask for an extension of such a parametric representation $w(z)$ to a continuous and schlicht mapping of the *whole* Riemann sphere onto itself. It is known that there exists such a mapping which is quasiconformal if and only if $\mathfrak{C}$ is a quasicircle; cf. [3], [5], [7], [10]. Obviously, the "optimal" case of an extension which is even conformal exists only in the trivial case of a circle $\mathfrak{C}$. Therefore, there arises the question for an "extremal quasiconformal parametric representation of the quasicircle $\mathfrak{C}$" (in the original sense of H.

Grötzsch called "möglichst konform"). That means a parametric representation for which the extremal quasiconformal extension to the whole sphere has as a dilatation bound which is as small as possible.
Of course, such an extremal quasiconformal parametric representation always yields again such a representation after an arbitrary Möbius transformation of the unit circle $|z| = 1$ onto itself.

In the following, we always denote by $Q_{\mathfrak{C}}$ the "reflection coefficient" of the quasicircle $\mathfrak{C}$. That is the smallest dilatation bound in the class of all quasiconformal reflections at $\mathfrak{C}$; cf. [5], [7], [10].

Our starting point is the

Theorem 1.1. *For every quasicircle $\mathfrak{C}$ there exists an extremal quasiconformal parametric representation. The smallest dilatation bound is $\sqrt{Q_{\mathfrak{C}}}$. If $\mathfrak{C}$ is, e.g., analytic then the extremal representation is unique up to a Möbius transformation of the unit circle $|z| = 1$ onto itself. It then has a constant dilatation and can be described by a quadratic differential.*

It is difficult to give concrete examples for curves $\mathfrak{C}$ with a known extremal quasiconformal parametric representation. We can offer here only the example of an ellipse $\mathfrak{C}$. Even in this case the solution does not look simply.

We can assume the ellipse $\mathfrak{C}$ in the form

$$\mathfrak{C}: \quad \frac{1}{Q^2} (\mathfrak{Re}\, w)^2 + (\mathfrak{Im}\, w)^2 = 1 \tag{1.1}$$

with $Q = Q_{\mathfrak{C}}$ because here the reflection coefficient is the quotient of the semiaxes [7].

Theorem 1.2. *We have for the by* (1.1) *defined ellipse $\mathfrak{C}$, under the normalization $w(1) = Q$, $w(i) = i$, $w(-1) = -Q$, $w(-i) = -i$, the (then uniquely determined) extremal quasiconformal parametric representation*

$$w\left(e^{it}\right) = Q \cos\left(\frac{\pi}{K(\kappa)} F(\kappa, \varphi)\right) + i \sin\left(\frac{\pi}{K(\kappa)} F(\kappa, \varphi)\right) \tag{1.2}$$

with the usual abbreviations

$$F(\kappa, \varphi) = \int_0^{\varphi} \frac{dt}{\sqrt{1 - \kappa^2 \sin^2 \varphi}}\,, \quad K(\kappa) = F\left(\kappa, \frac{\pi}{2}\right) \tag{1.3}$$

for the elliptic integral of the first kind, resp. for the complete elliptic integral of the first kind, further

$$\kappa = \left(\frac{2q^{1/4} + 2q^{9/4} + 2q^{25/4} + \cdots}{1 + 2q + 2q^4 + 2q^9 + \cdots}\right)^2, \quad q = \sqrt{\frac{\sqrt{Q} - 1}{\sqrt{Q} + 1}}, \tag{1.4}$$

$$\varphi = \arctan\left(\frac{1}{\sqrt[4]{1-\kappa^2}}\tan\frac{t}{2}\right). \qquad (1.5)$$

The dilatation of the extremal quasiconformal extension of this extremal quasiconformal parametric representation to the whole Riemann sphere is constant and equals $\sqrt{Q} = \sqrt{Q_{\mathfrak{C}}}$.

Here we have in the first equation (1.4) in the nominator and in the denominator a theta series.
In particular, the Theorem 1.2 means that in the case of an ellipse the corresponding affine mapping (with the dilatation $Q_{\mathfrak{C}}$) does *not* yield the extremal quasiconformal parametric representation.

In Section 3 we give a generalization and in Section 4 a discretization of our problem.

As an addendum, we will consider in the last Section 5 for an analytic quasicircle $\mathfrak{C}$ also a modified problem.

2. Proof of Theorem 1.1 and 1.2

The *Proof of Theorem 1.1* follows the idea in [7] (p. 95), [11] (p. 78). We start with an extremal quasiconformal reflection at $\mathfrak{C}$. Here we observe the infinitesimal ellipses which transform onto infinitesimal circles. If the dilatation is p (quotient of the semiaxis) we take now the ellipses with dilatation $\sqrt{p}$ and the same orientation of the axis. This means the solution of a Beltrami equation. In this manner, we obtain as in [7], [11] by some sort of factorization the Theorem 1.1 .

Remarks. (i) In the Theorem 1.1 we restrict ourself mainly to the clear case of an analytic quasicircle $\mathfrak{C}$. Of course, there remains the problem of a systematic inquiry of properties of the extremal quasiconformal parametric representation and its extension, in dependence of properties of $\mathfrak{C}$, e.g., asymptotical conformality or smoothness.

(ii) If we replace in this consideration p by a p_1, $1 \leq p_1 \leq Q_{\mathfrak{C}}$, we obtain in the same manner also a parametric representation of $\mathfrak{C}$ which is Q_1-quasiconformal in the exterior of the unit circle and Q_2-quasiconformal in the interior, with $1 \leq Q_1 \leq Q_{\mathfrak{C}}$, $1 \leq Q_2 \leq Q_{\mathfrak{C}}$, $Q_1 Q_2 = Q_{\mathfrak{C}}$; cf. again [11] (p. 78). Furthermore, it arises the question: Is it possible to arrange the "splitting" of the dilatation in a *function* p_1 (not necessarily constant) in the exterior of $\mathfrak{C}$ and p_2 in the interior with $p_1 p_2 = Q_{\mathfrak{C}}$ such that at the unit circle $|z| = 1$ a parametric representation with a prescibed property appears?

The case of an ellipse $\mathfrak{C}$ is a great stroke of luck, because here we obtain the mentioned Beltrami solution by a simple affine mapping. Namely, for the *Proof of Theorem 1.2* we start with the conformal mapping $W = W(z)$ of the unit disk $|z| < 1, z = e^{it}$, onto the interior of the stretched ellipse

$$\frac{1}{Q}\,(\mathfrak{Re}\,W)^2 + (\mathfrak{Im}\,W)^2 = 1 \tag{2.1}$$

with $W(1) = \sqrt{Q},\ W(i) = i,\ W(-1) = -\sqrt{Q},\ W(-i) = -i.$ By [4] (p. 318/319), we obtain the boundary values

$$W\left(e^{it}\right) = \sqrt{Q}\cos\left(\frac{\pi}{K(\kappa)}F(\kappa,\varphi)\right) + i\sin\left(\frac{\pi}{K(\kappa)}F(\kappa,\varphi)\right) \tag{2.2}$$

with (1.3), (1.4), (1.5). The following affine mapping

$$w = \sqrt{Q}\,\mathfrak{Re}\,W + i\,\mathfrak{Im}\,W$$

leaves us indeed with (1.2).

We add that in this exceptional case of an ellipse $\mathfrak{C}$ it is also possible to give explicit analytic expressions of the inverse mapping of the extremal quasiconformal parametric representation (1.2); cf. again [4], in another form using Tchebyshev polynomials in [12] (p. 258).

3. A generalization

Let n samples $K_1, \cdots, K_n$ of the unit disk be given in n complex planes. In addition, let be given a fixed system

$$S = \bigcup_{k,l=1}^{n} S_{l,k} \tag{3.1}$$

of $n(n-1)$ quasisymmetric and orientation preserving substitutions $S_{l,k}$ between the boundaries ∂K_l and ∂K_k (beside the trivial identity substitutions $S_{1,1}, \cdots,\ S_{n,n}$). We assume that this system of the substitutions

$$S_{l,k}:\ \ \partial K_k \to \partial K_l \tag{3.2}$$

satisfies for all k, l, m for the inverse and for the compositions

$$S_{k,l} = S_{l,k}^{-1}, \quad S_{m,k} = S_{m,l} \circ S_{l,k}. \tag{3.3}$$

In particular, this means that after prescribing the special substitutions $S_{2,1},\ S_{3,2}, \cdots,\ S_{n,n-1}$ we can obtain by composition with (3.3) also the other $S_{l,k}$.

As a welding function, every $S_{l,k}$ yields a quasicircle $\mathfrak{C}_{l,k} = \mathfrak{C}_{k,l}$ (unique up to a Möbius transformation), namely by conformal welding of K_k and K_l (resp. the at the unit circle reflected K_l). This leaves us with $\binom{n}{2}$ quasicircles.

Now we take an additional unit disk $\mathfrak{K}$ in the complex z-plane and a system $\mathfrak{S} = (\mathfrak{S}_1, \cdots, \mathfrak{S}_n)$ of quasisymmetric substitutions

$$\mathfrak{S}_k : \ \partial\mathfrak{K} \to \partial K_k, \ \ k = 1, \ \cdots, n, \tag{3.4}$$

which are linked by

$$\mathfrak{S}_l = S_{l,k} \circ \mathfrak{S}_k. \tag{3.5}$$

One can think also of defining only one substitution, e.g., $\mathfrak{S}_1$ and then generating the other by (3.3).
With this system $\mathfrak{S}$ of substitutions, we consider systems of n quasiconformal mappings $\mathfrak{w}_k(z)$ with dilatation bounds Q_k, satisfying

$$\mathfrak{w}_k(z) : \ \ \mathfrak{K} \ \to K_k \ \text{ which equals } \mathfrak{S}_k \text{ at the unit circle.} \tag{3.6}$$

We can state the

Problem. *For a given and fixed system S, what is the domain of variability of the systems $Q_1, \cdots, Q_n$ (in the n-dimensional space) if the substitutions $\mathfrak{S}_1, \cdots, \mathfrak{S}_n$ and the mappings $\mathfrak{w}_1, \cdots, \mathfrak{w}_n$ vary.*

This domain of variability will appear in dependence of the fixed system S.

Here we can give only the complete solution in the simplest case $n = 2$.

Theorem 3.1. *In the case $n = 2$, the exact domain of variability of all possible pairs (Q_1, Q_2) is given by*

$$Q_1 \geq 1, \quad Q_2 \geq 1, \quad Q_1 Q_2 \geq Q_{\mathfrak{C}_{1,2}} \tag{3.7}$$

where $Q_{\mathfrak{C}_{1,2}}$ is the reflection coefficient of the quasicircle $\mathfrak{C}_{1,2}$ arising by the substitution $S_{1,2}$. In the case of an analytic substitution $S_{1,2}$, that means in the case of an analytic Jordan curve $\mathfrak{C}_{1,2}$, there exists for every pair (Q_1, Q_2) with $Q_1 Q_2 = Q_{\mathfrak{C}_{1,2}}$, $Q_1 > 1$, $Q_2 > 1$ exactly one pair of corresponding substitutions $\mathfrak{S}_1, \mathfrak{S}_2$ (up to Möbius transformations of the unit circle onto itself).

For the *Proof*, we first observe that for every Q_1 with $1 \leq Q_1 \leq Q_{\mathfrak{C}_{1,2}}$ there exists a pair of admissible quasiconformal mappings with dilatation bounds Q_1 and $Q_2 = Q_{\mathfrak{C}_{1,2}}/Q_1$, that means $Q_1 Q_2 = Q_{\mathfrak{C}_{1,2}}$; cf. the Remark (ii) of Section 2. And a pair (Q_1, Q_2) in the domain of variability with $Q_1 Q_2 < Q_{\mathfrak{C}_{1,2}}$ is impossible because otherwise a $Q_1 Q_2$-quasiconformal reflection at $\mathfrak{C}_{1,2}$ would be follow.
Obviously, with an admissible pair (Q_1, Q_2) always also, e.g., tQ_1, tQ_2 or tQ_1, Q_2 or Q_1, tQ_2 with $t > 1$ is an admissible pair. This means that the

by (3.7) characterized domain is completely covered by the admissible pairs (Q_1, Q_2).

It seems that in the case $n > 2$ the situation is more involved (beside some special and obvious cases).

Remarks. (i) As in the case $n = 2$, it would be desirable, e.g. in the case $n = 3$, to construct an appropiate conformal welding of K_1, K_2, K_3 corresponding to the substitutions of the system S. This would be shape the whole situation more suggestive. Of course, this is impossible in the complex plane. Therefore, one has to try a welding in space, using three simply-connected surfaces with common boundary.

(ii) In the case $n = 3$, if $\mathfrak{C}_{1,2}$ and $\mathfrak{C}_{2,3}$ are given then also the substitutions $S_{1,2}$ and $S_{2,3}$, therefore by (3.3) also $S_{1,3}$ and then $\mathfrak{C}_{1,3}$ (up to a Möbius transformation). It would be desirable to have a direct procedure which yields for given $\mathfrak{C}_{1,2}$ and $\mathfrak{C}_{2,3}$ the third $\mathfrak{C}_{1,3}$. Of course, this is possible by cutting the planes along $\mathfrak{C}_{1,2}$ and $\mathfrak{C}_{2,3}$ followed by a suitable welding of the interior of $\mathfrak{C}_{1,2}$ and $\mathfrak{C}_{2,3}$. But it arises the question for a more direct procedure.

4. A discretization

Let n different points $w_1, \cdots, w_n$ be given and fixed in the complex w-plane. Additionally, we consider in the z-plane n different (non fixed) points $z_1, \cdots, z_n$ varying at the unit circle $|z| = 1$ and take into account those quasiconformal mappings $w(z)$ of the Riemann sphere onto itself with $w(z_k) = w_k, \; k = 1, \cdots, n$. Furthermore, we prescribe a homotopy class. That means, the image of the unit circle $|z| = 1$ has to be a Jordan curve through the w_k in a given and fixed homotopy class. Using essentially results of O. Teichmüller, in [8] was proven: In the class of these quasiconformal mappings (varying the z_k *and* the mappings) there exists an extremal quasiconformal mapping (that means with the smallest dilatation bound) which is unique up to a Möbius transformation of the unit circle $|z| = 1$ onto itself.

In [8] there was additionally given a description of the extremal mapping in more detail, e.g., with a quadratic differential.

Furthermore, in [9] the simplest non-trivial case $n = 4$ was studied explicitly, using elliptic functions and elliptic integrals.

5. Analytic Jordan curves

As an addendum, for the case of an analytic Jordan curve $\mathfrak{C}$ we will consider also a modified problem. Then there exists a parametric representation $w = w(z)$ which is a schlicht mapping of the unit circle $|z| = 1$ onto $\mathfrak{C}$ and has an analytic and schlicht *conformal* extension to an annulus $\frac{1}{R} < |z| < R,\ R > 1$. Here we can choose $R = \infty$ only in the case of a circle $\mathfrak{C}$.

For a given fixed $w(z)$ and then all possible values R we ask now for sup R. To this end, first we consider for $|z| = |\zeta| = 1$ the development

$$\log \frac{w(z) - w(\zeta)}{z - \zeta} = \sum_{i,k=-\infty}^{\infty} A_{ik} z^i \zeta^k \quad (A_{ik} = A_{ki}) \tag{5.1}$$

to obtain the coefficients A_{ik}. Then the solution of our problem is given by the following generalized Grunsky inequalities.

Theorem 5.1. *The desired sup R is the supremum of all R for which $w(z)$ is analytic in $\frac{1}{R} < |z| < R$ and satisfies with the corresponding coefficients A_{ik} the inequalities*

$$\left| \sum_{i,k=-n}^{n} A_{ik} X_i X_k + 2 \sum_{k=1}^{n} \frac{X_k X_{-k}}{k(R^{4k} - 1)} \right| \leq \sum_{k=1}^{n} \frac{|X_k|^2 + |X_{-k}|^2}{k(R^{2k} - R^{-2k})} \tag{5.2}$$

for all finite systems of complex numbers $X_{-n}, ..., X_n$ with $X_0 = 0$.

The *proof* follows by a slight transformation of Satz 2 in [6].

It remains the *desideratum*: Is it possible to prove in the style of [13] (middle of p. 59) the analyticity of $w(z)$ in $\frac{1}{R} < |z| < R,\ R > 1$, providing (5.2) and the analyticity in an annulus $\frac{1}{R'} < |z| < R'$ with some $R',\ 0 < R' < R$?

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