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Schottky theorem for algebroid functions

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Dedicated to Professor Cabiria Andreian Cazacu on her 85th Birthday

Abstract - In this article the Jensen-Poisson Formula is extended to algebroid functions. Some important consequences, analogous to the case of meromorphic functions in the complex plane, can be derived from this general version. In particular Schottky theorem for algebroid functions of order n, defined in a finite disc is obtained following the ideas of Yang Lo [8] , making use of the value distribution theory for algebroid functions developped by H.LSelberg.

Key words and phrases : Algebroid function, valued distribution, discriminant.

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1. Introduction

In this work we shall obtain the extension of Schottky Theorem for analytic functions in the plane to the wider class of holomorphic algebroid functions as a consequence of Selberg Main Theorem of the value distribution theory of algebroid function following the lines of Yang-Lo [8] in the plane. In order to do this we shall need previously some basic estimates for the maximum modulus of the function which, in the plane case, derive from Jensen-Poisson Formula. We shall show the validity of Jensen-Poisson Formula in the frame of algebroid functions. Schottky Theorem is a key step in the classical proof of the Great Picard Theorem for meromorphic functions in $\mathbb{C}$, so that its extension to algebroid functions should allow to obtain the corresponding result in this wider setting.

2. Algebroid functions. Notation and basic facts

An algebroid function $w = w(z)$ of order n is a mutivalued function which we shall assume to defined in the plane or in a finite disc $D(0, R)$ by an equation of the form

$$F(w, z) = A_n(z) w^n + A_{n-1}(z) w^{n-1} + ... + A_0(z) = 0 \ , \qquad (2.1)$$

where $A_0(z), .., A_n(z)$ are meromorphic functions in $D(0,R)$, $0 < R \leq \infty$, with no common zeros or poles and $F(w,z)$ is irreducible, that is, it cannot be decomposed as a product

$$F(w,z) = F_1(w,z) \cdot F_2(w,z),$$

where $F_1(w,z), F_2(w,z)$ are two non-constant functions of the same type as $F(w,z)$.

In these conditions (2.1) defines a $n-$valued meromorphic function $w(z)$ outside the critical points where $A_n(z)$ has a zero or one of the $A_i(z)$, $i = 1,..,n$, has a pole and also outside the set of points where the so-called discriminant of $w(z)$ vanishes, where two or more branches $w_i(z)$ of $w(z)$ ar equal. The algebroid functions are a natural extension of meromorphic functions which can be considered are algebroid functions of order one.

Next we present a description of algebroid functions in terms of holomorphic mappings on coverings of $\mathbb{C}$.

Given an algebroid function $w(z)$ of order n satisfying the equation $F(w,z) = 0$, we can associate to this function the Riemann surface $\mathcal{X}_F$

$$\mathcal{X}_F = \{(z, w_i(z)) \,|z \in \mathbb{C} \ , \ i = 1,..,n\},$$

where the $w_i\ 's$ are the branches of the algebroid function.

On $\mathcal{X}_F$ we can define the functions

$$\begin{array}{cccc} P & : & \mathcal{X}_F & \rightarrow & \mathbb{C} \\ & & (z, w_i(z)) & \rightarrow & z \end{array} ,$$

which is known as the canonical projection and also

$$\begin{array}{cccc} F & : & \mathcal{X}_F & \rightarrow & \widehat{\mathbb{C}} \\ & & (z, w_i(z)) & \rightarrow & w_i(z) \end{array} ,$$

which is a uniform holomorphic map on $\mathcal{X}_F$ and which we denote with the same symbol F as in the equation defining the algebroid function.

Next, we introduce some subsets of $\mathcal{X}_F$ which will play the role of the discs in the plane.

We denote by $\mathcal{X}_F[r]$ the subset of $\mathcal{X}_F$

$$\mathcal{X}_F[r] = \{(z,w) \in ||P(z,w)| \leq r\},$$

similarmente

$$\mathcal{X}_F(r) = \{(z,w) \in ||P(z,w)| < r\},$$

and

$$\mathcal{X}_F\langle r\rangle = \{(z,w) \in ||P(z,w)| = r\}.$$

We can also define the angle measure form $\sigma_{\mathcal{X}_F}$ and the area form $dA_{\mathcal{X}_F}$ by pulling back by P the corresponding normalized measures in $\mathbb{C}$, that is

$$\sigma_{\mathcal{X}_F} = P^* \left(\frac{d\theta}{2\pi} \right) \ , \ dA_{\mathcal{X}_F} = P^* \left(dA \right) \ ,$$

where dA is th area Lebesgue measure in the plane.

3. Value distribution theory of algebroid functions. The theory of H.L. Selberg

3.1. Notation and the First Main Theorem

The analogous functions to those of Nevanlinna are defined in this context as follows

$$\begin{aligned} m\left(r,a,w\right) &= \frac{1}{n} \int\limits_{\mathcal{X}_F\langle r\rangle} \log^+ \left| \frac{1}{w\left(re^{i\theta}\right) - a} \right| \sigma_{\mathcal{X}_F}, \\ m\left(r,\infty,w\right) &= \frac{1}{n} \int\limits_{\mathcal{X}_F\langle r\rangle} \log^+ \left| w\left(re^{i\theta}\right) \right| \sigma_{\mathcal{X}_F}, \end{aligned}$$

this is the proximity function, as for the counting function we call

$$n\left(r,a,w\right) = n^o \text{ of roots of } w\left(z\right) = a \text{ in } \mathcal{X}_F\left[r\right],$$

and the average counting function

$$N\left(r,a,w\right) = \frac{1}{n} \int_0^r \frac{n\left(t,a,w\right) - n\left(0,a,w\right)}{t} dt + \frac{n\left(0,a,w\right)}{n} \log r,$$

and the characteristic function is defined by

$$T\left(r,w\right) = m\left(r,\infty,w\right) + N\left(r,\infty,w\right).$$

Finally, we define the ramification terms

$$N_{Ram}\left(r,P\right), \ N_{Ram}\left(r,w\right),$$

the first one measures the ramifications corresponding to the surface $\mathcal{X}_F$, the second one measures the multiple values of the function $w\left(z\right)$.

In order to define these two terms, we consider the power expansion of one of the branches $w_i\left(z\right)$ near a point $z = z_0$

$$w_i\left(z\right) = w_i\left(z_0\right) + \gamma_\tau \left(z - z_0\right)^{\frac{\tau}{\lambda}} + \ldots \ , \quad \gamma_\tau \neq 0,$$

if $w_i\left(z\right)$ has not a pole at $z = z_0$ and

$$w_i\left(z\right) = \gamma_{-\tau} \left(z - z_0\right)^{-\frac{\tau}{\lambda}} + \ldots \ , \quad \gamma_{-\tau} \neq 0,$$

in the case that $w_i(z)$ has a pole at $z = z_0$.

Then we set

$$\begin{aligned} n_{Ram}(r,P) &= \sum_{\mathcal{X}_F[r]} (\lambda - 1)\,, \\ n_{Ram}(r,w) &= \sum_{\mathcal{X}_F[r]} (\tau - 1)\,, \end{aligned}$$

where in both sums, we are taking into account the numbers λ, τ corresponding to all the branches $w_i(z)$, $i = 1, 2, .., n$.

Now, we can already introduce the ramification terms as the average functions

$$\begin{aligned} N_{Ram}(r,P) &= \int_0^r \frac{n_{Ram}(t,P)}{t} dt, \\ N_{Ram}(r,w) &= \int_0^r \frac{n_{Ram}(t,w)}{t} dt, \end{aligned}$$

and it can be checked the following relation

$$N_{Ram}(r,w) - N_{Ram}(r,P) = 2N(r,w) + N\left(r, \frac{1}{w'}\right) - N\left(r, w'\right) \ . \quad (3.1)$$

Now we state the First Main Theorem of the value distribution theory for algebroid functions due to H.L.Selberg

Theorem 3.1. First Main Theorem. *Given al algebroid function $w(z)$ of order n , then for every $a \in \widehat{\mathbb{C}}$, it holds*

$$m(r,a) + N(r,a) = T(r,w) + \alpha(r,a) \ ,$$

where $\alpha(r,a)$ is a bounded term as $r \to \infty$ and we have the following estimate

$$\alpha(r,a) \le \log^+|a| + \log 2 + \frac{1}{k}\left|\sum_{\nu=1}^{k} \lambda_\nu \log|c_\nu|\right| \ ,$$

where $c_1, c_2, ..., c_\nu$ *are the first non-vanishing coefficients of the different power expansions at $z = 0$ of the different branches $w_i(z)$ and $\lambda_1, \lambda_2, ..., \lambda_\nu$ the corresponding $\lambda's$ as described above.*

3.2. The Second Main Theorem

Next we present a limited version of the Second Main Theorem where we shall consider the particular values $\infty, 0, 1, 2, ..., 2n$ though in the general statement we can take $q \ge 2n+1$ arbitrary values in $\widehat{\mathbb{C}}$. The proof is essentially due to Yang-Lo [8], and we keep track of all th constants involved.

Theorem 3.2. Second Main Theorem. *Let $w(z)$ be an algebroid function of order n. If $w_i(0) \neq 0, 1, 2, ..., 2n, \infty$ and $w_i'(0) \neq 0$ for every $i = 1, 2, .., n$, then we get for every r*

$$2nT(r,w) \leq N(r,w) + \sum_{k=0}^{2n-1} N\left(r, \frac{1}{w-k}\right) - N_{Ram}(r,P) + N_{Ram}(r,w) + S(r,w), \tag{3.2}$$

where

$$S(r,w) = m\left(r, \frac{w'}{w}\right) + \sum_{k=1}^{2n} m\left(r, \frac{w'}{w-i}\right) + C, \tag{3.3}$$

where $C = C(n, w_i(0), w_i'(0))$, $i = 1, .., n$.

3.3. The Selberg ramification theorem

The term $N_{Ram}(r,w)$ is estimated in terms of $T(r,w)$ by means of the Selberg Ramification Theorem. We make the assumption that the so-called discriminant D of $w(z)$, namely

$$D(z) = \prod_{i \neq j} (w_i(z) - w_j(z)),$$

does not vanish at zero, otherwise some further modifications would be required.

It can be proved, see E.Ullrich [7], H.L.Selberg [6] that

$$N_{Ram}(r,w) \leq \frac{1}{n} N\left(r, \frac{1}{D}\right), \tag{3.4}$$

and from the First Main Theorem we obtain

$$N\left(r, \frac{1}{D}\right) \leq T(r,D) + \frac{1}{n} \log^+ |D(0)|. \tag{3.5}$$

On the other hand

$$\frac{1}{n} N(r,D) \leq 2(n-1) N(r,w), \tag{3.6}$$

and also

$$\frac{1}{n} m(r,D) \leq 2(n-1) m(r,w) + (n-1)\log 2, \tag{3.7}$$

so that we conclude from $(3.5), (3.6)$ and (3.7)

$$\frac{1}{n} N\left(r, \frac{1}{D}\right) \leq 2(n-1) T(r,w) + (n-1)\log 2,$$

this together with (3.4) yields the

Theorem 3.3. Selberg Ramification Theorem. *Let $w(z)$ be an algebroid function of order n, then under the assumption $D(0) \neq 0$, we have the estimate*

$$N_{Ram}(r,w) \leq 2(n-1)\,T(r,w) + (n-1)\log 2 + \frac{1}{n}\log^{+}|D(0)|.$$

Selberg Ramification Theorem together with the Fundamental Inequality (3.2) yield

$$T(r,w) \leq N(r,w) + \sum_{k=0}^{2n-1} N\left(r, \frac{1}{w-k}\right) + S(r,w), \tag{3.8}$$

bearing in mind that $N_{Ram}(r,P)$ is a positive term and where now

$$S(r,w) = m\left(r, \frac{w'}{w}\right) + \sum_{k=1}^{2n} m\left(r, \frac{w'}{w-k}\right) + C\left(n, w_i(0), w_i'(0)\right) + \frac{1}{n}\log^{+}|D(0)|. \tag{3.9}$$

4. The Jensen-Poisson formula for algebroid functions

In this section we shall prove the Jensen-Poisson Formula fo algebroid functions.

Theorem 4.1. Jensen-Poisson Formula for algebroid functions. *Let $w(z)$ be an algebroid function of order n and let a_μ, $\mu = 1,..,M$, b_ν, $\nu = 1,..,N$ be the zeros and poles of $w(z)$ in $\mathcal{X}_F[R]$. Then, for $z = re^{i\theta}$, $0 \leq |z| < r < R$, such that $w_i(z) \neq 0, \infty$, for every $i = 1,..,n$, it holds*

$$\begin{aligned} \log \prod_{i=1}^{n} |w_i(z)| &= \frac{1}{2\pi} \int_{\mathcal{X}_F\langle R\rangle} \log w\left(\mathrm{Re}^{i\varphi}\right) \frac{R^2 - r^2}{R^2 + r^2 - 2Rr\cos(\varphi - \theta)} d\varphi \\ &\quad + \sum_{\mu=1}^{M} \log\left|\frac{R(z-a_\mu)}{R^2 - \overline{a_\mu}z}\right| - \sum_{\nu=1}^{N} \log\left|\frac{R(z-b_\nu)}{R^2 - \overline{b_\nu}z}\right|, \end{aligned}$$

with modifications if $w_i(z) = 0$ or ∞ for some $i = 1,..,n$.

Proof. For a fixed $z = re^{i\theta}$, which we can assume not to be a branch point, we shall map the disc $D(\zeta, R)$ onto the unit disc $D(0,1)$ by means of

$$\lambda = \frac{R(\zeta - z)}{R^2 - \overline{z}\zeta},$$

which sends $\zeta = z$ to $\lambda = 0$, and consider the algebroid function $w(\lambda) = w(\zeta(\lambda))$, which is given by the equation

$$F_\lambda(w,\lambda) = F(w, \zeta(\lambda)) = 0.$$

For this algebroid function, we can consider the associated Riemann surface $\mathcal{X}_{F_\lambda}$ in such a way that there is a biholomorphic fiber preserving map

$$\mathcal{X}_{F_\lambda}[1] \to \mathcal{X}_F[R].$$

We make initially the further assumption that neither the zeros nor the poles are branch points. Otherwise we might consider for $\epsilon > 0$, the modified function $w_\epsilon(z)$ given by the equation

$$F_\epsilon(w,z) = F(w, z+\epsilon),$$

so that for $\epsilon' s$ sufficiently close to zero our late assumption is fulfilled and then taking into account the continuity of all the terms in Jensen-Poisson formula, we let $\epsilon \to 0$.

Under these assumptions we cut the surface $\mathcal{X}_{F_\lambda}[1]$ along a ray c_j from every branch point of $\mathcal{X}_{F_\lambda}[1]$ up to the boundary of $\mathcal{X}_{F_\lambda}[1]$, taking care that these rays c_j do not go through any of the points $\lambda(a_n), \lambda(b_n)$ and that two of these rays do not intersect what clearly can be done since there are only a finite number of branch points, zeros and poles in $\mathcal{X}_F[R]$, and as a consequence in $\mathcal{X}_{F_\lambda}[1]$.

We obtain in this way a decomposition of $\mathcal{X}_{F_\lambda}[1]$ in a set of n disjoint simply connected regions $\mathcal{R}_i^\lambda$, all of them projecting biholomorphically onto $\overline{D}(0,1)$, excluding the projections of the slits c_j, which go from the projections of the branch points up to the boundary $C(0,1)$ of $\overline{D}(0,1)$.

In each of these regions $\mathcal{R}_i^\lambda$, $i = 1,..,n$, it is defined a uniform branch $w_i(\lambda)$ and these regions correspond through $\zeta(\lambda)$ biholomorphically to certain simply connected regions $\mathcal{R}_i^\zeta$, $i = 1,..,n$ in $\mathcal{X}_F[R]$, which project bilholomorphically onto $\overline{D}(0,R)$, excluding some slits, namely the projections of the images γ_j by $\zeta(\lambda)$ of the above described slits c_j in $\mathcal{X}_{F_\lambda}[1]$, these projections will be simple arcs going from the interior of $D(0,R)$ to $C(0,R)$, and in the regions $\mathcal{R}_i^\zeta$ are defined uniform branches $w_i(\zeta)$ of $w(\zeta)$.

Moreover the projection of each of these regions contains an entire disc $D(0,\delta)$.

Let us assume initially that the function $w(\zeta(\lambda))$ has no zeros nor poles in one of these regions $\mathcal{R}_i^\lambda$ of $\mathcal{X}_{F_\lambda}[1]$, and make another cross-cut C going from Γ_δ^i, the curve in $\mathcal{R}_i^\lambda$ projecting over $C(0,\delta)$, up to Γ_1^i, the part of the boundary $\partial\mathcal{R}_i^\lambda$ projecting over $C(0,1)$.

Now we obtain a new simply connected region $\mathcal{R}_i^{\lambda,C}$, where the function

$$\frac{\log w(\lambda)}{\lambda}$$

is analytic, so that if the boundary $\partial\mathcal{R}_i^\lambda$ is oriented positively, we conclude

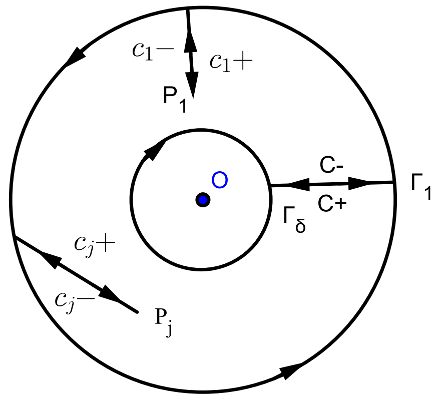

Figure 1.

from Cauchy Theorem

$$\begin{aligned}\int_{\partial\mathcal{R}_i^\lambda} \log w_i(\lambda)\frac{d\lambda}{\lambda} &= \int_{C(0,1)} \log w_i(\lambda)\frac{d\lambda}{\lambda} - \int_{C(0,\delta)} \log w_i(\lambda)\frac{d\lambda}{\lambda} \\ +\sum_j\left(\int_{c_j^+}-\int_{c_j^-}\right)+\int_{C^+}-\int_{C^-} &= 0. \end{aligned} \tag{4.1}$$

But, we clearly have

$$\int_{c_j^+}-\int_{c_j^-}=0 \text{ for every } j$$

and

$$\int_{C^+}-\int_{C^-}=0,$$

so that we conclude (4.1)

$$\begin{aligned}\int_{C(0,\delta)} \log w_i(\zeta(\lambda))\frac{d\lambda}{\lambda} &= \int_{C(0,\delta)} \log w_i(\lambda)\frac{d\lambda}{\lambda} = \int_{C(0,1)} \log w_i(\lambda)\frac{d\lambda}{\lambda} \\ &= \int_{C(0,1)} \log w_i(\zeta(\lambda))\frac{d\lambda}{\lambda},\end{aligned}$$

whence from the Residues Theorem

$$\begin{aligned}\log w_i(z) &= \log w_i(\zeta(0)) = \int_{C(0,\delta)} \log w_i(\zeta(\lambda))\frac{d\lambda}{\lambda} \\ &= \int_{C(0,1)} \log w_i(\zeta(\lambda))\frac{d\lambda}{\lambda} \\ &= \int_{C(0,R)} \log w_i(\zeta)\frac{\left(R^2-|z|^2\right)d\zeta}{\left(R^2-\overline{z}\zeta\right)(\zeta-z)} \\ &= \frac{1}{2\pi}\int_0^{\pi} \log w_i\left(\mathrm{Re}^{i\varphi}\right)\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}d\varphi.\end{aligned}$$

In the case where $w_i(z)$ has zeros and poles in the region $\mathcal{R}_i^{\zeta}$, say $a_1^i,..,a_{M_i}^i$, $b_1^i,..,b_{N_i}^i$, we consider the auxiliary meromorphic function in $\mathcal{X}_F$

$$\psi(\zeta) = w(\zeta)\frac{\prod_{\nu=1}^{N_i}\frac{R\left(\zeta-b_\nu^i\right)}{R^2-\overline{b_\nu^i}\zeta}}{\prod_{\mu=1}^{M_i}\frac{R\left(\zeta-a_\mu^i\right)}{R^2-\overline{a_\mu^i}\zeta}},$$

which has, as $w(\zeta)$, uniform branches in the simply connected regions $\mathcal{R}_i^{\zeta}$, $i=1,,..,n$.

In particular, in the region $\mathcal{R}_i^{\zeta}$ the function $\psi(\zeta)$ has no zeros nor poles and since for the corresponding branches holds $|\psi_i(\zeta)| = |w_i(\zeta)|$ on $|\zeta| = R$, we conclude

$$\begin{aligned}\log|w_i(z)| &= \frac{1}{2\pi}\int_0^{2\pi}\log w_i\left(\mathrm{Re}^{i\varphi}\right)\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}d\varphi \\ &\quad +\sum_{\mu=1}^{M_i}\log\left|\frac{R\left(z-a_\mu^i\right)}{R^2-\overline{a_\mu^i}z}\right| - \sum_{\nu=1}^{N_i}\log\left|\frac{R\left(z-b_\nu^i\right)}{R^2-\overline{b_\nu^i}z}\right|. \qquad (4.2)\end{aligned}$$

Finally, summing over all the branches we conclude the Jensen-Poisson Formula for algebroid functions, where at the left hand side we take into account all the zeros a_μ and all the poles b_ν in $\mathcal{X}_F[R]$.

$$\begin{aligned}\log\prod_{i=1}^{n}|w_i(z)| &= \frac{1}{2\pi}\int_{\mathcal{X}_F\langle R\rangle}\log w_i\left(\mathrm{Re}^{i\varphi}\right)\frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2}\sigma_{\mathcal{X}_F} \\ &\quad +\sum_{\mu=1}^{M_i}\log\left|\frac{R\left(z-a_\mu^i\right)}{R^2-\overline{a_\mu^i}z}\right| - \sum_{\nu=1}^{N_i}\log\left|\frac{R\left(z-b_\nu^i\right)}{R^2-\overline{b_\nu^i}z}\right|. \qquad \square\end{aligned}$$

As a consequence of (4.2), we obtain for analytic algebroid functions the following estimates, analogous to analytic functions in the plane

$$T(r,w) \le \log^+ M(r,w) \le n\frac{R+r}{R-r}T(R,w), \qquad (4.3)$$

where

$$M(r,w) = \max_{p\in\mathcal{X}_F[R]} |w(p)|.$$

The estimate at the left hand side follows from the definitions

$$T(r,w) = m(r,w) = \int\limits_{\mathcal{X}_F\langle r\rangle} \log^+ |w(p)|\, \sigma_{\mathcal{X}_F} \le \log^+ M(r,w).$$

To check the right hand side inequality, we assume p to be in the $i-$sheet of $\mathcal{X}_F$ so that $w(p) = w_i(z)$ and from (4.2) follows

$$\begin{aligned}
\log^+ |w(p)| &= \log^+ |w_i(z)| \\
&\le \frac{1}{2\pi}\int_{C(0,R)} \log\left|w_i\left(\mathrm{Re}^{i\varphi}\right)\right| \frac{R^2-r^2}{R^2-2Rr\cos(\theta-\varphi)+r^2} d\varphi \\
&\le \frac{1}{2\pi}\int_{C(0,R)} \log\left|w_i\left(\mathrm{Re}^{i\varphi}\right)\right| \frac{R^2-r^2}{(R-r)^2} d\varphi \\
&\le \frac{R-r}{R+r}\int_{\mathcal{C}(0,R)} \log^+ |w(p)|\, d\varphi \\
&\le \frac{R-r}{R+r}\int_{\mathcal{X}_F\langle R\rangle} \log^+ |w(p)|\, \sigma_{\mathcal{X}_F} \\
&= n\frac{R-r}{R+r} m(R,w) = n\frac{R-r}{R+r} T(R,w).
\end{aligned}$$

5. The Logarithmic Derivative Lemma

Making use of (4.2), then arguing as Yang-Lo [8] and finally summing over $i = 1,..,n$, we conclude the Logarithmic Derivative Lemma for algebroid functions

Lemma 5.1. Logarithmic Derivative Lemma for Algebroid Functions. *Let $w(z)$ be an algebroid function of order n then if $0 < r < \rho$, it holds*

$$\begin{aligned}
m\left(r, \frac{w'}{w}\right) &< 10n + 4n\log^+ \sum_{i=1}^{n} \log^+ \frac{1}{|w_i(0)|} + 2n\log^+ \frac{1}{r} + 3n\log^+ \frac{1}{\rho - r} \\
&\quad +4\log^+ \rho + \log^+ T(\rho, f).
\end{aligned} \tag{5.1}$$

As a consequence of (3.9) and (5.1) we also obtain

$$\begin{aligned} S(r,w) &= m\left(r,\frac{w'}{w}\right)+\sum_{k=1}^{2n} m\left(r,\frac{w'}{w-k}\right) \\ &+ C\left(n,w_i(0),w_i'(0)\right)++\frac{1}{n}\log^+|D(0)| \\ &\leq 2n(2n+1)\log^+\frac{1}{r}+3n(2n+1)\log^+\frac{1}{\rho-r}+4(2n+1)\log^+\rho \\ &\quad +(2n+1)\log^+\rho+(2n+1)\log^+T(\rho,f) \\ &\quad +C\left(n,w_i(0),w_i'(0)\right)+\frac{1}{n}\log^+|D(0)|. \qquad (5.2) \end{aligned}$$

6. Schottly theorem for algebroid functions

We again make use of the ideas of Yang-Lo [8] to derive from (3.8) and (5.2) the following estimate for $T(r,w)$ in the case of an algebroid function $w(z)$ of order n in $D(0,R)$ omitting the values $0,1,2,...,2n$.

Theorem 6.1. *Let $w(z)$ an holomorphic algebroid function of order n in $D(0,R)$ such that $w_i(0)\neq 0$, $w_i'(0)\neq 0$, $i=1,..,n$, $D(0)\neq 0$, and assume that it omits the values $0,1,2,...,2n$, then for every $r\in 0<r<R$ we have*

$$T(r,w) < C(n)\left\{\sum_i^n \log^+|w_i(0)|+\sum_i^n \log^+\frac{1}{R|w_i'(0)|}+\log^+|D(0)|+\log\frac{2R}{R-r}\right\},$$

where $C(n)$ is a positive constant depending of the order of the algebroid function $w(z)$.

Making use of (4.3), we can also derive from Theorem 6.1

Theorem 6.2. Schottky Theorem for Algebroid Functions. *Let $w(z)$ be an algebroid function of order n in $D(0,R)$ and let $w_i(0)\neq 0$, $w_i'(0)\neq 0$, $i=1,..,n$, $D(0)\neq 0$, and assume that it omits the values $0,1,2,...,2n$, then we have for every r, $0<r<R$*

$$\log M(r,w) < \frac{C(n)R}{R-r}\left\{\sum_i^n \log^+|w_i(0)|+\log^+|D(0)|+\log\frac{2R}{R-r}\right\},$$

where $C(n)$ is a positive constant depending of the order of the algebroid function $w(z)$.

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