

Fine properties of weighted Carnot–Carathéodory spaces under minimal assumptions on smoothness

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Dedicated to Professor Cabiria Andreian Cazacu on her 85th birthday

Abstract - For weighted Carnot–Carathéodory spaces, we deduce the relation between basis vector fields and nilpotentized vector fields. We apply this result to obtain comparison estimates for a weighted Carnot–Carathéodory space and its local homogeneous groups. All these problems are solved under minimal assumptions on smoothness of the basis vector fields.

Key words and phrases : weighted Carnot–Carathéodory space, local homogeneous group, minimal smoothness, local geometry.

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1. Introduction

The goal of the paper is to study local geometry of equiregular weighted Carnot–Carathéodory spaces, i.e., spaces with arbitrary degrees of basis vector fields under assumption that these fields belong to the class C^1 or $C^{1,\alpha}$, $\alpha > 0$.

The case of a weighted Carnot–Carathéodory space is new in comparison to the classical one where a degree of a vector field is defined by the order of the commutator of horizontal vector fields (see also Definition 2.1 below). See, e.g., papers [42, 16] for some results on this new case.

Carnot–Carathéodory geometry is applied for studying hypoelliptic operators (see, e.g., [10, 19, 41]), and it is also extensively used in the theory of partial differential equations (see, e.g., [13, 7, 48, 4]).

2. Basic definitions and results

First of all, recall the definition of an «ordinary» Carnot–Carathéodory space.

Definition 2.1. (see [4]; cf. [17, 23, 37]) *Fix a connected Riemannian C^∞ -manifold \mathbb{M} of topological dimension N . The manifold \mathbb{M} is called the Carnot–Carathéodory space if the tangent bundle $T\mathbb{M}$ has a filtration*

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \dots \subsetneq H_i\mathbb{M} \subsetneq \dots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles such that every point $p \in \mathbb{M}$ has a neighborhood $U \subset \mathbb{M}$

equipped with a collection of C^1 -smooth vector fields X_1, \dots, X_N enjoying the following two properties.

- (1) At every point $v \in U$ we have a subspace

$$H_i\mathbb{M}(v) = H_i(v) = \text{span}\{X_1(v), \dots, X_{\dim H_i}(v)\} \subset T_v\mathbb{M}$$

of the dimension $\dim H_i$ independent of v , $i = 1, \dots, M$.

- (2) The inclusion $[H_i, H_j] \subset H_{i+j}$, $i + j \leq M$, holds.

Moreover, if the third condition holds then the Carnot–Carathéodory space is called the Carnot manifold:

- (3) $H_{j+1} = \text{span}\{H_j, [H_1, H_j], [H_2, H_{j-1}], \dots, [H_k, H_{j+1-k}]\}$, where $k = \lceil \frac{j+1}{2} \rceil$, $H_0 = \{0\}$, $j = 1, \dots, M - 1$.

The subbundle $H\mathbb{M}$ is called horizontal.

The number M is called the depth of the manifold \mathbb{M} .

Properties of Carnot–Carathéodory spaces and Carnot manifolds under assumptions of regularity mentioned in Definition 2.1 can be found in [23, 22, 50, 4, 16]. Many classical and modern results, development trends of the theory of Carnot–Carathéodory spaces and their applications can be found in [1, 2, 3, 5, 6, 7, 8, 9, 11, 12, 14, 15, 17, 18, 19, 20, 21, 27, 29, 30, 31, 32, 33, 34, 35, 36, 37, 39, 41, 44, 45, 46, 47, 49, 51].

Recall now the definition of a weighted Carnot–Carathéodory space.

Definition 2.2. ([25]; cf. Definition 2.1 and [42, 43]) *Fix a connected Riemannian C^∞ -manifold \mathbb{M} of topological dimension N . The manifold \mathbb{M} is called the weighted Carnot–Carathéodory space if the tangent bundle $T\mathbb{M}$ has a filtration*

$$H\mathbb{M} = H_1\mathbb{M} \subsetneq \dots \subsetneq H_i\mathbb{M} \subsetneq \dots \subsetneq H_M\mathbb{M} = T\mathbb{M}$$

by subbundles with assigned natural numbers $l_1 < l_2 < \dots < l_M$ such that every point $p \in \mathbb{M}$ has a neighborhood $U \subset \mathbb{M}$ equipped with a collection of C^1 -smooth vector fields X_1, \dots, X_N enjoying the following two properties.

- (1) At every point $v \in U$ we have a subspace

$$H_i\mathbb{M}(v) = H_i(v) = \text{span}\{X_1(v), \dots, X_{\dim H_i}(v)\} \subset T_v\mathbb{M}$$

of the dimension $\dim H_i$ independent of v , $i = 1, \dots, M$.

- (2) The inclusion $[H_i, H_j] \subset H_m$ holds where $m = \max\{p : l_i + l_j \geq l_p\}$.

The number M is called the depth of the manifold \mathbb{M} .

Remark 2.1. Emphasize that depths M from Definitions 2.1 and 2.2 may differ. Moreover, in Definition 2.2 M may coincide with the topological dimension N .

Remark 2.2. To this end, we assume that weighted Carnot–Carathéodory spaces under consideration have the same collection of basis vector fields for all points.

Example 2.1. In [22], an example of a 4-dimensional Carnot manifold of a depth 3 with C^1 -smooth basis vector fields is given. If we put in that example $\text{wgt } Y = 1$, $\text{wgt } X = 2$, $\text{wgt } Z = 3$, $\text{wgt } T = 4$ then we obtain a weighted Carnot — Carathéodory space.

Definition 2.3. Consider the initial value problem

$$\dot{\gamma}(t) = \sum_{i=1}^N y_i X_i(\gamma(t)), \quad t \in [0, 1], \quad \gamma(0) = x,$$

where the vector fields X_1, \dots, X_N are C^1 -smooth. Then, for the point $y = \gamma(1)$, we write $y = \exp\left(\sum_{i=1}^N y_i X_i\right)(x)$.

The mapping $(y_1, \dots, y_N) \mapsto \exp\left(\sum_{i=1}^N y_i X_i\right)(x)$ is called the exponential mapping.

Definition 2.4. Consider $u \in \mathbb{M}$ and $(v_1, \dots, v_N) \in B_E(0, r)$, where $B_E(0, r)$ is a Euclidean ball in \mathbb{R}^N . Define a mapping $\theta_u : B_E(0, r) \rightarrow \mathbb{M}$ as follows:

$$\theta_u(v_1, \dots, v_N) = \exp\left(\sum_{i=1}^N v_i X_i\right)(u).$$

It is known that θ_u is a C^1 -diffeomorphism if $0 < r \leq r_u$ for some $r_u > 0$. The collection $\{v_i\}_{i=1}^N$ is called the normal coordinates or the coordinates of the 1st kind (with respect to $u \in \mathbb{M}$) of the point $v = \theta_u(v_1, \dots, v_N)$.

Proposition 2.1. (see, e.g., [26]) There exists a compactly embedded neighborhood $\mathcal{U} \Subset \mathbb{M}$ such that $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$.

Definition 2.5. The weight $\text{wgt } X_k$ equals $\min\{l_m \mid X_k \in H_m\}$, $k = 1, \dots, N$.

Remark 2.3. The condition (2) of Definition 2.2 implies

$$[X_i, X_j](v) = \sum_{k: \text{wgt } X_k \leq \text{wgt } X_i + \text{wgt } X_j} c_{ijk}(v) X_k(v), \quad (2.1)$$

$i, j = 1, \dots, N$.

For weighted spaces, the following result is true.

Theorem 2.1. (cf. [23]) Fix $u \in \mathbb{M}$. The collection

$$\bar{c}_{ijk} = \begin{cases} c_{ijk}(u) \text{ of (2.1)} & \text{if } \text{wgt } X_i + \text{wgt } X_j = \text{wgt } X_k, \\ 0 & \text{otherwise} \end{cases}$$

constitutes a structure of a nilpotent Lie algebra.

We construct the Lie algebra \mathfrak{g}^u of Theorem 2.1 as a nilpotent Lie algebra of vector fields $\{(\widehat{X}_i^u)'\}_{i=1}^N$ on \mathbb{R}^N such that the exponential mapping $(x_1, \dots, x_N) \mapsto \exp\left(\sum_{i=1}^N x_i (\widehat{X}_i^u)'\right)(0)$ is the identity [40, 6, 17, 28, 38]. As soon as this mapping is the identity, we have $x_i = \exp(x_i (\widehat{X}_i^u)')(0)$. It follows that derivative with respect to x_i at 0 of the left-hand side, equal to the vector e_i of the canonical basis in \mathbb{R}^N , coincides with the derivative of the right-hand side, equal to $(\widehat{X}_i^u)'(0)$. Thus the condition for the exponential mapping to be identical one implies the initial value

$$(\widehat{X}_i^u)'(0) = e_i \quad (2.2)$$

for the vector fields $(\widehat{X}_i^u)'$, $i = 1, \dots, N$.

By Definition 2.4, we have $[(\theta_u)_* \langle e_i \rangle](0) = D\theta_u(0) \langle e_i \rangle = X_i(u)$. From here and (2.2) it follows

$$[(\theta_u)_* \langle (\widehat{X}_i^u)' \rangle](0) = X_i(u), \quad i = 1, \dots, N. \quad (2.3)$$

By the construction, the vector fields $\{(\widehat{X}_i^u)'\}_{i=1}^N$ satisfy

$$[(\widehat{X}_i^u)', (\widehat{X}_j^u)'] = \sum_{\text{wgt } X_k = \text{wgt } X_i + \text{wgt } X_j} c_{ijk}(u) (\widehat{X}_k^u)' \quad (2.4)$$

everywhere on \mathbb{R}^N .

Other properties of this Lie algebra of vector fields and the construction of the corresponding nilpotent Lie group $\mathcal{G}^u\mathbb{M}$ are similar to those described in [26] with the new notions of weights instead of degrees, see Definitions 5 and 6 and Remark 4 there.

Below the necessary properties, notations and assumptions are cited (see also [23, 26]).

Notation 2.1. We use the following standard notation: for each N -dimensional multi-index $\mu = (\mu_1, \dots, \mu_N)$, its *homogeneous norm* equals $|\mu|_h = \sum_{i=1}^N \mu_i \text{wgt } X_i$.

Theorem 2.2. (see [12]) *If $\{\frac{\partial}{\partial x_l}\}_{l=1}^N$ is a standard basis in \mathbb{R}^N then the j th coordinate of a vector field $(\widehat{X}_i^u)'(x) = \sum_{j=1}^N z_i^j(u, x) \frac{\partial}{\partial x_j}$ equals*

$$z_i^j(u, x) = \begin{cases} \delta_{ij} & \text{if } j \leq \dim H_{\text{wgt } X_i}, \\ \sum_{\substack{|\mu+e_i|_h = \text{wgt } X_j, \\ \mu > 0}} F_{\mu, e_i}^j(u) x^\mu & \text{if } j > \dim H_{\text{wgt } X_i}, \end{cases}$$

$i = 1, \dots, N$.

Remark 2.4. Theorems 2.1 and 2.2 imply that the collection $\{l_1, \dots, l_M\}$ defines the algebraic structure corresponding to local groups and vice versa. I.e., if we have a collection of vector fields with weights $\{l_1, \dots, l_M\}$ and a corresponding filtration like the one in Definition 2.2 then we have a «triangular» table of commutators similar to (2.1) and constants defining the structure of the corresponding Lie algebra. Moreover, if we have a table of commutators then it defines a filtration of the tangent bundle and the collection of the weights of the vector fields.

Definition 2.6. Let \mathbb{M} be a weighted Carnot–Carathéodory space of topological dimension N and depth M , and $u \in \mathbb{M}$. For $x, v \in \mathcal{U} \subset \mathcal{G}^u \mathbb{M}$ such that $x = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(v)$, we define the quasimetric $d_\infty^u(x, v)$ as follows:

$$d_\infty^u(x, v) = \max_{i=1, \dots, N} \{|x_i|^{\frac{1}{\text{wgt } X_i}}\}.$$

Denote the ball $\{v \in \mathcal{G}^u \mathbb{M} : d_\infty^u(x, v) < r\}$ of radius r centered at x by $\text{Box}^u(x, r)$.

Remark 2.5. The function d_∞^u is homogeneous with respect to dilatations corresponding to the filtration and to the collection of the weights $\{l_1, \dots, l_N\}$.

It follows from the definition of $\mathcal{G}^u \mathbb{M}$ that function d_∞^u is a quasimetric, i.e., the generalized triangle inequality is valid for it locally on \mathbb{M} (see. e.g., [26]).

Definition 2.7. Let \mathbb{M} be a weighted Carnot–Carathéodory space. Given $u \in \mathcal{U}$ and $v \in \mathcal{U}$ such that $v = \exp\left(\sum_{i=1}^N v_i X_i\right)(u)$, define the mapping Δ_ε^u as

$$\Delta_\varepsilon^u(v) = \exp\left(\sum_{i=1}^N v_i \varepsilon^{\text{wgt } X_i} X_i\right)(u)$$

for $\varepsilon > 0$ such that the right-hand side of this relation is well-defined.

Definition 2.8. Let \mathbb{M} be a weighted Carnot–Carathéodory space of topological dimension N and depth M , and put $x = \exp\left(\sum_{i=1}^N x_i X_i\right)(u)$. Define the metric function $d_\infty(x, u)$:

$$d_\infty(x, u) = \max_{i=1, \dots, N} \{|x_i|^{\frac{1}{\text{wgt } X_i}}\}.$$

Denote the ball $\{u : d_\infty(x, u) < r\}$ of radius r centered at x by $\text{Box}(x, r)$.

Note that Propositions 1–3 of [26] hold on weighted Carnot–Carathéodory spaces. It means that for each point $z \in \mathbb{M}$ there exists a neighborhood $\mathcal{U} \Subset \mathbb{M}$, $\mathcal{U} \ni z$, such that $d_\infty(v, w)$ and $d_\infty^u(v, w)$ are well-defined for all $u, v, w \in \mathcal{U}$.

The following assertion is true.

Theorem 2.3. (see [24, 25]) *Let \mathbb{M} be a weighted Carnot–Carathéodory space with $C^{1,\alpha}$ -smooth basis vector fields, $\alpha \in [0, 1]$ (if $\alpha = 0$ then the vector fields are just C^1 -smooth). For each $w \in \mathbb{M}$, there exists a neighborhood $\mathcal{O} \ni w$, $\mathcal{O} \Subset \mathbb{M}$, such that for $u, x \in \mathcal{O}$ the representations*

$$X_q(\Delta_\varepsilon^u x) = \sum_{p=1}^N a_{q,p}^u(\Delta_\varepsilon^u x) \widehat{X}_p^u(\Delta_\varepsilon^u x),$$

where

$$a_{q,p}^u(\Delta_\varepsilon^u x) = \begin{cases} O(\varepsilon^{l_1}), & \text{wgt } X_p < \text{wgt } X_q, \\ \delta_{pq} + O(\varepsilon^{l_1}), & \text{wgt } X_p = \text{wgt } X_q, \\ O(\varepsilon^{\min\{\alpha l_1, 1\} + \text{wgt } X_p - \text{wgt } X_q}), & \text{wgt } X_p > \text{wgt } X_q \text{ and } \alpha > 0, \\ o(\varepsilon^{\text{wgt } X_p - \text{wgt } X_q}), & \text{wgt } X_p > \text{wgt } X_q \text{ and } \alpha = 0 \end{cases} \quad (2.5)$$

hold, $q = 1, \dots, N$, and the above estimates are uniform on \mathcal{O} .

Its proof repeats the proof of Theorems 4 and 5 of [24] (see also [25]) almost verbatim taking into account the facts that $|\Delta_\varepsilon^u x - u| = O(\varepsilon^{l_1})$ and that the minimal degree of ε for $\Delta_\varepsilon^u x$ is l_1 instead of 1. More exactly, if $\alpha > 0$ then in the proof of the first statement of Theorem 4 of [24] (see also [25]) the estimates of the summands in the formula

$$\begin{aligned} & (C^u - C_{t\delta_\varepsilon x}) \left(\sum_{p=1}^N \widehat{y}_i^p(\delta_\varepsilon x, t) e_p, \sum_{q=1}^N \varepsilon^{\text{wgt } X_q} x_q e_q \right) \\ &= \sum_{p,q=1}^N \sum_{k: \text{wgt } X_k = \text{wgt } X_p + \text{wgt } X_q} \varepsilon^{\text{wgt } X_q} (c_{pqk}(u) - c_{pqk}(\theta_u(t\delta_\varepsilon x))) \widehat{y}_i^p(\delta_\varepsilon x, t) x_q e_k \\ & \quad - \sum_{p,q=1}^N \sum_{k: \text{wgt } X_k < \text{wgt } X_p + \text{wgt } X_q} \varepsilon^{\text{wgt } X_q} c_{pqk}(\theta_u(t\delta_\varepsilon x)) \widehat{y}_i^p(\delta_\varepsilon x, t) x_q e_k. \end{aligned}$$

are $O(\varepsilon^{l_1(1+\alpha)})$ and $O(\varepsilon^{l_1})$ instead of $O(\varepsilon^{1+\alpha})$ and $O(\varepsilon^1)$, respectively. In the proof of the second statement of Theorem 4 of [24] for $\alpha > 0$ the estimate of the first summand in the formula

$$\begin{aligned} (C^u - C_{tx}^\varepsilon)(e_p, e_q) &= \sum_{k: \text{wgt } X_k = \text{wgt } X_p + \text{wgt } X_q} (c_{pqk}(u) - c_{pqk}(\theta_u(t\delta_\varepsilon x))) e_k \\ & - \sum_{k: \text{wgt } X_k < \text{wgt } X_p + \text{wgt } X_q} \varepsilon^{\text{wgt } X_p + \text{wgt } X_q - \text{wgt } X_k} c_{pqk}(\theta_u(x)) e_k, \end{aligned}$$

is $O(\varepsilon^{\alpha l_1})$ instead of $O(\varepsilon^\alpha)$. Nevertheless, the estimate of the second summand is $O(\varepsilon^{\min\{k,p,q\}})$, and the rough estimate is $O(\varepsilon)$.

Arguments of the proof of Theorem 5 of [24] (see also [25]) are the same with obvious changes.

Remark 2.6. The proof of Theorem 2.3 uses results of [16] that are formulated for weighted Carnot–Carathéodory spaces where «degrees» of vector fields vary from 1 to N . It turns out that the restriction $\text{wgt } X_i \leq N$, $i = 1, \dots, N$, is not essential.

Indeed, let $l_1 < l_2 < \dots < l_N$ be weights of the basis vector fields of \mathbb{M} . Consider the Cartesian product $\mathbb{M} \times \mathbb{R}^{l_N - N}$ with the collection $\{\mathcal{X}_1, \dots, \mathcal{X}_N, e_{j_1}, \dots, e_{j_{l_N - N}}\}$, where

$$\mathcal{X}_i(x, y) = X_i(x), \quad x \in \mathbb{M}, y \in \mathbb{R}^{l_N - N}, \quad i = 1, \dots, N,$$

and assign to each e_{j_k} the weight being equal to k -th element of the ordered set

$$\{1, 2, 3, \dots, l_N - 1, l_N\} \setminus \{l_1, l_2, \dots, l_N\},$$

$k = 1, \dots, l_N - N$. Thus, we have an l_N -dimensional weighted Carnot–Carathéodory space enjoying all properties mentioned in [16]. Simple algebraic arguments show that Lemma 3.1 and Proposition 3.2 of [16], and Theorem 4 of [24] (see also [25]) hold for initial vector fields X_1, \dots, X_N .

Theorem 2.3 implies immediately Gromov type Convergence Theorem [17] in the coordinates of the 1st kind (recall that here dilatations are constructed with respect to weights).

Theorem 2.4. (see [16, 26] for «ordinary» Carnot–Carathéodory spaces) *Let \mathbb{M} be a weighted Carnot–Carathéodory space with C^1 -smooth basis vector fields. For each point, there exist a neighborhood $\mathcal{O} \subset \mathbb{M}$ containing it and a positive number $r > 0$ such that the uniform convergence $X_i^\varepsilon \rightarrow \widehat{X}_i^u$ as $\varepsilon \rightarrow 0$, $i = 1, \dots, N$, holds on $\text{Box}(u, r)$, $u \in \mathcal{O}$, and this convergence is uniform in $u \in \mathcal{O}$.*

To make the understanding of the paper easier, we formulate all the assumptions on a neighborhood $\mathcal{U} \Subset \mathbb{M}$.

Assumption 2.1. (see also [26]) To this end, we consider a compactly embedded neighborhood $\mathcal{U} \Subset \mathbb{M}$ such that

- 1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ for all $u \in \mathcal{U}$;
- 2) $\mathcal{U} \subset \mathcal{G}^u \mathbb{M}$ for all $u \in \mathcal{U}$;
- 3) $\widehat{\theta}_v^u(B_E(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$ (here one denotes $\widehat{\theta}_v^u(x_1, \dots, x_N) = \exp\left(\sum_{i=1}^N x_i \widehat{X}_i^u\right)(v)$ and $r_{u,v} = \sup\{r : \widehat{\theta}_v^u \text{ is a diffeomorphism on } B_E(0, r)\}$);

- 4) $\mathcal{U} \Subset \mathcal{O}$, where \mathcal{O} is a neighborhood from Theorem 2.3.
- 5) $\|D\theta_u\langle e_i \rangle\|, \|D\hat{\theta}_v^u\langle e_i \rangle\| \geq T > 0$ on \mathcal{U} for all $u, v \in \mathcal{U}$, $i = 1, \dots, N$.

Theorem 2.3 has following corollaries. The first one is the generalized triangle inequality.

Theorem 2.5. (see [23, 26] for «ordinary» case) *Let \mathbb{M} be a weighted Carnot–Carathéodory space with C^1 -smooth basis vector fields. Assume that $\mathcal{U} \Subset \mathbb{M}$ is a compactly embedded neighborhood small enough such that*

- 1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ for all $u \in \mathcal{U}$;
- 2) $\mathcal{U} \subset \mathcal{G}^u \mathbb{M}$ for all $u \in \mathcal{U}$;
- 3) $\hat{\theta}_v^u(B_E(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$;
- 4) $\mathcal{U} \Subset \mathcal{O}$, where \mathcal{O} is a neighborhood from Theorem 2.3.

The value d_∞ is a quasimetric; i.e., for $u, v, w \in \mathcal{U}$, the generalized triangle inequality

$$d_\infty(v, w) \leq c(d_\infty(v, u) + d_\infty(u, w))$$

holds, where the constant $0 < c < \infty$ depends only on \mathcal{U} .

Statements of the following Local Approximation Theorem for d_∞ -quasimetrics for the case of C^1 -smooth basis vector fields are the second corollary of Theorem 2.3. See Remark 3.1 for comments regarding the case of $\alpha > 0$.

Theorem 2.6. (see [23, 26] for «ordinary» case) *Let \mathbb{M} be a weighted Carnot–Carathéodory space with C^1 -smooth basis vector fields. Assume that $\mathcal{U} \Subset \mathbb{M}$ is a compactly embedded neighborhood small enough such that*

- 1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ for all $u \in \mathcal{U}$;
- 2) $\mathcal{U} \subset \mathcal{G}^u \mathbb{M}$ for all $u \in \mathcal{U}$;
- 3) $\hat{\theta}_v^u(B_E(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$;
- 4) $\mathcal{U} \Subset \mathcal{O}$, where \mathcal{O} is a neighborhood from Theorem 2.3.

Suppose that $\text{Box}(u, \varepsilon) \subset \mathcal{U}$. Then for any points $v, w \in \text{Box}(u, \varepsilon)$ the following relation is valid:

$$|d_\infty(v, w) - d_\infty^u(v, w)| = \begin{cases} O(\varepsilon^{1 + \frac{\min\{\alpha l_1, 1\}}{l_M}}), & \{X_i\}_{i=1}^N \in C^{1, \alpha}, \alpha > 0, \\ o(\varepsilon), & \{X_i\}_{i=1}^N \in C^1, \end{cases} \quad (2.6)$$

as $\varepsilon \rightarrow 0$, and if $u' \in \text{Box}(u, \varepsilon)$ then

$$|d_\infty^{u'}(v, w) - d_\infty^u(v, w)| = \begin{cases} O(\varepsilon^{1 + \frac{\min\{\alpha l_1, 1\}}{l_M}}), & \{X_i\}_{i=1}^N \in C^{1, \alpha}, \alpha > 0, \\ o(\varepsilon), & \{X_i\}_{i=1}^N \in C^1, \end{cases}$$

as $\varepsilon \rightarrow 0$.

Moreover, $o(1)$ and $O(1)$ are uniform in $u \in \mathcal{W}$, where $\mathcal{W} \Subset \mathcal{U}$, and in $v, w, u' \in \text{Box}(u, \varepsilon) \Subset \mathcal{U}$.

Proofs of these two theorems are similar to those in [26] with the new notions of weights.

3. Main result

The goal of this section is to prove the following geometric property of weighted Carnot — Carathéodory spaces.

Theorem 3.1. *Let \mathbb{M} be a weighted Carnot–Carathéodory space with $C^{1,\alpha}$ -smooth, $\alpha > 0$, or C^1 -smooth basis vector fields. Then for each point of \mathbb{M} , there exists a sufficiently small neighborhood $\mathcal{U} \subseteq \mathbb{M}$ such that*

- 1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ for all $u \in \mathcal{U}$;
- 2) $\mathcal{U} \subset \mathcal{G}^u \mathbb{M}$ for all $u \in \mathcal{U}$;
- 3) $\hat{\theta}_v^u(B_E(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$;
- 4) $\mathcal{U} \in \mathcal{O}$, where \mathcal{O} is a neighborhood from Theorem 2.3.

Moreover, this neighborhood \mathcal{U} possesses the following property: for $u, v \in \mathcal{U}$ and $w = \gamma(1)$ and $\hat{w} = \hat{\gamma}(1)$, where $\gamma, \hat{\gamma} : [0, 1] \rightarrow \mathbb{M}$ are absolutely continuous (in the classical sense) curves contained in $\text{Box}(u, \varepsilon)$ such that

$$\dot{\gamma}(t) = \sum_{i=1}^N b_i(t) X_i(\gamma(t)), \quad \gamma(0) = v, \quad \text{and} \quad \dot{\hat{\gamma}}(t) = \sum_{i=1}^N b_i(t) \hat{X}_i^u(\gamma(t)), \quad \hat{\gamma}(0) = v,$$

and each measurable function $b_i(t)$ meets the property

$$\int_0^1 |b_i(t)| dt < S \varepsilon^{\text{wgt } X_i}, \quad (3.1)$$

$S < \infty$, $i = 1, \dots, N$, we have

$$\max\{d_\infty(w, \hat{w}), d_\infty^u(w, \hat{w})\} = \begin{cases} O(\varepsilon^{1 + \frac{\min\{\alpha l_1, 1\}}{l_M}}), & \{X_i\}_{i=1}^N \in C^{1,\alpha}, \quad \alpha > 0, \\ o(\varepsilon), & \{X_i\}_{i=1}^N \in C^1, \end{cases}$$

with $O(1)$ and $o(1)$ to be uniform in $u \in \mathcal{U}$ and all collections of functions $\{b_i(t)\}_{i=1}^N$ with the property (3.1) as $\varepsilon \rightarrow 0$.

Proof. Consider the case of $C^{1,\alpha}$ -smooth fields, $\alpha > 0$. The arguments for the case of C^1 -smooth vector fields are similar.

Apply the normal coordinates θ_u^{-1} with respect to the point u to curves γ and $\hat{\gamma}$. To simplify notation, we set the field $D\theta_u^{-1}\langle X_i \rangle$ to be equal Y_i and denote $D\theta_u^{-1}\langle \hat{X}_i^u \rangle$ by \hat{Y}_i^u , $i = 1, \dots, N$. We also set $\gamma_u(t) = \theta_u^{-1}(\gamma(t))$ and $\hat{\gamma}_u(t) = \theta_u^{-1}(\hat{\gamma}(t))$. Let us rewrite the tangent vector to the curve γ_u at the point $\gamma_u(t)$ as

$$\begin{aligned} \sum_{i=1}^N b_i(t) Y_i(\gamma_u(t)) &= \\ &= \sum_{i=1}^N b_i(t) \hat{Y}_i^u(\gamma_u(t)) + \sum_{i=1}^N b_i(t) \left(\sum_{j=1}^N [a_{i,j}^u(\gamma_u(t)) - \delta_{ij}] \hat{Y}_j^u(\gamma_u(t)) \right) \end{aligned}$$

The tangent vector to the curve $\delta_u(t) = \widehat{\gamma}_u(1) + \gamma_u(t) - \widehat{\gamma}_u(t)$, which joins $\gamma_u(1)$ and $\widehat{\gamma}_u(1)$, can be written as

$$\begin{aligned} \dot{\delta}_u(t) &= \\ &= \sum_{i=1}^N b_i(t) [\widehat{Y}_i^u(\gamma_u(t)) - \widehat{Y}_i^u(\widehat{\gamma}_u(t))] + \sum_{i=1}^N b_i(t) \left(\sum_{j=1}^N [a_{i,j}^u(\gamma_u(t)) - \delta_{ij}] \widehat{Y}_j^u(\gamma_u(t)) \right) \\ &= \sum_{i=1}^N b_i(t) [\widehat{Y}_i^u(\gamma_u(t)) - \widehat{Y}_i^u(\widehat{\gamma}_u(t))] + \sum_{j=1}^N \left(\sum_{i=1}^N b_i(t) [a_{i,j}^u(\gamma_u(t)) - \delta_{ij}] \right) \widehat{Y}_j^u(\gamma_u(t)), \end{aligned}$$

where the coefficients $\{a_{i,j}^u\}_{i,j=1}^N$ coincide with those in (2.5). Taking into account the coordinate representations of the vector fields $\{\widehat{Y}_i^u\}_{i=1}^N$ (see Theorem 2.2) we obtain the following ODE system for $1 \leq k \leq \dim H_1$:

$$[\dot{\delta}_u]_k(t) = \sum_{j=1}^{\dim H_1} \sum_{i=1}^N b_i(t) [a_{i,j}^u(\gamma_u(t)) - \delta_{ij}] \delta_{kj}.$$

What has been said above and the estimate $|a_{i,j}^u(\gamma_u(t)) - \delta_{ij}| = O(\varepsilon^{l_1})$ for $j \leq \dim H_1$ (see Theorem 2.3) imply

$$|[\delta_u]_k(t) - [\delta_u]_k(0)| \leq \int_0^1 \sum_{j=1}^{\dim H_1} \sum_{i=1}^N |b_i(\tau)| |a_{i,j}^u(\gamma_u(\tau)) - \delta_{ij}| d\tau = O(\varepsilon^{2l_1})$$

$t \in [0, 1]$. Next, for $\dim H_1 < k \leq \dim H_2$ we have

$$\begin{aligned} [\dot{\delta}_u]_k(t) &= \sum_{i=1}^{\dim H_1} \sum_{|\mu|_h=l_2-l_1} b_i(t) F_{\mu, e_i}^k(u) [\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu] \\ &+ \sum_{j=1}^{\dim H_1} \left(\sum_{i=1}^N b_i(t) [a_{i,j}^u(\gamma_u(t)) - \delta_{ij}] \right) \sum_{|\mu|_h=l_2-l_1} F_{\mu, e_j}^k(u) \gamma_u(t)^\mu \\ &+ \sum_{j=\dim H_1+1}^{\dim H_2} \sum_{i=1}^N b_i(t) [a_{i,j}^u(\gamma_u(t)) - \delta_{ij}] \delta_{kj}. \end{aligned}$$

Note that in the first sum the multi-index μ may consist of more than one element. Numbers of all these elements do not exceed $\dim H_1$, thus, we can use the previous estimate: if $\mu = (\mu_1, \dots, \mu_s, 0, \dots, 0)$ then

$$\begin{aligned} |\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu| &= \left| \prod_{k \leq \dim H_1} (\widehat{\gamma}_u(t)_k + S_k \varepsilon^{2l_1})^{\mu_k} - \prod_{k \leq \dim H_1} (\widehat{\gamma}_u(t)_k)^{\mu_k} \right| \\ &\leq S_0 \varepsilon^{2l_1 + \left(\sum_{k=1}^{\dim H_1} \mu_k - 1 \right) \cdot l_1} = S_0 \varepsilon^{l_2 - l_1 + l_1} = S_0 \varepsilon^{l_2}. \end{aligned}$$

Next, taking into account the facts that $\delta_u(t) - \delta_u(0) = \gamma_u(t) - \widehat{\gamma}_u(t)$ and that, by Theorem 2.3, in the last sum we have $|a_{i,j}^u(\gamma_u(t))| = O(\varepsilon^{l_2 - l_1 + \min\{\alpha l_1, 1\}})$ for $\text{wgt } X_i = 1$ and $|a_{i,j}^u(\gamma_u(t)) - \delta_{ij}| = O(\varepsilon^{l_1})$ for $\text{wgt } X_i \geq 2$, similarly to previous cases we have $O(\varepsilon^{l_1})$ in the middle sum, and finally we obtain

$$|[\delta_u]_k(t) - [\delta_u]_k(0)| = O(\varepsilon^{\min\{\alpha l_1, 1\} + l_2}), \quad t \in [0, 1].$$

Arguing by induction, we suppose that we have proved the inequality

$$|[\delta_u]_k(t) - [\delta_u]_k(0)| = O(\varepsilon^{\min\{\alpha l_1, 1\} + l_r}),$$

for $\dim H_{r-1} < k \leq \dim H_r$, $r = 3, \dots, Q-1$, $t \in [0, 1]$. Then, using this assumption and the preceding estimates, we obtain the following relation for $\dim H_{Q-1} < k \leq \dim H_Q$:

$$\begin{aligned} [\dot{\delta}_u]_k(t) &= \sum_{i=1}^{\dim H_{Q-1}} \sum_{|\mu|_h = l_Q - \text{wgt } X_i} b_i(t) F_{\mu, e_i}^k(u) [\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu] \\ &+ \sum_{j=1}^{\dim H_{Q-1}} \left(\sum_{i=1}^N b_i(t) [a_{i,j}^u(\gamma_u(t)) - \delta_{ij}] \right) \sum_{|\mu|_h = l_Q - \text{wgt } X_j} F_{\mu, e_j}^k(u) \gamma_u(t)^\mu \\ &+ \sum_{j=\dim H_{Q-1}+1}^{\dim H_Q} \sum_{i=1}^N b_i(t) [a_{i,j}^u(\gamma_u(t)) - \delta_{ij}]. \quad (3.2) \end{aligned}$$

From our assumption and by the standard arguments (see, e.g. [26]) it follows that in the first sum for all μ with $|\mu|_h = l_Q - \text{deg } X_i$ we have

$$|\gamma_u(t)^\mu - \widehat{\gamma}_u(t)^\mu| = O(\varepsilon^{\min\{\alpha l_1, 1\} + |\mu|_h}) = O(\varepsilon^{\min\{\alpha l_1, 1\} + l_Q - \text{wgt } X_i}).$$

Next, in the last sum in (3.2) we have

$$|a_{i,j}(\gamma_u(t)) - \delta_{ij}| = O(\varepsilon^{l_Q - \text{wgt } X_i + \min\{\alpha l_1, 1\}})$$

for $\text{wgt } X_i < l_Q$. If $\text{wgt } X_i \geq l_Q$ then $|a_{i,j}(\gamma_u(t)) - \delta_{ij}| = O(\varepsilon^{l_1})$. Finally, in the middle sum in (3.2) for $\text{wgt } X_j > \text{wgt } X_i$ we obtain

$$|a_{i,j}(\gamma_u(t)) \gamma_u(t)^\mu| = O(\varepsilon^{l_Q - \text{wgt } X_i + \min\{\alpha l_1, 1\}}).$$

For $\text{wgt } X_j \leq \text{wgt } X_i$ we have $|[a_{i,j}(\gamma_u(t)) - \delta_{ij}] \gamma_u(t)^\mu| = O(\varepsilon^{l_Q - \text{wgt } X_j + l_1})$. This implies the following estimate for $|[\delta_u]_k(t) - [\delta_u]_k(0)|$:

$$\begin{aligned}
|[\delta_u]_k(t) - [\delta_u]_k(0)| &= O(\varepsilon^{l_Q - \text{wgt } X_i + \min\{\alpha l_1, 1\}}) \sum_{i=1}^{\dim H_{Q-1}} \int_0^1 |b_i(\tau)| d\tau \\
&+ \sum_{j=1}^{\dim H_{Q-1}} O(\varepsilon^{l_Q - \text{wgt } X_j + l_1}) \sum_{i: \text{wgt } X_i \geq \text{wgt } X_j} \int_0^1 |b_i(\tau)| d\tau \\
&+ O(\varepsilon^{l_Q - \text{wgt } X_i + \min\{\alpha l_1, 1\}}) \sum_{i: \text{wgt } X_i < \text{wgt } X_j} \int_0^1 |b_i(\tau)| d\tau \\
&+ O(\varepsilon^{l_Q - \text{wgt } X_i + \min\{\alpha l_1, 1\}}) \sum_{i: \text{wgt } X_i < l_Q} \int_0^1 |b_i(\tau)| d\tau \\
&+ O(\varepsilon^{l_1}) \sum_{i: \text{wgt } X_i \geq l_Q} \int_0^1 |b_i(\tau)| d\tau = O(\varepsilon^{l_Q + \min\{\alpha l_1, 1\}}).
\end{aligned}$$

Let us estimate $d_\infty^u(\gamma_u(1), \widehat{\gamma}_u(1))$. Recall that $\delta_u(t) = \widehat{\gamma}_u(1) + \gamma_u(t) - \widehat{\gamma}_u(t)$, $[\delta_u(t)]_k - [\delta_u(0)]_k = [\gamma_u(t)]_k - [\widehat{\gamma}_u(t)]_k$, $k = 1, \dots, N$, and the coordinates of $\{[\gamma_u(1)]_k\}_{k=1}^N$ and $\{[\widehat{\gamma}_u(1)]_l\}_{l=1}^N$ with respect to zero in the system $\{\widehat{Y}_i^u\}$ coincide with Cartesian ones. To obtain our estimate, we apply the group operation in $\mathbb{G}_u\mathbb{M}$: if $\gamma_u(1) = \exp\left(\sum_{i=1}^N w_i \widehat{Y}_i^u\right)(\widehat{\gamma}_u(1))$ then

$$\begin{aligned}
w_i &= [\delta_u(1)]_i - [\delta_u(0)]_i \\
+ \sum_{|\mu + \beta + e_k + e_l|_h = \text{wgt } X_i} G_{\mu, \beta, e_k, e_l}^i(u) \gamma_u(1)^\mu \widehat{\gamma}_u(1)^\beta ([\gamma_u(1)]_k [\widehat{\gamma}_u(1)]_l - [\gamma_u(1)]_l [\widehat{\gamma}_u(1)]_k).
\end{aligned}$$

We have $|[\delta_u(1)]_i - [\delta_u(0)]_i| = O(\varepsilon^{\text{wgt } X_i + \min\{\alpha l_1, 1\}})$ and

$$\begin{aligned}
&|[\gamma_u(1)]_k [\widehat{\gamma}_u(1)]_l - [\gamma_u(1)]_l [\widehat{\gamma}_u(1)]_k| \\
&\leq |[\gamma_u(1)]_k| \cdot |[\widehat{\gamma}_u(1)]_l - [\gamma_u(1)]_l| + |[\gamma_u(1)]_l| \cdot |[\gamma_u(1)]_k - [\widehat{\gamma}_u(1)]_k|.
\end{aligned}$$

Therefore,

$$|w_i| = O(\varepsilon^{\text{wgt } X_i + \alpha}) + O(\varepsilon^{|\mu + \beta|_h + \text{wgt } X_k + \text{wgt } X_l + \min\{\alpha l_1, 1\}}) = O(\varepsilon^{\text{wgt } X_i + \min\{\alpha l_1, 1\}}).$$

This implies $d_\infty^u(\gamma_u(1), \widehat{\gamma}_u(1)) = O(\varepsilon^{1 + \frac{\min\{\alpha l_1, 1\}}{l_M}})$. Hence, the same estimate holds for $w = \gamma(1)$ and $\widehat{w} = \widehat{\gamma}(1)$. Next, we apply the arguments from Remark 3.1 (they are similar to those in the remark after [25, Theorem 9]) and deduce

$$d_\infty(\gamma(1), \widehat{\gamma}(1)) = O(\varepsilon^{1 + \frac{\min\{\alpha l_1, 1\}}{l_M}}).$$

Since all the coefficients at $O(1)$ are uniform on \mathcal{U} , then so do $O(1)$ in both estimates. The theorem follows. \square

Corollary 3.1. *Let \mathbb{M} be a weighted Carnot–Carathéodory space with $C^{1,\alpha}$ -smooth, $\alpha > 0$, or C^1 -smooth basis vector fields. For each point of \mathbb{M} , there exists a sufficiently small neighborhood $\mathcal{U} \Subset \mathbb{M}$ such that*

- 1) $\theta_u(B_E(0, r_u)) \supset \mathcal{U}$ for all $u \in \mathcal{U}$;
- 2) $\mathcal{U} \subset \mathcal{G}^u \mathbb{M}$ for all $u \in \mathcal{U}$;
- 3) $\widehat{\theta}_v^u(B_E(0, r_{u,v})) \supset \mathcal{U}$ for all $u, v \in \mathcal{U}$;
- 4) $\mathcal{U} \Subset \mathcal{O}$, where \mathcal{O} is a neighborhood from Theorem 2.3.

Moreover, for $u, u', v \in \mathcal{U}$ such that $d_\infty(u, u') \leq C\varepsilon$ for some $0 < C < \infty$, and points $w = \gamma(1)$ and $\widehat{w} = \widehat{\gamma}(1)$, where $\gamma, \widehat{\gamma} : [0, 1] \rightarrow \mathbb{M}$ are absolutely continuous (in the classical sense) curves lying in $\text{Box}(u, \varepsilon)$ such that

$$\dot{\gamma}(t) = \sum_{i=1}^N b_i(t) \widehat{X}_i^{u'}(\gamma(t)), \quad \gamma(0) = v, \quad \text{and} \quad \dot{\widehat{\gamma}}(t) = \sum_{i=1}^N b_i(t) \widehat{X}_i^u(\gamma(t)), \quad \widehat{\gamma}(0) = v,$$

and (3.1) holds, we have

$$\max\{d_\infty^{u'}(w, \widehat{w}), d_\infty^u(w, \widehat{w})\} = \begin{cases} O(\varepsilon^{1 + \frac{\min\{\alpha l_1, 1\}}{l_M}}), & \{X_i\}_{i=1}^N \in C^{1,\alpha}, \alpha > 0, \\ o(\varepsilon), & \{X_i\}_{i=1}^N \in C^1, \end{cases}$$

with $O(1)$ and $o(1)$ to be uniform in $u \in \mathcal{U}$ and all collections of functions $\{b_i(t)\}_{i=1}^N$ with the property (3.1) as $\varepsilon \rightarrow 0$.

Remark 3.1. Note that statements of Theorem 2.6 follow directly from Theorem 2.3 only for C^1 -smooth vector fields. If the basis vector fields belong to the class $C^{1,\alpha}$, $\alpha > 0$, then the scheme is the following. First of all, we obtain the estimate of d_∞^u in Theorem 3.1, and the same estimate for the particular case of constant functions b_i , $i = 1, \dots, N$. After that we apply the scheme of proof of Local Approximation Theorem from [23] and derive all estimates in Theorem 2.6. Finally, we use it for the estimate of d_∞ from Theorem 3.1.

Remark 3.2. The following sharper result is true. Let \mathbb{M} be a weighted Carnot–Carathéodory space with $C^{1,\alpha}$ -smooth basis vector fields, $\alpha \in (0, 1]$. For each $w \in \mathbb{M}$, there exists a neighborhood $\mathcal{O} \ni w$, $\mathcal{O} \Subset \mathbb{M}$, such that for $u, x \in \mathcal{O}$ the representations

$$X_q(\Delta_\varepsilon^u x) = \sum_{p=1}^N a_{q,p}^u(\Delta_\varepsilon^u x) \widehat{X}_p^u(\Delta_\varepsilon^u x),$$

where

$$a_{q,p}^u(\Delta_\varepsilon^u x) = \begin{cases} O(\varepsilon^{l_1}), & \text{wgt } X_p < \text{wgt } X_q, \\ \delta_{pq} + O(\varepsilon^{l_1}), & \text{wgt } X_p = \text{wgt } X_q, \\ O(\varepsilon^{\max\{l_1, \min\{\alpha l_1, 1\} + \text{wgt } X_p - \text{wgt } X_q\}}), & \text{wgt } X_p > \text{wgt } X_q \end{cases}$$

hold, $q = 1, \dots, N$, and the above estimates are uniform on \mathcal{O} .

This assertion is convenient for the cases where l_1 is much bigger than $l_M - l_1$.

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