Annals of the University of Bucharest (mathematical series)
5 (LXIII) (2014), 71–82

On admissible limits of holomorphic functions of several complex variables

Peter V. Dovbush

Dedicated to 85 years of Academician C. Andreian Cazacu

Abstract - The aim of the present article is to establish the connection between the existence of the limit along the normal and the admissible limit at a fixed boundary point for holomorphic functions of several complex variables.

Key words and phrases : Holomorphic function, boundary behavior, admissible limit.

Mathematics Subject Classification (2010) **:** 32A40.

1. Introduction

The connection between the existence of a radial limit and an angular limit for a holomorphic function defined on the unit disc is described by Lehto and Virtanen [6, Theorem 5] in terms of the growth of the spherical derivative.

For a precise description we introduce several terms and notation.

We introduce the notation $U := \{z \in \mathbb{C} : |z| < 1\}$ for the unit disc in $\mathbb{C}$. Let $\alpha > 1$. A non-tangential region $\Gamma_\alpha(\xi)$ for $\alpha > 1$ and an angular region $A_\theta(\xi)$ for $\theta \in (0, 2\pi)$ at $\xi \in \partial U$ are defined as follows:

$$\Gamma_\alpha(\xi) := \{z \in U : |1 - z\bar{\xi}| < \frac{\alpha}{2}(1 - |z|^2)\},$$

$$A_\theta(\xi) := \{z \in U : \pi - \theta < \arg(z - \xi) < \pi + \theta\}.$$

It is to be noted that non-tangential regions and angular regions are equivalent: For every $\alpha > 1$ there is a $\theta \in (0, \frac{\pi}{2})$ such that $\Gamma_\alpha(\xi) \subset A_\theta(\xi)$ and for every $\theta \in (0, \frac{\pi}{2})$ there is an $\alpha > 1$ and a disk d centered at ξ such that $A_\theta(\xi) \cap d \subset \Gamma_\alpha(\xi)$.

To see this let d_1 be the be the unit disk with center ξ, $z \in U$ and $\varphi = \pi - \arg(z - \xi)$. From the law of cosines

$$|z|^2 = 1 - 2\cos\varphi|\xi - z| + |\xi - z|^2.$$

Since $|\xi| = 1$ we have $|\xi - z| = |1 - z\overline{\xi}|$ and

$$\frac{|1 - z\overline{\xi}|}{1 - |z|^2} = \frac{1}{2\cos\varphi - |1 - z\overline{\xi}|}.$$

Thus,

$$\frac{1}{2\cos\varphi} \leq \frac{|1 - z\overline{\xi}|}{1 - |z|^2} \quad \text{for } z \in U,$$

and

$$\frac{|1 - z\overline{\xi}|}{1 - |z|^2} \leq \frac{2}{\cos\varphi} \quad \text{for } z \in U \cap d_1.$$

We say that a holomorphic function function f in U [notation $f \in \mathcal{O}(U)$] has the non-tangential limit L at $\xi \in \partial U$ if $f(z) \to L$ as $z \to \xi$, $z \in \Gamma_\alpha(\xi)$; has radial limit L at ξ if $\lim_{t \to 1} f(t\xi) = L$.

Define the spherical derivative of $f(z)$ to be

$$f^\sharp(z) = \frac{|f'(z)|}{1 + |f(z)|^2}.$$

Now we can reformulate Theorem 5 in [6] as follows:

Theorem 1.1. *If $f \in \mathcal{O}(U)$ has a radial limit at the point $\xi \in \partial U$, then it has an non-tangential limit at this point if and only if for any fixed $\alpha > 1$ in the non-tangential region $\Gamma_\alpha(\xi)$*

$$(1 - |z|) f^\sharp(z) \leq O(1). \tag{1.1}$$

Let $B^n := \{z \in \mathbb{C}^n : |z| < 1\}$ and let

$$D_\alpha(\xi) := \Big\{ z \in B^n : |1 - (z, \xi)| < \frac{\alpha}{2}(1 - |z|^2) \Big\},$$

where $(z, \xi) = z_1\overline{\xi}_1 + \ldots + z_n\overline{\xi}_n$ and $|z|^2 = (z, z)$.

Following Koranyi [4], we say that a holomorphic function f in B^n (henceforth, in symbols, $f \in \mathcal{O}(B^n)$) has admissible limit L at ξ if for every $\alpha > 1$ for every sequence $\{z^j\}$ in $D_\alpha(\xi)$ that converges to ξ, $f(z^j) \to L$ as $j \to \infty$. (The case $L = \infty$ is not excluded.)

It is clear that the notions of admissible limit and non-tangential limit coincides when $n = 1$.

We call

$$f^\sharp(z, v) = \frac{\left| \sum_{j=1}^n \frac{\partial f}{\partial z_j}(z) v_j \right|}{1 + |f(z)|^2}$$

the spherical derivative of function f in the direction $v \in C^n$. It is clear that $f^\sharp(z, v) = f^\sharp(z)|v|$ when $n = 1$.

An analogue of Theorem 1.1 for admissible limit fails in several complex variables. The function $f(z_1, z_2) = z_2^2(2(1-z_1))^{-1}$ is holomorphic in B^2 and bounded there, since $|f(z)| < (1-|z_1|)^2(2(1-|z_1|))^{-1} \leq 1$.

Applying Schwarz'z lemma to the restriction of the map $f : B^n \to \Delta$ to the complex line passing through the point $z \in B^n$ in the direction of the vector $v \in C^n$, is not difficult to get the estimate

$$f^\sharp(z, v) < \frac{|v|}{1-|z|}.$$

Set $z^j = (1-1/j, 1/\sqrt{j})$ for $j = 4, 5, \ldots$. It is clear that $z^j \to \zeta = (1, 0)$ as $j \to \infty$. A simple calculation shows that $z^j \subset \mathcal{D}_\alpha(\zeta)$ if j is sufficiently large. Notice that $\lim_{r\to 1-} f(r\zeta) = \lim_{r\to 1-} 0 = 0$ and $f(z^j) = \frac{1/j}{2/j} = \frac{1}{2}$, and so f does not have admissible limit at ζ.

In the present paper, we prove a criterion of existence of admissible limits of holomorphic functions of several complex variables. We also give an extension of Lindelöf's principle.

Let D be a bounded domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary ∂D, then at each $\xi \in \partial D$ the tangent space $T_\xi^c(\partial D)$ and the unit outward normal vector ν_ξ are well-defined. We denote by $T_\xi^c(\partial D)$ and $N_\xi^c(\partial D)$ the complex tangent space and the complex normal space, respectively. The complex tangent space at ξ is defined as the $(n-1)$ dimensional complex subspace of $T_\xi(\partial D)$ and given by $T_\xi^c(\partial D) = \{z \in \mathbb{C}^n : (z, w) = 0, \forall w \in N_\xi^c(\partial D)\}$, where $(\cdot, \cdot)$ denotes canonical Hermitian product of $\mathbb{C}^n$. Let $p(z, T_\xi(\partial D))$ is the Euclidean distance from z to the real tangent plane $T_\xi(\partial D)$.

An admissible domain $\mathcal{A}_\alpha(\xi)$ with vertex $\xi \in \partial D$ and aperture $\alpha > 0$ is defined as follows [8]:

$$\mathcal{A}_\alpha(\xi) = \{\, z \in D \ : |(z-\xi, \nu_\xi)| < (1+\alpha) r_\xi(z), |z-\xi|^2 < \alpha r_\xi(z) \,\}, \quad (1.2)$$
$$\text{where } r_\xi(z) = \min\{r(z), p(z, T_\xi(\partial D))\}.$$

For the ball B^n the set $D_\alpha(\xi)$ essentially coincides with (1.2). The region $\mathcal{A}_\alpha(\xi)$ allows parabolic approach to ∂D at ξ in the "$T_\xi^c(\partial D)$ directions," and non-tangential approach in the "$N_\xi^c(\partial D)$ direction."

The existence of admissible limits (in Fatou's theorem in the space $\mathbb{C}^n$, $n > 1$) was discovered by Koranyi [1] and Stein [2]; the complex geometrical nature of this phenomenon has been investigated by Chirka [3].

In Fatou's theorem for strongly pseudoconvex domains it is essentially impossible to replace the class of admissible domains by a wider class of sets. However, in arbitrary domains each boundary point has its own optimal approach, in general, wider than over the admissible domains and Fatous theorem is still true for such maximal admissible domains.

The function f, defined in a domain D in $\mathbb{C}^n$ has a limit L, $L \in \overline{\mathbb{C}}$, along the normal ν_ξ to ∂D at the point ξ iff $\lim_{t\to 0} f(\xi - t\nu_\xi) = L$; f has an *admissible limit* L, at $\xi \in \partial D$ iff

$$\lim_{\mathcal{A}_\alpha(\xi)\ni z\to\xi} f(z) = L$$

for every $\alpha > 0$; f is admissible bounded at ξ if $\sup_{z\in\mathcal{A}_\alpha(\xi)} |f(z)| < \infty$ for every $\alpha > 0$.

We can now state our main result:

Theorem 1.2. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary. If $f \in \mathcal{O}(D)$ has a limit L along the normal to ∂D at the point ξ, then at the point $\xi \in \partial D$ the function f has an admissible limit L if and only if in every admissible domain with vertex ξ the spherical derivative of f in the normal and complex tangent directions increases like $o(1/r_\xi(z))$ and $o(1/\sqrt{r_\xi(z)})$, respectively.*

The example above shows that the Lindelöf principle for bounded functions – formulated in terms of admissible convergence – fails. However the following refinement of Lindelöf's theorem holds.

Theorem 1.3. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary. If a function f in D has a limit L, $L \in \overline{\mathbb{C}}$, along the normal ν_ξ at a point $\xi \in \partial D$, and in every admissible domain with vertex ξ the function f is holomorphic, L is his omitted value and the spherical derivative of f in the normal and complex tangent directions grows no faster than $K/d(z)$ and $K/\sqrt{d(z)}$, respectively, then f has an admissible limit L at ξ.*

2. A criterion of existence of admissible limits

Our main concern for the moment will be the proof of Proposition 2.1. For this it will be convenient to introduce a slightly modified family of approach regions which also will be useful in the rest of the paper.

Let x_j, y_j be the real coordinates of $z \in \mathbb{C}^n$ such that $z_j = x_j + iy_j$. At times it will be convenient to use real variable notations by identifying z with $(x_1, \zeta) \in \mathbb{R}^{2n}$, where $\zeta = (y_1, x_2, y_2, \ldots, x_n, y_n) \in \mathbb{R}^{2n-1}$. After a unitary transformation of $\mathbb{C}^n$, if necessary, we may assume the inner normal to ∂D at 0 points the positive x_1 direction, $T_0^c(\partial D) = \{z \in \mathbb{C}^n : z_1 = 0\}$. Let π denote the map which projects $\mathbb{C}^n$ onto N_0, i.e., if $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ then $\pi(z) = (z_1, 0, \ldots, 0)$.

Without loss of generality, there is a real valued C^2 function ψ defined on $T_0(\partial D) = \{(0, \zeta), \zeta \in \mathbb{R}^{2n-1}\}$ so that $\partial D = \{(\psi(\zeta), \zeta), \zeta \in \mathbb{R}^{2n-1}\}$ and $D = \{(x_1, \zeta), x_1 > \psi(\zeta)\}$. (This is certainly true in the neighborhood of 0 by the implicit function theorem, and our concerns are purely local here.) The fact that $T_0(\partial D)$ is tangent to ∂D at 0 implies $\nabla\phi(0) = 0$.

For $z = (x_1, \zeta) \in D$ we set

$$d(z) := \min\{x_1, x_1 - \psi(\zeta)\},$$

and define an approach region

$$A_\alpha(\xi) := \{\, z \in D : |z|^2 < \alpha d(z), |y_1| < \alpha x_1\}. \tag{2.1}$$

Observe that

$$\lim_{D \ni z \to 0} \frac{r(z)}{x_1 - \psi(\zeta)} = 1,$$

and hence

$$\lim_{\mathcal{A}_\alpha(0) \ni z \to 0} \frac{r_0(z)}{d(z)} = 1.$$

It follows that the regions $A_\alpha(\xi)$ are "equivalent" to the admissible regions $\mathcal{A}_{\gamma(\alpha)}(\xi)$ in the sense that

$$\mathcal{A}_{\beta(\alpha)}(\xi) \subseteq A_\alpha(\xi) \subseteq \mathcal{A}_{\gamma(\alpha)}(\xi).$$

As in [8, p. 59] set

$$|\bigtriangledown F(z)|^2 := d(z)^2 \cdot |\bigtriangledown_1 F(z)|^2 + d(z) \cdot |\bigtriangledown_{2,n} F(z)|^2,$$

where

$$|\bigtriangledown_1 F(z)|^2 := \Big|\frac{\partial F}{\partial z_1}(z)\Big|^2, \quad |\bigtriangledown_{2,n} F(z)|^2 := \sum_{j=2}^{n} \Big|\frac{\partial F}{\partial z_j}(z)\Big|^2.$$

Proposition 2.1. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary. Suppose that $f \in \mathcal{O}(D)$ has a limit L along the normal ν_ξ to ∂D at the point ξ equal to L, $L \neq \infty$. If*

$$\frac{|\nabla f(z)|}{1 + |f(z)|^2}$$

is admissible bounded at ξ, then f admissible bounded at ξ.

Proof. Since the domain D has C^2-smooth boundary, then there is a constant $r > 0$ such that the ball $B_r(-r\nu_0) \subset D$ and $\partial B_r(-r\nu_0) \cap \partial D = \{\xi\}$. Without restriction we may suppose that $\xi = (1,' 0)$ and $r = 1$.

Let f has the finite limit L along the normal ν_0 to ∂D at the point 0. Since $d(z) \geq d_{B_1(0)}(\pi(z)) = 1 - |z_1|$ for all $z \in B_1(\xi)$ we have

$$\frac{(1 - |z_1|) \cdot |\frac{\partial f}{\partial z_1}(\pi(z))|}{1 + |f(\pi(z))|^2} < \frac{|\nabla f(\pi(z))|}{1 + |f(\pi(z))|^2} < O(1), z \in A_\alpha(0) \cap N_0^c(\partial D).$$

Therefore $f(\pi(z))$ fulfills all the hypotheses of Theorem 1.1. Hence $f(\pi(z)) \to L$ as $z \to 0$, $z \in A_\alpha(0) \cap N_0^c(\partial D)$.

Assume, to reach a contradiction, that f is not admissible bounded at 0. Let $\{z^m\}$ be any sequence of points from $A_\alpha(0)$ such that $z^m \to 0$ as $m \to \infty$ and $f(z^m) \to \infty$ as $m \to \infty$.

For the biholomorphic mapping $\Phi_b(z) = (w_1(z), \ldots, w_n(z))$, where $b \in \mathbb{C}^n$, $w_1(z) = \frac{z_1 - b_1}{cd(b)}$, $w_\mu(z) = \frac{z_\mu - b_\mu}{c\sqrt{d(b)}}$, $\mu = 2, \ldots, n$, the polydisc

$$P(b,c) := \{z \in \mathbb{C}^n : |z_1 - b_1| < cd(b), |z_\mu - b_\mu| < c\sqrt{d(b)}, \mu = 2, \ldots, n, \}$$

is mapped to the unit polydisc $U^n = \{w \in \mathbb{C}^n : |w_\mu| < 1, \mu = 1, \ldots, n\}$. By [7, Lemma 7.2] the exists $c = c(\alpha)$ such that $P(b,c) \subset A_{2\alpha}(0)$ for all sufficiently small $b \in A_\alpha(0)$. Therefore with each point $b \in A_\alpha(0)$ sufficiently close to 0 we can associate a function $g_b = f(\Phi_b^{-1}(w))$ which is well defined and holomorphic in the polydisc U^n.

By [7, Lemma 5.2] there exists $c = c(\alpha)$, $c < 1/2$, so that if $z = (x_1, \zeta)$ sufficiently small and $|z| < \alpha d(z)$ we have $d(z) \geq cx_1$. Let t be an arbitrary point of the interval $[z^m, \pi(z^m)]$. Note that $x_1^m \geq d(t) \geq cx_1^m$.

Choose an integer N such that $\alpha < cN/2$. From the definition of the set $A_\alpha(0)$ it follows that $|z^m - \pi(z^m)|^2 < cNx_1^m/2$. Then the interval $[z^m, \pi(z^m)]$ may be covered by k_m polydiscs, where $k_m < N+1$,

$$P_{m,k}(c) := P(b^{m,k}, c) =$$
$$\{z \in \mathbb{C}^n : |z_1 - b_1^{m,k}| < cd(b^{m,k}), |z_\mu - b_\mu^{m,k}| < c\sqrt{d(b^{m,k})}, \mu = 2, \ldots, n\}$$

such that $b^{m,1} = (z_1^m, '0)$, $b^{m,k_m} = z^m$, $b^{m,k} \in [z^m, \pi(z^m)]$, $k = 2, \ldots, k_m - 1$, $P_{m,k}(c/2) \ni b^{m,k+1}$ (and hence $P_{m,k}(c/2) \cap P_{m,k+1}(c/2) \neq \emptyset$) for all $m \geq 1$, $k < k_m$. To each point $b^{m,k}$ we associate a function $g_{m,k} = g_{b^{m,k}}$ as above.

Set $G^m = g_{m,k_m}$, $m \geq 1$. Since $f(z^m) \to \infty$ as $m \to \infty$ and $P_{m,k_m}(c) \ni z^m$ we have $g_{m,k_m}(0) = f(z^m) \to \infty$ as $m \to \infty$. Suppose that there is a sequence of points $\{w^m\}$ which belongs to some polydisc P_2, $\overline{P}_2 \subset U^n$, such that $G^m(w^m) \not\to \infty$ as $m \to \infty$. It follows that the family $\{G^m\}$ is not normal in U^n and by Marty's criterion (see, e.g., [2]) there are points $p^m \in \overline{P}_2$ and vectors $v^m \in \mathbb{C}^n$ with $|v^m| = 1$ such that

$$\frac{\left|dG^m_{p^m}(v^m)\right|}{1 + |G^m(p^m)|^2} > m, \quad (m = 1, 2, \ldots), \tag{2.2}$$

where

$$dG^m_{p^m}(v^m) = \sum_{\mu=1}^{n} \frac{\partial G^m}{\partial w_\mu}(p^m) v_\mu^m.$$

According to the rule of differentiation of composite functions

$$\frac{\partial G^m}{\partial w_1}(p^m) = cd(b^{m,1})\frac{\partial f}{\partial z_1}(t^m)$$
$$\frac{\partial G^m}{\partial w_\mu}(p^m) = c\sqrt{d(b^{m,1})}\frac{\partial f}{\partial z_\mu}(t^m), (\mu = 2, \ldots, n),$$

where $t^m = \Phi^{-1}_{b^{m,1}}(p^m) \in P_{m,1}(c) \subset A_{2\alpha}(0)$. By [7, Lemma 5.2] there exists $c_1 = \min\{1/2, 1/2K\alpha\}$ so that if $z = (x_1, \zeta) \in A_{2\alpha}(0)$ is sufficiently small then $x_1 > d(z) \geq c_1 x_1$. Since $b_1^{m,1} = x_1^m$ and $(1-c)x_1^m \leq Re\, t_1^m \leq (1+c)x_1^m$ we have

$$\frac{c_1}{1+c} \leq \frac{d(b^{m,1})}{d(t^m)} \leq \frac{1}{(1-c)}.$$

This, together with the Bunyakovskiǐ-Schwarz inequality, implies from (2.2) that

$$\frac{|\nabla f(t^m)|}{1+|f(t^m)|^2} > O(1)m.$$

This is in contradiction to the assumption that $|\nabla f(z)|/(1+|f(z)|^2)$ is admissible bounded in 0. This suggests that the sequence $\{G^m\}$ converges to ∞ uniformly on compact subsets of U^n. Set $G^m = g_{m,\{k_m-1\}}$, $m \geq 1$. (Note that we set $g_{m,\{k_m-1\}} \equiv g_{m,k_m}$ if $k_m - 1 \leq 0$.) Since

$$P_{m,k_m-1}(c/2) \cap P_{m,k_m}(c/2) \neq \emptyset$$

we have $G^m(0) \to \infty$ as $m \to \infty$ and we may repeat the above argument. After finite number of steps the proof will be completed since $P_{m,1}(c) \ni \pi(z^m)$ and $f(\pi(z^m)) \to L$ as $m \to \infty$. We get $|\nabla f(z)|/(1+|f(z)|^2)$ is not admissible bounded in 0, contrary to the hypothesis on $|\nabla f(z)|/(1+|f(z)|^2)$. This contradiction proves our claim. □

With this result we can obtain the following theorem

Theorem 2.1. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary. If a function f holomorphic in D has a limit along the normal ν_ξ at a point $\xi \in \partial D$, then it has an admissible limit at this point if and only if for every $\alpha > 0$*

$$\frac{|\nabla f(z)|}{1+|f(z)|^2} \to 0 \tag{2.3}$$

as $z \to \xi$, $z \in A_\alpha(\xi)$.

Proof. *Necessity.* Assume $\xi = 0$, without loss of generality. First, let f has finite admissible limit L at 0. Without loss of generality, assume $L = 0$ at 0. Let $P_1(z)$ denote the polydisc centered at z, whose radii are essentially cx_1, $c\sqrt{x_1}$, $\ldots$, $c\sqrt{x_1}$, with c sufficiently small. By [7, Lemma 7.2] exists $c = c(\alpha)$ such that $P_1(z) \subset A_{2\alpha}(\xi)$. Let $P(z)$ denote the polydisc centered at z, whose radii are essentially $cd(z)$, $c\sqrt{d(z)}$, …, $c\sqrt{d(z)}$. Since $d(z) = \min\{x_1, x_1 - \psi(\zeta)\} \le x_1$ we have $P(z) \subseteq P_1(z) \subset D$. The one variable Cauchy's estimate shows that

$$\triangledown_1 f(z) \le \frac{\sup_{\{w\in P(z)\}}|f(w)|}{cd(z)}, \quad \triangledown_{2,n} f(z) \le \frac{\sup_{\{w\in P(z)\}}|f(w)|}{c\sqrt{d(z)}}.$$

Since $f(z) \to 0$ as $z \to 0$, $z \in A_\alpha(0)$, we have

$$|\nabla f(z)| \to 0$$

as $z \to 0$, $z \in A_\alpha(0)$. It remains to observe that $|\nabla f(z)| \ge |\nabla f(z)|/(1 + |f(z)|^2)$.

If the function f has an admissible limit at the point 0 equal to infinity, then for any $\alpha > 0$ there is a $\varepsilon > 0$ such that $1/f \in \mathcal{O}(A_\alpha(0)) \cap B_\varepsilon(0))$. The function $F = 1/f$ has an admissible limit equal to zero at the point 0, so, as we have proved, F satisfies (2.3). It remains to observe that outside the zeros of f we obviously have $(1+|F(z)|^2)^{-1}|\nabla F(z)| = (1+|f(z)|^2)^{-1}|\nabla f(z)|$.

Sufficiency. **(a)** Suppose that the function f has a limit L along the normal ν_0 to ∂D at the point 0 equal to L, $L \ne \infty$.

We may assume, without loss of generality, that $L = 0$. Write

$$f(z) = \{f(z) - f(z_1, 0, \ldots, 0)\} + f(z_1, 0, \ldots, 0).$$

The first term on the right side is dominated by

$$|z(1) - z(0)| \sup_{\{0<t<1\}} |\triangledown_{2,n} f(z(t))|,$$

where $z(t) = (z_1, z_2 t, \ldots, z_n t)$, $t \in [0, 1]$, If $z \in A_\alpha(0)$, then by [7, Lemma 7.3] $z(t) \in A_\alpha(0)$, $t \in [0, 1]$, and there $d(z(t)) \approx d(z)$ while $|z(1) - z(0)| < \alpha\sqrt{d(z)}$. (The expression $A \approx B$ means that there are positive constants c_1 and c_1 such that $c_1 A < B < c_2 A$.) By Proposition 2.1 f is admissible bounded in 0 and therefore

$$|z(1) - z(0)| \sup_{\{0<t<1\}} |\triangledown_{2,n} f(z(t))| \le O(1) \frac{|\triangledown f(z(t_0))|}{1 + |f(z(t_0)|^2},$$

where $0 \le t_0 \le 1$. Since $|\triangledown f(z(t_0))|/(1+|f(z(t_0))|) \to 0$ as $z(t_0) \to 0$ we have that $f(z) - f(z_1, 0, \ldots, 0) \to 0$ as $z \to 0$ in $A_\alpha(0)$. Since $f(z_1, 0, \ldots, 0) \to 0$ as $z \to 0$ in $A_\alpha(0)$ we conclude that

$$\lim_{A_\alpha(0) \ni z \to 0} f(z) = 0.$$

The above proof is quite analogous to the proof in [8, p, 68].

(b) Let the function f has the infinite limit along the normal ν_0 to ∂D at the point 0. Let $\{z^m\}$ be any sequence of points from $A_\alpha(0)$ such that

$z^m \to 0$ as $m \to \infty$. As in the proof of Proposition 2.1 let $\{G^m\}$, be a sequence of function defined on U^n. Then as in Proposition 2.1 we obtain $f(z^m) \to \infty$ as $m \to \infty$. Since the sequence of points $\{z^m\}$ was arbitrary, by definition this means that f has the admissible limit equal to infinity at the point 0. The theorem is proved. ◻

Now we can give the proof of Theorem 1.2: For each z near ∂D denote by $\zeta(z)$ the point on ∂D closest to z. Choose the coordinate system $\widetilde{z}_1, \ldots, \widetilde{z}_n$ in $\mathbb{C}^n$ such that $\zeta(z) = 0$, and $\{\widetilde{z} \in \mathbb{C}^n : (\widetilde{z}_1, 0 \ldots, 0)\} = N_0^c(\partial D)$, and $\{\widetilde{z} \in \mathbb{C}^n : (0, \widetilde{z}_2, \ldots, \widetilde{z}_n)\} = T_0^c(\partial D)$, and $\nu_0 = (i, 0, \ldots, 0)$. Denote by $grad_{\mathbb{C}} F = \left(\frac{\partial F}{\partial \widetilde{z}_1}, \ldots, \frac{\partial F}{\partial \widetilde{z}_n}\right)$ the complex gradient of function F. Write also

$$|\widetilde{\bigtriangledown}_1 F|^2 = \Big|\frac{\partial F}{\partial \widetilde{z}_1}\Big|^2, \quad |\widetilde{\bigtriangledown}_{2,n} F|^2 = \sum_{j=2}^{n} \Big|\frac{\partial F}{\partial \widetilde{z}_j}\Big|^2.$$

Then $|grad_{\mathbb{C}} F|^2 = |\widetilde{\bigtriangledown}_1 F|^2 + |\widetilde{\bigtriangledown}_{2,n} F|^2$ but this splitting varies (with the decomposition $\mathbb{C}^n = N_{\zeta(z)} \oplus T^c_{\zeta(z)}$) as z varies in $A_\alpha(\xi)$).

Observe that if $z \in A_\alpha(\xi)$ we have

$$d(z)^2 \cdot |\bigtriangledown_1 F(z)|^2 + d(z) \cdot |\bigtriangledown_{2,n} F(z)| \approx \\ d(z)^2 \cdot |\widetilde{\bigtriangledown}_1 F(z)|^2 + d(z) \cdot |\widetilde{\bigtriangledown}_{2,n} F(z)|^2 \quad (2.4)$$

(see [8, pp. 61-62]). We write $A \approx B$ if the ration $|A|/|B|$ is bounded between two positive constants. From (2.4) follows that Theorem 2.1 is actually equivalent to Theorem 1.2.

We also can obtain the following Theorem.

Theorem 2.2. *Let D be a domain in $\mathbb{C}^n$, $n > 1$, with C^2-smooth boundary. Let in every admissible domain with vertex ξ the function f is holomorphic and its spherical derivative in the normal and complex tangent directions grows no faster than $K/d(z)$ and $K/\sqrt{d(z)}$, respectively. If*

$$\lim_{A_\beta(\xi) \ni z \to \xi} f(z) = L \text{ for some } \beta > 0,$$

then f has an admissible limit at ξ.

Proof. Fix $\alpha > \beta$. Let $\{z^m\}$ be an arbitrary sequence of $A_\alpha(\xi)$. Let $G^m = g_{m,1}$, $m \geq 1$, be the sequence of function defined as in proof of Proposition 2.1. The family $\{g_{m,1}\}$ is normal on P (this was proved in Theorem 1.2). Since $f(z) \to L$ as $z \to 0$ in $A_\beta(0)$, without lost a generality, we may assume that $P_{m,1}(c) \subset A_\beta(0)$ for all $m = 1, 2, \ldots$. Hence G^m tends to L uniformly on every compact subset of P.

By [7, Lemma 5.2] there exists $c_1 = \min\{1/2, 1/2K\alpha\} < 1/2$ so that if $z = (x_1, \zeta) \in A_{2\alpha}(0)$ is sufficiently small then $x_1 > d(z) \geq c_1 x_1$. Since $b_1^{m,1} = b_1^{m,2} = x_1^m$ we have

$$c_1 \leq \frac{d(b^{m,2})}{d(b^{m,1})} \leq \frac{1}{c_1}.$$

Since

$$\Phi_{b^{m,2}}^{-1}(w) = (cd(b^{m,2})w + b_1^{m,2}, c\sqrt{d(b^{m,2})}w + b_2^{m,2}, \dots, c\sqrt{d(b^{m,2})}w + b_n^{m,2}),$$

$|b_1^{m,2} - b_1^{m,1}| < cd(b^{m,1})/2$, and $|b_\mu^{m,2} - b_\mu^{m,1}| < c\sqrt{d(b^{m,1})}/2$, $\mu = 1, 2, \dots, n$, the little calculation shows that for for all $w \in P(0, c_1/4) \subset P$

$$|w_1 cd(b^{m,2}) + b_1^{m,2} - b_1^{m,1}| < \frac{cc_1}{4}\frac{d(b^{m,2})}{d(b^{m,1})}d(b^{m,1}) + \frac{c}{2}d(b^{m,1}) < \frac{3c}{4}d(b^{m,1})$$

and

$$|w_\mu cd(b^{m,2}) + b_\mu^{m,2} - b_\mu^{m,1}| < \frac{cc_1}{4}\frac{d(b^{m,2})}{d(b^{m,1})}d(b^{m,1}) + \frac{c}{2}\sqrt{d(b^{m,1})} < \frac{3c}{4}\sqrt{d(b^{m,1})}$$

for all μ, $\mu = 1, 2, \dots, n$. It follows $g_{m,2}$ takes the same values on $P(0, c_1/4)$ as f on $\Phi_{b^{m,2}}^{-1}(P(0, c_1/4)) \subset P_{m,1}(c)$ hence $g_{m,2} \to L$ on $\overline{P(0, c_1/5)} \subset P$.

The family $\{g_{m,2}\}$ is normal on P (this was proved in Theorem 1.2) hence the family $\{g_{m,2}\}$ also tends to L uniformly on compact subsets of P. After finite steps we obtain that $f(z^m) \to L$ as $m \to \infty$. Since the sequence of points $\{z^m\}$ chosen from $A_\beta(0)$ is arbitrary, this completes the proof that the function f has the admissible limit L at the point ξ. The theorem is proved. □

Remark 2.1. For bounded holomorphic functions this theorem appears in Chirka's paper [1], with the proof sketched there relying on certain estimates on harmonic measures. A proof based on a different method was given by Ramey [7, Theorem 2].

3. The proof of Theorem 1.3

By hypothesis of the theorem $L \notin f(D)$ then $(f(z) - L)^{-1}$ is holomorphic on D and has a radial limit at ξ equal to ∞. It is thus sufficient to consider the case $L = \infty$.

By Theorem 1.1 and hypothesis on f we have $f(\pi(z)) \to \infty$ as $z \to \xi$, $z \in A_\alpha(\xi) \cap N_\xi^c(\partial D)$. Let $\{z^m\}$ be any sequence of points from $A_\alpha(\xi)$ such that $z^m \to \xi$ as $m \to \infty$. Since the spherical derivative of f in the normal and complex tangent directions grows no faster than $K/d(z)$ and $K/\sqrt{d(z)}$, respectively, from (2.4) follows

$$d(z)^2 \cdot |\bigtriangledown_1 F(z)|^2 + d(z) \cdot |\bigtriangledown_{2,n} F(z)|^2 \leq O(1) \quad (z \in A_\alpha(\xi)).$$

Using the notation introduced in the proof of Proposition 2.1, the Bunyakovskiĭ-Schwarz inequality and the fact that $d(b^{m,1}) \approx d(z)$ for all $z \in P_{m,1}$ it follows that

$$\frac{|dG_p^m(v)|^2}{(1+|G^m(p)|^2)^2} \leq O(1) \quad (m = 1, 2, \dots)$$

for all $p \in P$ and all $v \in \mathbb{C}^n$, $|v| = 1$.

By Marty's criterion (see, e.g., [2]) the family $\{G^m\}$ are normal in U^n. Since $G^m(\pi(z^m)) = g_{m,1}(0) \to \infty$ as $m \to \infty$ it follows that the sequence $\{G^m\}$ converges uniformly on compact subsets of U^n to ∞. Then as in Theorem 2.1 we obtain $f(z^m) \to \infty$ as $m \to \infty$.

Since the sequence of points $\{z^m\}$ chosen from $A_\beta(\xi)$ is arbitrary, this completes the proof that the function f has the admissible limit L at the point ξ. The theorem is proved. □

References

[1] E.M. Chirka, The theorems of Lindelof and Fatou in $\mathbb{C}^n$, *Math. USSR-Sb.*, **21**, *4* (1973), 619-639, http://dx.doi.org/10.1070/sm1973v021n04abeh002039.

[2] P.V. Dovbush, Normal functions of many complex variables, *Vestnic Moskov Univ. Ser. I, Mat. Mekh*, **1** (1981), 38-42 [English transl. in *Moscow Univ. Math. Bull.*, **36**, *1* (1981), 44-48].

[3] P.V. Dovbush, Existence of admissible limits of functions of several complex variables, *Sibirskii Matematicheskii Zhurnal*, **28**, *3* (1987), 73-77 [English transl. in *Siberian Mathematical Journal*, **28**, *3* (1987), 411-414, http://dx.doi.org/10.1007/bf00969572].

[4] A. Koranyi, Harmonic functions on Hermitian hyperbolic space, *Trans. Amer. Math. Soc.*, **135** (1969), 507-516, http://dx.doi.org/10.1090/s0002-9947-1969-0277747-0.

[5] P. Montel, Sur les familles de fonctions analytiqyes, qui admettent des valeurs exceptionnelles dans un domaine, *Ann. Sci. Éc. Norm. Supér. (3)* **29** (1912), 487-535.

[6] O. Lehto and V.I. Virtanen, Boundary behavior and normal meromorphic functions, *Acta. Math.*, **97** (1957), 47-65, http://dx.doi.org/10.1007/bf02392392.

[7] W.C. Ramey, Boundary behavior of bounded holomorphic functions along maximally complex submanifolds, *Amer. J. Math.*, **106**, *4* (1984), 974-1001, http://dx.doi.org/10.2307/2374329.

[8] E.M. STEIN, *Boundary Behavior of Holomorphic Functions of Several Complex Variables*, Princeton Univ. Press, 1972.

Peter V. Dovbush

Institute of Mathematics and Computer Science of Academy of Sciences of Moldova

5 Academy Street, MD-2028, Kishinev, The Republic of Moldova

E-mail: `dovbush@math.md`