Annals of the University of Bucharest (mathematical series)
5 (LXIII) (2014), 131–135

New modulus estimates in Orlicz-Sobolev classes

Denis Kovtonyuk and Vladimir Ryazanov

Dedicated to 85 years of Academician C. Andreian Cazacu

Abstract - Under a condition of the Calderon type on φ, we show that a homeomorphism f of finite distortion in $W^{1,\varphi}_{\rm loc}$ and, in particular, $f \in W^{1,p}_{\rm loc}$ for $p > n-1$ in $\mathbb{R}^n$, $n \geq 3$, is a lower Q-homeomorphisms with $Q(x) = [K_I(x,f)]^{\frac{1}{n-1}}$ and a ring Q-homeomorphism with $Q(x) = K_I(x,f)$ where $K_I(x,f)$ is its inner dilatation. Similar statements are valid also for finitely bi-Lipschitz mappings that a far–reaching extension of the well-known classes of isometric and quasiisometric mappings. This makes possible to apply our theories on the local and boundary behavior for the lower and ring Q-homeomorphisms to all given classes.

Key words and phrases : Sobolev and Orlicz-Sobolev classes, mappings of finite distortion, lower and ring Q-homeomorphisms, finitely bi-Lipschitz mappings.

Mathematics Subject Classification (2010) **:** Primary 30C85, 30D40, 31A15, 31A20, 31A25, 31B25; Secondary 37E30.

1. Introduction

It is well–known that the concept of moduli with weights essentially due to Andreian Cazacu, see, e.g., [1]–[3], see also recent works [4]–[6] of her learner. Here we give new modulus estimates for space mappings that essentially improve the corresponding estimates in our last paper [12]. Since this note is short, we refer readers for definitions, history and explanations either to it or in addition to papers [8]–[11] and [16] and monographs [13] and [14]. Many applications will be published elsewhere.

2. Lower Q-homeomorphisms and Orlicz–Sobolev classes

Given a mapping $f : D \to \mathbb{R}^n$ with partial derivatives a.e., recall that $f'(x)$ denotes the Jacobian matrix of f at $x \in D$ if it exists, $J(x) = J(x,f) = \det f'(x)$ is the Jacobian of f at x, and $\|f'(x)\|$ is the operator norm of $f'(x)$, i.e.,

$$\|f'(x)\| = \max\{|f'(x)h| : h \in \mathbb{R}^n\}, |h| = 1\}. \tag{2.1}$$

We also let

$$l(f'(x)) = \min\{|f'(x)h| : h \in \mathbb{R}^n\}, |h| = 1\}. \tag{2.2}$$

The **outer dilatation** of f at x is defined by

$$K_O(x) = K_O(x,f) = \begin{cases} \frac{\|f'(x)\|^n}{|J(x,f)|} & \text{if } J(x,f) \neq 0, \\ 1 & \text{if } f'(x) = 0, \\ \infty & \text{otherwise,} \end{cases} \tag{2.3}$$

the **inner dilatation** of f at x by

$$K_I(x) = K_I(x,f) = \begin{cases} \frac{|J(x,f)|}{l(f'(x))^n} & \text{if } J(x,f) \neq 0, \\ 1 & \text{if } f'(x) = 0, \\ \infty & \text{otherwise,} \end{cases} \tag{2.4}$$

Note that, see, e.g., Section 1.2.1 in [15],

$$K_O(x,f) \le K_I^{n-1}(x,f) \qquad \text{and} \qquad K_I(x,f) \le K_O^{n-1}(x,f)\ , \tag{2.5}$$

in particular, $K_O(x,f) < \infty$ a.e. if and only if $K_I(x,f) < \infty$ a.e. The latter is equivalent to the condition that a.e. either $\det f'(x) > 0$ or $f'(x) = 0$.

Recall that a homeomorphism f between domains D and D' in $\mathbb{R}^n$, $n \geqslant 2$, is called of **finite distortion** if $f \in W^{1,1}_{\text{loc}}$ and

$$\|f'(x)\|^n \leqslant K(x) \cdot J_f(x) \tag{2.6}$$

for some a.e. finite function K. The term is due to Tadeusz Iwaniec. In other words, (2.6) just means that dilatations $K_O(x,f)$ and $K_I(x,f)$ are finite a.e.

In view of (2.5), the next statement says on a stronger modulus estimate than the obtained in [12], Theorem 4.1, in terms of the outer dilatation $K_O(x,f)$.

Theorem 2.1. *Let D and D' be domains in $\mathbb{R}^n$, $n \geqslant 3$, and let $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a nondecreasing function such that, for some $t_* \in \mathbb{R}^+$,*

$$\int\limits_{t_*}^{\infty} \left[\frac{t}{\varphi(t)}\right]^{\frac{1}{n-2}} dt < \infty\,. \tag{2.7}$$

Then each homeomorphism $f : D \to D'$ of finite distortion in the class $W^{1,\varphi}_{\text{loc}}$ is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q(x) = [K_I(x,f)]^{\frac{1}{n-1}}$.

Proof. Let B be a (Borel) set of all points $x \in D$ where f has a total differential $f'(x)$ and $J_f(x) \neq 0$. Then, applying Kirszbraun's theorem and uniqueness of approximate differential, see, e.g., 2.10.43 and 3.1.2 in [7], we see that B is the union of a countable collection of Borel sets B_l, $l = 1, 2, \dots$, such that $f_l = f|_{B_l}$ are bi-Lipschitz homeomorphisms, see, e.g., 3.2.2 as well as 3.1.4 and 3.1.8 in [7]. With no loss of generality, we may assume that the B_l are mutually disjoint. Denote also by B_* the rest of all points $x \in D$ where f has the total differential but with $f'(x) = 0$.

By the construction the set $B_0 := D \setminus (B \bigcup B_*)$ has Lebesgue measure zero, see Theorem 1 in [11]. Hence $\mathcal{A}_S(B_0) = 0$ for a.e. hypersurface S in $\mathbb{R}^n$ and, in particular, for a.e. sphere $S_r := S(x_0, r)$ centered at a prescribed point $x_0 \in \overline{D}$, see Theorem 2.11 in [9] or Theorem 9.1 in [14]. Thus, by Corollary 4 in [11] $\mathcal{A}_{S_r^*}(f(B_0)) = 0$ as well as $\mathcal{A}_{S_r^*}(f(B_*)) = 0$ for a.e. S_r where $S_r^* = f(S_r)$.

Let Γ be the family of all intersections of the spheres S_r, $r \in (\varepsilon, \varepsilon_0)$, $\varepsilon_0 < d_0 = \sup\limits_{x \in D} |x - x_0|$, with the domain D. Given $\varrho_* \in \operatorname{adm} f(\Gamma)$ such that $\varrho_* \equiv 0$ outside of $f(D)$, set $\varrho \equiv 0$ outside of D and on $D \setminus B$ and, moreover,

$$\varrho(x) := \Lambda(x) \cdot \varrho_*(f(x)) \qquad \text{for } x \in B$$

where

$$\Lambda(x) \;=\; \left[\; J_f(x) \cdot K_I^{\frac{1}{n-1}}(x, f) \;\right]^{\frac{1}{n}} \;=\; \left[\; \frac{\det f'(x)}{l(f'(x))} \;\right]^{\frac{1}{n-1}} \;=$$

$$=\; [\; \lambda_2 \cdot \ldots \cdot \lambda_n \;]^{\frac{1}{n-1}} \;\geqslant\; [\; J_{n-1}(x) \;]^{\frac{1}{n-1}} \qquad \text{for a.e. } x \in B\;;$$

here as usual $\lambda_n \geqslant \ldots \geqslant \lambda_1$ are principal dilatation coefficients of $f'(x)$, see, e.g., Section I.4.1 in [15], and $J_{n-1}(x)$ is the $(n-1)-$dimensional Jacobian of $f|_{S_r}$ at x where $r = |x - x_0|$, see Section 3.2.1 in [7].

Arguing piecewise on B_l, $l = 1, 2, \dots$, and taking into account Kirszbraun's theorem, by Theorem 3.2.5 on the change of variables in [7], we have that

$$\int\limits_{S_r} \varrho^{n-1}\, d\mathcal{A} \geqslant \int\limits_{S_*^r} \varrho_*^{n-1}\, d\mathcal{A} \geqslant 1$$

for a.e. S_r and, thus, $\varrho \in \operatorname{ext}\operatorname{adm}\Gamma$.

The change of variables on each B_l, $l = 1, 2, \dots$, see again Theorem 3.2.5 in [7], and countable additivity of integrals give also the estimate

$$\int\limits_{D} \frac{\varrho^n(x)}{K_I^{\frac{1}{n-1}}(x)}\, dm(x) \leqslant \int\limits_{f(D)} \varrho_*^n(x)\, dm(x)$$

and the proof is complete. $\square$

Corollary 2.1. *Each homeomorphism f with finite distortion in $\mathbb{R}^n$, $n \geqslant 3$, of the class $W^{1,p}_{\rm loc}$ for $p > n-1$ is a lower Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q = K_I^{\frac{1}{n-1}}$.*

Moreover, by Corollary 5 in [11] on a connection between lower and ring Q-homeomorphisms, we also obtain the following consequence of Theorem 2.1.

Theorem 2.2. *Let $f : D \to \mathbb{R}^n$, $n \geqslant 3$, be a homeomorphism with $K_I \in L^1_{\rm loc}$ in $W^{1,\varphi}_{\rm loc}$ where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a nondecreasing function with condition (2.7). Then f is a ring Q-homeomorphism at every point $x_0 \in \overline{D}$ with $Q = K_I$.*

In view of (2.5), the latter statement says on a stronger modulus estimate than the obtained in [11], Corollary 9, in terms of the outer dilatation K_O.

Remark 2.1. By Remark 8 in [11] the conclusion of Theorem 2.2 is valid if K_I is integrable only on almost all spheres of small enough radii centered at x_0 assuming that the function K_I is extended by zero outside of D.

3. On finitely bi–Lipschitz mappings

Given an open set $\Omega \subseteq \mathbb{R}^n$, $n \geqslant 2$, following Section 5 in [9], see also Section 10.6 in [14], we say that a mapping $f : \Omega \to \mathbb{R}^n$ is **finitely bi-Lipschitz** if

$$0 \ < \ l(x,f) \ \leqslant \ L(x,f) \ < \ \infty \qquad \forall \ x \in \Omega \tag{3.1}$$

where

$$L(x,f) \ = \ \limsup_{y \to x} \ \frac{|f(y) - f(x)|}{|y - x|} \tag{3.2}$$

and

$$l(x,f) \ = \ \liminf_{y \to x} \ \frac{|f(y) - f(x)|}{|y - x|} \ . \tag{3.3}$$

By the classic Rademacher–Stepanov theorem, we obtain from the right hand inequality in (3.1) that finitely bi-Lipschitz mappings are differentiable a.e. and from the left hand inequality in (3.1) that $J_f(x) \neq 0$ a.e. Moreover, such mappings have $(N)-$property with respect to each Hausdorff measure, see, e.g., either Lemma 5.3 in [9] or Lemma 10.6 [14]. Thus, the proof of the following theorems is perfectly similar to one of Theorem 2.1 and hence we omit it, cf. also similar but weaker Corollary 5.15 in [9] and Corollary 10.10 in [14] formulated in terms of the outer dilatation K_O.

Theorem 3.1. *Every finitely bi-Lipschitz homeomorphism $f : \Omega \to \mathbb{R}^n$, $n \geqslant 2$, is a lower Q-homeomorphism with $Q = K_I^{\frac{1}{n-1}}$.*

References

[1] C. Andreian Cazacu, Influence of the orientation of the characteristic ellipses on the properties of the quasiconformal mappings, *Proceedings of Romanian-Finnish Seminar on Teichmüller Spaces and Riemann Surfaces*, Bucharest, 1971, 65-85.

[2] C. Andreian Cazacu, Moduli inequalities for quasiregular mappings, *Ann. Acad. Sci. Fenn. Math.*, **2** (1976), 17-28, http://dx.doi.org/10.5186/aasfm.1976.0204.

[3] C. Andreian Cazacu, On the length-area dilatation, *Complex Var. Theory Appl.*, **50**, *7-11* (2005), 765-776, http://dx.doi.org/10.1080/02781070500087840.

[4] M. Cristea, Local homeomorphisms satisfying generalized modular inequalities, *Complex Var. Elliptic Equ.*, **59**, *10* (2014), 1363-1387, http://dx.doi.org/10.1080/17476933.2013.845176.

[5] M. Cristea, On generalized quasiconformal mappings, *Complex Var. Elliptic Equ.*, **59**, *2* (2014), 232-246, http://dx.doi.org/10.1080/17476933.2012.731398.

[6] M. Cristea, On generalized quasiregular mappings, *Complex Var. Elliptic Equ.*, **58**, *12* (2013), 1745-1764, http://dx.doi.org/10.1080/17476933.2012.713946.

[7] H. Federer, *Geometric measure theory*, Springer-Verlag, New York, 1969.

[8] D.A. Kovtonyuk and V.I. Ryazanov, Toward the theory of lower Q-homeomorphisms, *Ukr. Mat. Visn.*, **5**, *2* (2008), 159-184 [in Russian]; transl. in *Ukr. Math. Bull.*, **5**, *2* (2008), 157-181.

[9] D. Kovtonyuk and V. Ryazanov, On the theory of mappings with finite area distortion, *J. Anal. Math.*, **104** (2008), 291-306, http://dx.doi.org/10.1007/s11854-008-0025-5.

[10] D. Kovtonyuk and V. Ryazanov, On the boundary behavior of generalized quasi-isometries, *J. Anal. Math.*, **115** (2011), 103-119, http://dx.doi.org/10.1007/s11854-011-0025-8.

[11] D.A. Kovtonyuk, V.I. Ryazanov, R.R. Salimov and E.A. Sevost'yanov, Toward the theory of classes of Orlicz-Sobolev, *Algebra and Analysis*, **25**, *6* (2013), 49-101 [in Russian]; transl. in *St. Petersburg Math. J.*

[12] D.A. Kovtonyuk, V.I. Ryazanov, R.R. Salimov and E.A. Sevost'yanov, On mappings in the Orlicz-Sobolev classes, *Ann. Univ. Buchar. Math. Ser.*, **3 (LXI)** (2012), 67-78.

[13] D.A. Kovtonyuk, R.R. Salimov and E.A. Sevost'yanov (ed. V.I. Ryazanov), *Toward the Theory of Mappings in Classes of Sobolev and Orlicz–Sobolev*, Naukova dumka, 2013 [in Russian].

[14] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, *Moduli in Modern Mapping Theory*, Springer Monogr. Math., Springer, New York, 2009, http://dx.doi.org/10.1007/978-0-387-85588-2.

[15] Y.G. Reshetnyak, Space mappings with bounded distortion, Nauka, 1982; transl. in *Transl. Math. Monogr.*, 73, AMS, Providence, RI, 1989.

[16] V. Ryazanov and E. Sevost'yanov, Equicontinuons classes of ring Q-homeomorphisms, *Sibirsk. Mat. Zh.*, **48**, *6* (2007), 1361-1376 [in Russian]; transl. in *Sib. Math. J.*, **48**, *6* (2007), 1093-1105, http://dx.doi.org/10.1007/s11202-007-0111-4.

Denis Kovtonyuk and Vladimir Ryazanov

Institute of Applied Mathematics and Mechanics

National Academy of Sciences of Ukraine

74 Roze Luxemburg str., 83114 Donetsk, Ukraine

E-mail: `vl.ryazanov1@gmail.com`