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Ahlfors' question and beyond

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Abstract - Some qualitatively complete results concerning Ahlfor's question on characterization of conformal maps with quasiconformal extensions are established.

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1. Preamble and main theorems

1.1. Introductory remarks

In 1963, Ahlfors posed in [1] (and repeated in his book [2]) the following question which gave rise to various investigations of quasiconformal extendibility of univalent functions.

Question. *Let f be a conformal map of the disk (or half-plane) onto a domain with quasiconformal boundary (quasicircle). How can this map be characterized?*

He conjectured that the characterization should be in analytic properties of the invariant (logarithmic derivative) f''/f'. Many results were established on quasiconformal extensions of holomorphic maps in terms of f''/f' and other invariants (see, e.g., the survey [10] and the references there).

The purpose of this note is to give a somewhat complete answer by applying the Grunsky coefficient inequalities.

Recall that, due to the classical Grunsky theorem [6], a holomorphic function $f(z) = z + \text{const} + O(z^{-1})$ in a neighborhood of $z = \infty$ is continued to a univalent holomorphic function on the disk

$$\Delta_r^* = \{z \in \widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\} : \ |z| > r\}$$

if and only if its Grunsky coefficients α_{mn} defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn} r^{m+n} z^{-m} \zeta^{-n}, \quad (z, \zeta) \in (\Delta_r^*)^2, \tag{1.1}$$

satisfy the inequality

$$\Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} r^{m+n} x_m x_n \Big| \le 1, \tag{1.2}$$

for any sequence $\mathbf{x} = (x_n)$ from the unit sphere $S(l^2)$ of the Hilbert space l^2 with norm $\|\mathbf{x}\| = \big(\sum_1^{\infty} |x_n|^2\big)^{1/2}$. Here the principal branch of the logarithmic function is chosen and $0 < r < \infty$. The quantity

$$\varkappa_r(f) = \sup \Big\{ \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} r^{m+n} x_m x_n \Big| :\ \mathbf{x} = (x_n) \in S(l^2) \Big\} \le 1 \tag{1.3}$$

is called the **Grunsky norm** of f in Δ_r^*, and the minimal dilatation $k(f) = \inf\{k(w^\mu) = \|\mu\|_\infty :\ w^\mu|\partial D^* = f\}$ is called the **Teichmüller norm** of f.

Consider the class $\Sigma_r(0)$ of nonvanishing univalent hydrodynamically normalized functions

$$f(z) = z + b_0 + b_1 z^{-1} + \dots$$

on D_r^* with quasiconformal extensions to $\widehat{\mathbb{C}}$ satisfying $f(0) = 0$ (hence, with Beltrami coefficients $\mu_f(z) = \overline{\partial} f / \partial f$ supported on the disk $\Delta_r = \{|z| < r\}$.

More generally, let L be a quasiconformal curve in C with the interior and exterior domains $D,\ D^* \ni \infty$. Consider the Beltrami coefficients supported in D, i.e., the elements of the unit ball

$$\mathrm{Belt}(D)_1 = \{\mu \in L_\infty(C) :\ \mu(z)|D^* = 0, \quad \|\mu\|_\infty < 1\}.$$

Any such μ determines a linear functional $\langle \mu, \psi \rangle_D = \iint_D \mu \psi dx dy$ on $L_1(D)$ whose norm equals $\|\mu\|_\infty$.

The subspace $A_1(D)$ in $L_1(D)$ formed by holomorphic functions in D is intrinsically connected with exremal $\mu \in \mathrm{Belt}(D)_1$. Let $A_1^2(D)$ be its subset consisting of $\psi = \omega^2$ with ω holomorphic on D, and

$$\alpha_D(f^\mu) = \sup\{|\langle (\mu/\|\mu\|_\infty) \psi \rangle_D| :\ \psi \in A_1^2(D),\ \|\psi\|_{A_1} = 1\}.$$

In the case $r = 1$, we shall use the notations $\Delta,\ \Delta^*,\ \Sigma(0)$ and apply to $f \in \Sigma(0)$ the root transform

$$\mathcal{R}_p :\ f(z) \mapsto f_p(z) := f(z^p)^{1/p} = z + \frac{b_0}{p} z^{-p+1} + \dots, \tag{1.4}$$

where $p \ge 2$ is an integer. Every image $f_p = \mathcal{R}_p f$ is p-symmetric with respect to rotation around the origin, i.e.,

$$f_p(e^{2n\pi i/p}z) = e^{2n\pi i/p} f_p(z), \quad n = 0, 1, \dots, p-1; \quad p \ge 2,$$

for any $z \in \Delta^*$, and is connected with its original $f \in \Sigma^0$ by the commutative diagram

$$\begin{array}{ccc} \widetilde{\mathbb{C}}_p & \xrightarrow{\mathcal{R}_p f} & \widetilde{\mathbb{C}}_p \\ {\scriptstyle \pi_p}\big\downarrow & & \big\downarrow{\scriptstyle \pi_p} \\ \widehat{\mathbb{C}} & \xrightarrow{f} & \widehat{\mathbb{C}} \end{array}$$

where $\widetilde{\mathbb{C}}_p$ denotes the p-sheeted sphere $\widehat{\mathbb{C}}$ branched over 0 and ∞, and the projection $\pi_p(z) = z^p$. In view of the assumption $f(0) = 0$, the covering map $\mathcal{R}_p f$ is well defined and commutes with the projection π_p on the sphere $\widehat{\mathbb{C}}$.

By the Kühnau-Schiffer theorem, the Grunsky norm $\varkappa(f)$ of any $f \in \Sigma(0)$ is reciprocal to the least positive Fredholm eigenvalue ϱ_L of the curve $L = f(|z| = 1)$ given by

$$\frac{1}{\varrho_L} = \sup \frac{|\mathcal{D}_G(u) - \mathcal{D}_{G^*}(u)|}{\mathcal{D}_G(u) + \mathcal{D}_{G^*}(u)}, \tag{1.5}$$

where G and G^* are, respectively, the interior and exterior of L; $\mathcal{D}$ denotes the Dirichlet integral, and the supremum is taken over all functions u continuous on $\widehat{\mathbb{C}}$ and harmonic on $G \cup G^*$ (see [15], [21]). This yields, in particular, that

$$\varkappa(\mathcal{R}_p f) \ge \varkappa(f) \quad \text{for any} \quad p > 1.$$

Note that the Taylor and Grunsky coefficients of any $\mathcal{R}_p f$ are represented as polynomials of the coefficients $b_0, b_1, b_2, \dots$ of the original functions f.

1.2. Main results

Consder for $\mu \in \mathrm{Belt}(\Delta)_1$ the truncated Beltrami coefficients

$$\mu_\rho(z) = \begin{cases} \mu(z), & |z| < \rho, \\ 0, & |z| > \rho \quad (\rho < 1) \end{cases} \tag{1.6}$$

and define for the corresponding function $f^\mu \in \Sigma(0)$,

$$\widehat{\varkappa}(f^\mu) = \lim_{\rho \to 1} \sup_p \sup_{\psi \in A_1^2(\Delta_\rho), \|\psi\|_{A_1(\Delta_\rho)} = 1} |\langle \mathcal{R}_p^* \mu, \psi \rangle_{\Delta_\rho}|. \tag{1.7}$$

This quantity can be regarded as the **outer limit Grunsky norm** of f on Δ (cf. the equality (2.6) below).

Theorem 1.1. *Every nonvanishing conformal map f of the disk Δ^* onto a generic quasiconformal disk, with $f(\infty) = \infty$, admits k-quasiconformal extensions with dilatations $k \ge \widehat{\varkappa}(f_*)$, where f_* is obtained from f by admissible translation and rescaling so that*

$$f_*(z) = (f(z) - c)/f'(\infty) \in \Sigma(0). \tag{1.8}$$

The lower bound $\widehat{\varkappa}(f_*)$ cannot be replaced by a smaller one for any $f \in \Sigma(0)$.

In addition, the curve $L = f(S^1)$ is a k'-quasicircle (i.e., the image of the unit circle S^1 under k-quasiconformal maps of $\widehat{\mathbb{C}}$ with $k \ge k'$), and its reflection coefficient q_L equals $\widehat{\varkappa}(f_*)$ and relates to the minimal dilatation k' by

$$\frac{1 + q_L}{1 - q_L} = \left(\frac{1 + k'}{1 - k'}\right)^2. \tag{1.9}$$

For odd univalent functions, this theorem yields a quantitatively complete solution of Ahlfors' problem.

Theorem 1.2. *For any odd univalent function $f(z) = z + b_1 z^{-1} + b_3 z^{-3} + \dots$ in Δ^*, either $\varkappa(f) = 1$ or f admits quasiconformal extensions across the unit circle with dilatations $k \ge \widehat{\varkappa}(f)$. The lower bound $\widehat{\varkappa}(f)$ is sharp for each f. The reflection coefficient of the curve $f(S^1)$ relates to its quasiconformal dilatation similar to* (1.9).

Theorem 1.2 can be extended to arbitrary centrally symmetric quasidisks $D^* \ni \infty$ by applying the generalized Grunsky coefficients defined by

$$\log \frac{f(z) - f(\zeta)}{z - \zeta} = - \sum_{m,n=1}^{\infty} \alpha_{mn}\ \chi(z)^{-m} \chi(\zeta)^{-n}, \tag{1.10}$$

for univalent f in D^* with exp[ansions $f(z) = z + \text{const} + O(z^{-1})$ near $z = \infty$. Here χ denotes the conformal map of D^* onto Δ^* with $\chi(\infty) = \infty,\ \chi'(\infty) > 0$, and

$$\varkappa_{D^*}(f) = \sup_{\mathbf{x}} \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn} x_m x_n \Big| \le 1, \quad \mathbf{x} = (x_n) \in S(l^2)$$

(cf. [6], [18], [19], [20], [22], [12]). Assume also that the origin $z = 0$ is placed in the complementary domain $D = \mathbb{C} \setminus \overline{D^*}$ and that quasiconformal extensions of our functions f preserve the origin. The collection of such f is denoted by $\Sigma(D^*, 0)$. Then we have

Theorem 1.3. *Every quasiconformal extension w^μ of an odd $f \in \Sigma(D^*, 0)$ in a centrally symmetric domain D^* has dilatation $k(w^\mu) \ge \widehat{\varkappa}(f_\chi)$, where $f_\chi(t) = \chi'(\infty) f \circ \chi^{-1}(t)$. This lower bound is sharp for any f.*

2. Proof of Theorem 1.1

We accomplish the proof in three stages.

1^0. We renormalize a given function f to get $f_* \in \Sigma(0)$ with the same Beltrami coefficient and then will deal only with such f_*. For simplicity of notations, the subscript $*$ will be omitted.

Every coefficient $\alpha_{mn}(f)$ in (1.2) is represented as a polynomial of a finite number of the initial coefficients $b_1, b_2, \dots, b_s$ of f. Hence it depends holomorphically on Beltrami coefficients μ of quasiconformal extensions of f running over the ball $\mathrm{Belt}(D)_1$. The same is true also for coefficients of $\mathcal{R}_p f$.

This implies the holomorphy of maps

$$h_{\mathbf{x},p}(\mu) = \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}(\mathcal{R}_p f^\mu)\ r^{m+n} x_m x_n :\ \mathrm{Belt}(D_r)_1 \to \Delta \tag{2.1}$$

for any fixed $p,\ r$ and $\mathbf{x} = (x_n) \in l^2$ with $\|\mathbf{x}\| = 1$ (see, e.g. [11]). Note also that

$$\sup_{\mathbf{x} \in S(l^2)} h_{\mathbf{x},p}(\mu) = \varkappa_r(\mathcal{R}_p f^\mu), \tag{2.2}$$

and each norm $\varkappa_r(\mathcal{R}_p f^\mu)$ is a continuous plurisubharmonic function of $\mu \in \mathrm{Belt}(\Delta_r)_1$.

We will use the following lemmas.

Lemma 2.1. *For any quasidisk D^* containing $z = \infty$, the Grunsky norm $\varkappa_{D^*}(f)$ of every function $f \in \Sigma(D^*, 0)$ is dominated by its Teichmüller norm as follows*

$$\varkappa_{D^*}(f) \le k \frac{k + \alpha_D(f)}{1 + \alpha_D(f) k}, \tag{2.3}$$

and $\varkappa_{D^*}(f) < k$ unless

$$\sup\ \Big\{ \Big| \iint_D \mu(z)\psi(z) dx dy \Big| :\ \psi \in A_1^2(D),\ \|\psi\|_{A_1} = 1 \Big\} = \|\mu_0\|_\infty. \tag{2.4}$$

The last equality is equivalent to $\varkappa_{D^*}(f) = k(f)$.

If, in addition, the equivalence class of f (the collection of maps equal f on ∂D^*) is a Strebel point, then μ_0 is necessarily of the form

$$\mu_0(z) = \|\mu_0\|_\infty |\psi_0(z)|/\psi_0(z) \quad \textit{with} \quad \psi_0 \in A_1^2(D). \tag{2.5}$$

This key lemma has been partially proved in [8] for the maps of circular disks and extended in such form to generic quasiconformal disks in [12] using Milin's results on special orthonormal systems and univalence (such disks arise in the proof of Theorem 1.3). For functions $f \in \Sigma_r(0)$ holomorphic in the closed disk $\overline{\Delta_r^*}$, the equality (2.5) was obtained by a different method in [16].

Moreover, due to [12], the bound (2.6) can be strengthened for small $\|\mu\|_\infty$ as follows.

Lemma 2.2. *The Grunsky norm of any $f \in \Sigma(D^*, 0)$ with Teichmüller quasiconformal extension satisfies the asymptotic equality*

$$\varkappa_{D^*}(f) = \sup \Big| \iint\limits_{D} \mu(z)\psi(z)dxdy \Big| + O(\|\mu\|_\infty^2), \quad \|\mu\|_\infty \to 0, \tag{2.6}$$

with the same supremum as in (2.4).

Lemma 2.3. *For a fixed $\rho < 1$, the restriction map $\iota_\rho : \ \mu \mapsto \mu_\rho$ given by* (1.6) *is holomorphic in L_∞ norm; hence, it induces a holomorphic map of the ball* $\mathrm{Belt}(\Delta)_1$ *into itself.*

Proof. The assertion of the lemma easily follows from the linearity of ι_ρ on $\mathrm{Belt}(\Delta)_1$.

We proceed to the proof of the theorem and first consider the maps $f^\mu \in \Sigma(0)$ having Teichmüller extremal extension, i.e., with Beltrami coefficients in Δ of the form

$$\mu(z) = \kappa|\psi_0(z)|/\psi_0(z), \quad \psi_0 \in A_1(\Delta).$$

If all zeros of ψ_0 in Δ are of even order, the theorem immediately follows from Lemma 2.2. Similarly, if ψ_0 has a single zero of odd order at the origin, this lemma can be applied to the squared map $\mathcal{R}_2 f = f(z^2)^{1/2}$ giving the assertion of Theorem 1.1 with $\widehat{\varkappa}(f) = \varkappa(\mathcal{R}_2 f)$. Thus, starting if needed with $\mathcal{R}_2 f$, one only needs to consider $f^\mu \in \Sigma(0)$ with μ defined by even $\psi_0 \in A_1(\Delta)$ of the form

$$\psi_0(z) = c_n z^n + c_{n+2} z^{n+2} + \dots \quad (n \geq 0 \ \text{ even}, \quad c_n > 0),$$

having at least two zeros of odd order in Δ.

After applying the transform (1.4), we get the Teichmüller map $\mathcal{R}_p f = f^{k|\mathcal{R}_p^*\psi_0|/\mathcal{R}_p^*\psi_0}$ determined by the quadratic differential

$$\mathcal{R}_p^*\psi_0 = \psi_0(z^p)p^2 z^{2p-2}. \tag{2.7}$$

Fix r_j arbitrarily close to 1 and pick p_j so large that all zeros of odd order of $\mathcal{R}_{p_j}^*\psi_0$ are placed in the annulus $\{r_j < |z| < 1\}$. Then, taking the truncated Beltrami coefficients for

$$\mathcal{R}_{p_j}^*\mu = k|\mathcal{R}_{p_j}^*\psi_0|/\mathcal{R}_{p_j}^*\psi_0$$

by (1.6) with $\rho = r_j$ and applying to these Lemma 2.1, one obtains that the corresponding maps $f^{[\mathcal{R}^*_{p_j}\mu]_{r_j}}$ satisfy

$$\varkappa_{r_j}\big(f^{[\mathcal{R}^*_{p_j}\mu]_{r_j}}\big) = \sup_{(x_n)\in S(l^2)} \Big| \sum_{m,n=1}^{\infty} \sqrt{mn}\ \alpha_{mn}\big(f^{[\mathcal{R}^*_{p_j}\mu]_{r_j}}\big)\ r_j^{m+n} x_m x_n \Big| = k. \tag{2.8}$$

Put $\mu* = \mu/\|\mu\|_\infty$ and note that the Beltrami coefficient μ and its truncations are extremal in their classes. Thus the equalities (2.5) and (2.8) imply that there exist holomorphic functions $h_{\mathbf{x}_j,p_j}(\nu)$ of type (2.1) on the ball $\mathrm{Belt}(\Delta)_1$ mapping this ball into the unit disk Δ and such that on the extremal coefficients $t\mu^* \in \mathrm{Belt}(\Delta)_1$ these functions are estimated by

$$h_{\mathbf{x}_j,p_j}(t\mu^*) = |t| - \gamma_j, \quad |t| < 1, \tag{2.9}$$

with $\gamma_j \to 0$ as $j \to \infty$.

It follows from (2.9) that the quantity

$$\lim_{j\to\infty} \tanh^{-1} h_{\mathbf{x}_j,p_j}(t\mu^*) = \tanh^{-1} \widehat{\varkappa}(f^{t\mu^*})$$

coincides with the Carathéodory distance on the ball $\mathrm{Belt}(\Delta)_1$ between $t\mu^*$ and $\mathbf{0}$ as well as with the Kobayashi and Teichmüller distances between these points, because all these distances are equal to the hyperbolic length

$$d_\Delta(0, t\mu^*) = \tanh^{-1} \|t\mu^*\|_\infty = \tanh^{-1} |t|$$

of the radial segment $[0, |t|]$ in the unit disk.

This implies, taking $t = \|\mu\|_\infty$, the desired equality $\widehat{\varkappa}(f^\mu_*) = k(f^\mu_*) = k$. For any other extension $f^{\widetilde{\mu}}_*$ of f_* preserving 0, we have $\|\widetilde{\mu}\|_\infty > k$, thus $\|\widetilde{\mu}\|_\infty \geq \widehat{\varkappa}(f^{\widetilde{\mu}}_*)$. Thereby the first assertion of the theorem is proved for $f \in \Sigma(0)$ admitting Teichmüller extensions.

$\mathbf{2}^0$. Now consider generic $f \in \Sigma(0)$, and let μ be one of its exremal Beltrami coefficients in Δ (i.e., with minimal L_∞ norm). Truncate this μ by (1.6) and pick $\rho' > \rho$ close to ρ. Restricting μ to the disk $\Delta_{\rho'} = \{|z| < \rho'\}$, one obtains, in view of conformality of μ_ρ on the annulus

$$U_{\rho,\rho'} = \{\rho < |z| < \rho'\}$$

that the equivalence class of $f^{\mu_\rho}|_{|z|=\rho'}$ is a Strebel point [23], i.e., the map f^{μ_ρ} admits Teichmüller extension across the circle $\{|z| = \rho'\}$ onto the disk $\Delta_{\rho'}$. It has the Beltrami coefficient

$$\mu_{\rho,\rho'}(z) = k_{\rho,\rho'} |\psi_{\rho,\rho'}| / \psi_{\rho,\rho'}$$

which satisfies

$$k_{\rho,\rho'} = \|\mu_\rho\|_\infty - o(1) = \|\mu\|_\infty - o(1) < \|\mu\|_\infty,$$

where $o(1) = \gamma_1(\rho' - \rho) \to 0$ as $\rho' \to \rho$ and both $\rho,\ \rho' \to 1$. The last estimate follows from the properties of extremal quasiconformal maps (and their Beltrami coefficients).

Applying to $f^{\mu_{\rho,\rho'}}$ the arguments from the previous step, one gets the equality

$$\varkappa_p\big(f^{\mathcal{R}_p\mu_{\rho,\rho'}}\big) = k\big(f^{\mathcal{R}_p\mu_{\rho,\rho'}}\big),$$

provided that $p \ge p_0(\rho)$ is so large that the quadratic differential $\mathcal{R}_p^*\psi_{\rho,\rho'}(z)$ has no zeros of odd order in the annulus $U_{\rho,\rho'}$. Since every $\mathcal{R}_p^*\psi_{\rho,\rho'}$ also is of the form (2.7), the corresponding Beltrami coefficient $\mathcal{R}_p^*\mu_{\rho,\rho'}$ is of Teichmüller type, hence extremal in its class.

Using the same properties of extremal quasiconformal maps and the density of Strebel points (see [4]), one concludes that

$$\|\mu - \mu_{\rho,\rho'}\|_\infty = \gamma(\rho' - \rho),$$

with $\gamma(\rho' - \rho) \to 0$ as $\rho' \to \rho$ and both $\rho,\ \rho'$ approach 1. This yields the estimate

$$\iint\limits_{|z|<\rho'} \mu(z) z^n dxdy - \iint\limits_{|z|<\rho'} \mu_{\rho,\rho'}(z) z^n dxdy \asymp \gamma(\rho - \rho')$$

for all $n = 0, 1, 2, \dots$ (and then for any integrable holomorphic ψ in the disk $\Delta_{\rho'}$). Since

$$\widehat{\varkappa}(f^{\mu_{\rho,\rho'}}) \le k(f^{\mu_{\rho,\rho'}}) \le k(f^{\mu}),$$

one gets in the limit the desired bound $\widehat{\varkappa}(f^\mu) \le k = k(f^\mu)$ for the initial map f^μ.

3^0. Finally, any quasiconformal reflection σ with respect to the curve $L = f(|z| = 1)$ (i.e., an orientation reversing quasiconformal homeomorphism of the sphere $\widehat{\mathbb{C}}$ which interchanges the interior and exterior domains of L and keeps this curve pointwise fixed) can be represented in the form

$$\sigma = \widehat{f} \circ \sigma_0 \circ f^{-1},$$

where σ_0 is the conformal symmetry $w \mapsto 1/\overline{w}$ and $\widehat{f}$ is a quasiconformal extension of the given function f across the unit circle. The reflection coefficient of L, being equal to the minimal dilatation among such σ, is connected with extremal dilatation among quasiconformal maps of the plane moving the unit circle into L by (1.9). This equality is one of the basic relations in the theory of quasiconformal reflections (see, e.g., survey [10] and the references given there).

3. Proofs of Theorems 1.2 and 1.3

The odd functions

$$f(z) = z + b_1 z^{-1} + b_3 z^{-3} + \dots \in \Sigma(0) \tag{3.1}$$

(with odd quasiconformal extensions) have some specific properties; thus Theorems 1.2 and 1.3, being in fact the consequences of the key Theorem 1.1, provide somewhat more.

Such functions (different from the identity) have even Beltrami coefficients, and $f(0) = 0$. If f admits a Teichmüller extension, its defining quadratic differential ψ_0 also is even, i.e., of the form $\psi_0(z) = c_0 + c_2 z^2 + c_4 z^4 + \dots$. So, for the functions with expansion(3.1), f_* given by (1.8) coincides with its original f.

Proof of Theorem 1.2. Since f is univalent in D^*, the Grunsky theorem yields $\varkappa(f) \le 1$. If $\varkappa(f) = \kappa < 1$, the function f admits, due to Pommerenke and Zhuravlev, a κ_1-quasiconformal extension to Δ with $\kappa_1 = \kappa_1(\kappa) \ge \kappa$ (see [20]; [13, pp.82-84], [24]). This bound κ_1 can be given explicitly, but we do not need it here.

The remaining assertions of Theorem 1.2 follow from Theorem 1.1.

The proof of Theorem 1.3 follows the same lines, since the conformal map χ and its inverse both are centrally symmetric, and the same holds for all composed maps $f \circ \chi^{-1}$.

4. Additional remarks

1. Passing from $f \in \Sigma(0)$ to their inversions

$$F(z) = 1/f(1/z) = z - b_0 z^2 + (b_0^2 - b_1) z^3 + a_4 z^4 + \dots, \quad |z| < 1,$$

one gets the analogs of Theorems 1.1-1.3 for univalent functions in the unit disk.

2. The restriction map ι_ρ of Lemma 2.2 does not descend to a holomorphic map of the universal Teichmüller space obtained by appropriate factorization of the ball $\mathrm{Belt}(\Delta)_1$.

3. The homotopies $f_r(z) = rf(z/r) : \Delta^* \times [0,1]$ of odd functions $f \in \Sigma(0)$ also admit some specific properties and have been investigated in [17] and [11].

In [17], there was considered the action of the root transform $\mathcal{R}_p$ on odd f holomorphic in the closed disk $\overline{\Delta^*}$ and with $b_1 \ne 0$ and established that for all sufficiently small r,

$$\varkappa(\{\mathcal{R}_2 f\}_r) = \varkappa(f_r). \tag{4.1}$$

For example, this holds when $r^2 + O(r^4) < 0.36\sqrt{|b_1|}$.

Some close (though not explicit) results ensuring the equality $\varkappa(f_r) = k(f_r)$ (and thereby (4.1)) for sufficiently small r are obtained in [11]. The underlying features concern the distribution of zeros of the defining holomorphic quadratic differentials for f and f_r.

4. It was conjectured in [5] (in an equivalent form) that for any $f \in \Sigma(0)$, we have

$$\limsup_{p\to\infty} \varkappa(\mathcal{R}_p f) = k,$$

where k stands for the minimal dilatation of quasiconformal extensions of f (preserving the origin).

The arguments in the proof of Theorem 1.1 provide simulatneously a result of such type though it involves some other functions from $\Sigma(0)$ outwardly related to a given f.

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