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Some metric relations of the homeomorphisms satisfying generalized modular inequalities (II)

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Dedicated to Professor Cabiria Andreian Cazacu on her 85th birthday

Abstract - We continue the study from [13] of the metric relations which hold for some homeomorphisms $f : D \to D^{'}$ between two domains from $\mathbb{R}^n$ satisfying a generalized modular inequality. This relations are in connection with the well known property of local quasisymetry of quasiconformal mappings.

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If $D \subset \mathbb{R}^n$ is a domain, we set $A(D)$ the set of all path families Γ from D and if $\Gamma \in A(D)$, we put $F(\Gamma) = \{\rho : \mathbb{R}^n \to [0, \infty]$ Borel maps$| \int\limits_{\gamma} \rho ds \geq 1$ for every $\gamma \in \Gamma$ locally rectifiable$\}$. We define for $p > 1$, $\Gamma \in A(D)$ and $\omega \in L^1_{loc}(D)$ the p modulus of weight ω, $M^p_\omega(\Gamma) = \inf\limits_{\rho \in F(\Gamma)} \int\limits_{\mathbb{R}^n} \rho^p(x) dx$. The systematic utilization of the arbitrary weight p modulus in the mapping theory was initiated by Cabiria Andreian in [1] and [2].

Let $D, D^{'}$ be domains in $\mathbb{R}^n$ and $f : D \to D^{'}$ a homeomorphism. We say that f satisfies condition (N) if $\mu_n(f(A)) = 0$ for every $A \subset D$ with $\mu_n(A) = 0$ (here μ_n denotes the Lebesgue measure in $\mathbb{R}^n$). If $x \in D$, we set $L(x, f) = \limsup\limits_{h \to 0} \frac{|f(x+h) - f(x)|}{|h|}$ and if $\overline{B}(x, r) \subset D$, we set $L(x, f, r) = \sup\limits_{|y-x|=r} |f(y) - f(x)|$, $l(x, f, r) = \inf\limits_{|y-x|=r} |f(y) - f(x)|$ and $H_f(x) = \limsup\limits_{r \to 0} \frac{L(x,f,r)}{l(x,f,r)}$. We say that f is K quasiconformal if $\frac{M_n(\Gamma)}{K} \leq M_n(f(\Gamma)) \leq K M_n(\Gamma)$ for every $\Gamma \in A(D)$ (the geometric definition of the quasiconformality). This definition is equivalent with the metric definition of the quasiconformality which says that f is quasiconformal if there exists $H \geq 1$ such that $H_f(x) \leq H$ for every $x \in D$. We recommend the reader the book [33] for some basic facts concerning the theory of quasiconformal mappings.

We say that f is η-quasisymetric if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that $\frac{|f(z) - f(x)|}{|f(y) - f(x)|} \leq \eta(\frac{|z-x|}{|y-x|})$ if $x, y, z \in D$, $x \neq y$, $x \neq z$.

If f is η-quasisymetric, then $H_f(x) \leq \eta(1)$ for every $x \in D$ and hence f is quasiconformal. Also, if f is quasiconformal and $\overline{B}(x, 3r) \subset D$, then $f|B(x,r) : B(x,r) \to f(B(x,r))$ is η-quasisymetric and if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a homeomorphism, then f is quasiconformal if and only if f is η-quasisyemtric.

An important class of continuous, open, discrete mappings which generalizes the class of quasiconformal mappings is the class of quasiregular mappings (see [22], [23], [34] for some basic facts concerning this theory). In the last 20 years were studied in [5-21], [24-32] more general classes of continuous, open discrete mappings (the so called mappings of finite distortion). For all of them a modular inequality of type "$M_n(f(\Gamma)) \leq M^n_{K_{I,n}(f)}(\Gamma)$" holds for every $\Gamma \in A(D)$ and this is the main instrument which permits to prove that a lot of the important facts from the theory of quasiregular mappings still hold in this new classes of mappings. We must say that using this modular method, Cabiria Andreian proved with 50 years before in 1959 in [1] that analogues of the theorems of Fatou, Nevanlinna-Frostman and Beurling still hold for very general classes of plane homeomorphisms.

In some recent papers [9-13] we studied classes of continuous, open, discrete mappings $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ for which a modular inequality of type "$M_q(f(\Gamma)) \leq \gamma(M^p_\omega(\Gamma))$" holds for every $\Gamma \in A(D)$ and some $p > 1$, $q > n-1$, $\omega \in L^1_{loc}(D)$ and a strictly increasing function $\gamma : [0,\infty) \to [0,\infty)$ with $\lim\limits_{t\to 0} \gamma(t) = 0$. We extended partially basic facts from the theory of quasiregular mappings and from the classes of mappings of finite distortion mentioned before. Using the modulus method, we gave Liouville, Picard, Montel type theorems, equicontinuity results, eliminability and boundary extension results and we gave estimates of the modulus of continuity. In the last paper [13] we studied the metric relations which hold for some homeomorphisms $f : D \to D^{'}$ between two domains in $\mathbb{R}^n$ satisfying a modular inequality of type "$M_q(f(\Gamma)) \leq \gamma(M^p_\omega(\Gamma))$". We continue this researches in the present paper.

Let $D \subset \mathbb{R}^n$ be a domain and $f : D \to \mathbb{R}^n$ a map. We say that f is ACL if f is continuous and for every cube $Q \subset\subset D$ with the sides parallel to coordinate axes and for every face S of Q it results that $f|P_S^{-1}(y) \cap Q : P_S^{-1}(y) \cap Q \to \mathbb{R}^n$ is absolutely continuous for a.e. $y \in S$, where $P_S : \mathbb{R}^n \to S$ is the projection on S. An ACL map has a.e. first partial derivatives and if $q > 1$, we say that f is ACL^q if f is ACL and the first partial derivatives are locally in L^q. If $q > 1$, we denote by $W^{1,q}_{loc}(D, \mathbb{R}^n)$ the Sobolev space of all functions $f : D \subset \mathbb{R}^n \to \mathbb{R}^n$ which are locally in L^q together with their first order distributional derivatives. We see from Proposition 1.2 page 6 in [23] that if $f \in C(D, \mathbb{R}^n)$, then f is ACL^q if and only if $f \in W^{1,q}_{loc}(D, \mathbb{R}^n)$.

If $f \in L^1(D)$ and $A \subset D$ is measurable, we set $\displaystyle\fint_A f(x)dx = \int_A f(x)dx/\mu_n(A)$ and we also denote it by f_A. We say that a map $f \in L^1(D)$ is in the $BMO(D)$ class if there exists $M > 0$ such that $\fint_{B(x,r)} |f(z) - f_{B(x,r)}|dz \leq$

M for every ball $B(x,r) \subset D$. If $D \subset \mathbb{R}^n$ is open, $E, F \subset \overline{D}$, we set $\Delta(E,F,D) = \{\gamma : [0,1] \to \mathbb{R}^n \text{ path} | \gamma(0) \in E, \gamma(1) \in F \text{ and } \gamma((0,1)) \subset D\}$ and if $x \in \mathbb{R}^n$ and $0 < r < R$, we set $\Gamma_{x,r,R} = \Delta(\overline{B}(x,r), S(x,R), B(x,R) \setminus \overline{B}(x,r))$. We denote by V_n the volume of the unit ball in $\mathbb{R}^n$, by ω_{n-1} the area of the unit sphere in $\mathbb{R}^n$ and if $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, we set $l(A) = \inf\limits_{|x|=1} |A(x)|$.

If $f : D \to D^{'}$ is a homeomorphism between two domains from $\mathbb{R}^n$ and g is its inverse, we see that $\mu_f : \mathcal{B}(D) \to [0,\infty]$ given by $\mu_f(A) = \mu_n(f(A))$ for every $A \in \mathcal{B}(D)$ is a set function and hence $\mu_f^{'}(x)$ exists a.e. and $\mu_f^{'}$ is a Borel function and we define a.e. the q inner dilation of f, $K_{I,q}(f)$ by $K_{I,q}(f)(x) = L(f(x), g)\mu_f^{'}(x)$. If f is differentiable in x and $J_f(x) \neq 0$, we see that $K_{I,q}(f)(x) = \frac{|J_f(x)|}{l(f^{'}(x))^q}$. (See also [13] for more details).

Given $r > 0$, let $\Phi_n(r)$ be the set of all rings $A = R(C_0, C_1)$ such that $0 \in C_0$ and there exists $a \in C_0 \cap S(0,1)$ and such that $\infty \in C_1$ and there exists $b \in S(0,r) \cap C_1$ (see also [33], page 33-36). As in [4], we set $H_{n,q}(r) = \inf\limits_{A \in \Phi_n(r)} M_q(\Gamma_A)$ for $q > 1$ and $r > 0$. Here $\Gamma_A = \Delta(C_0, C_1, \mathbb{R}^n)$ if A is a ring $A = R(C_0, C_1)$. Then $H_{n,q} : (0,\infty) \to (0,\infty)$ is decreasing and if $n-1 < q < n$ there exists a constant $Q(n,q) > 0$ such that $H_{n,q}(r) \geq Q(n,q)(1 - r^{n-q})$ for every $0 < r < 1$ and $H_{n,q}(r) \geq \frac{Q(n,q)}{2^n}((r^2+2)^{\frac{n-q}{2}} - r^{n-q})$ if $r \geq 1$. If $q = n$, we set $H_{n,n} = H_n$ and we see from [34] that $H_n : (0,\infty) \to (0,\infty)$ is a decreasing homeomorphism. We see from Theorem 9 in [4] that if $n - 1 < q \leq n$ and $A = R(C_0, C_1)$ is a ring such that $a, b \in C_0$, $c, \infty \in C_1$, then $M_q(\Gamma_A) \geq |b-a|^{n-q} H_{n,q}(\frac{|c-a|}{|b-a|})$. We see from [3] that if $x \in \mathbb{R}^n$, $0 < a < b$, $D = B(x,b) \setminus \overline{B}(x,a)$, $n - 1 < q$, $E, F \subset D$ are disjoint such that $S(x,t) \cap E \neq \phi$, $S(x,t) \cap F \neq \phi$ for every $a < t < b$, then $M_q(\Delta(E,F,D)) \geq C(n,q)(b^{n-q} - a^{n-q})$ if $q \neq n$ and $M_n(\Delta(E,F,D)) \geq C(n)\ln(\frac{b}{a})$, where $C(n,q)$ is a constant depending only on n and q and $C(n)$ is a constant depending only on n. Throughout this paper we shall keep the notations $Q(n,q)$, $C(n,q)$, $C(n)$ for the constants from the papers of Caraman [3-4]. We also denote by $l_n = \sum\limits_{k=1}^{\infty} \frac{1}{k^n}$ and $t_n = \sum\limits_{k=1}^{\infty} \frac{1}{k^{n-1}}$ for $n \geq 2$.

In [7] and [8] we gave the following estimates of the modulus of $\Gamma_{x,r,R}$:

Lemma A. *Let $n \geq 2$, $p > 1$, $x \in \mathbb{R}^n$, $0 < r < R$, $\omega \in L^1(B(x,r))$ and suppose that one of the following conditions hold:*

1) $\sup\limits_{0<\delta<R} \int\limits_{B(x,\delta)} \omega(z)dz/\delta^p = M < \infty$.

2) $\sup\limits_{0<\delta<R} ⨍\limits_{B(x,\delta)} |\omega(z) - \omega_{B(x,r)}|dz = M < \infty$.

Then, if condition 1) holds, it results that $M_\omega^p(\Gamma_{x,r,R}) \leq \frac{Me^p l_p}{(\ln\ln(\frac{Re}{r}))^p}$.

If condition 2) holds and $n \geq 3$, then

$$M_\omega^n(\Gamma_{x,r,R}) \leq \frac{V_n e^n((M + ⨍_{B(x,R)} \omega(z)dz)l_n + Me^n t_n)}{(\ln\ln(\frac{Re}{r}))^n}.$$

If condition 2) *holds and* $n = 2$, *then*

$$M^2_\omega(\Gamma_{x,r,R}) \le \frac{V_2 e^2 (M + \fint_{B(x,R)} \omega(z)dz) l_2}{(\ln\ln(\frac{Re}{r}))^2} + \frac{V_2 e^4 M}{\ln\ln(\frac{Re}{r})}.$$

The following theorem from [13] ensures very general conditions in order that a homeomorphism $f : D \to D^{'}$ between two domains from $\mathbb{R}^n$ to satisfy some generalized modular inequalities:

Lemma B. *Let* $n \ge 2$, $q > 1$, $D, D^{'}$ *be domains in* $\mathbb{R}^n$, *let* $f : D \to D^{'}$ *a homeomorphism satisfying condition* (N), *let* g *be its inverse and suppose that* g *is* ACL^q. *Then* $M_q(f(\Gamma)) \le M^q_{K_{I,q}(f)}(\Gamma)$ *for every* $\Gamma \in A(D)$ *and if* $\int_{G^{'}} L(y,g)^q dy < \infty$ *for every open set* $G^{'} \subset\subset D^{'}$, *then* $\int_G K_{I,q}(f)(x)dx < \infty$ *for every open set* $G \subset\subset D$. *Also, if* $1 < q < p$ *and*

$$C = \left(\int_D K_{I,q}(f)(x)^{p/(p-q)} dx \right)^{\frac{p-q}{p}} < \infty,$$

then $M_q(f(\Gamma)) \le C M_p(\Gamma)^{\frac{q}{p}}$ *for every* $\Gamma \in A(D)$.

Theorem 1. *Let* $n \ge 2$, $n - 1 < q \le n$, $p > 1$, $D, D^{'}$ *be domains in* $\mathbb{R}^n$, $\omega \in L^1_{loc}(D)$, $\gamma : [0,\infty) \to [0,\infty)$ *be increasing with* $\lim_{t\to 0} \gamma(t) = 0$ *and let* $f : D \to D^{'}$ *be a homeomorphism such that* $M_q(f(\Gamma)) \le \gamma(M^p_\omega(\Gamma))$ *for every* $\Gamma \in A(D)$. *Then, if* $|y - x| \le |z - x| \le R$ *and* $\overline{B}(x,R) \subset D$, *we see that*

$$H_n\left(\frac{|f(z) - f(x)|}{|f(y) - f(x)|}\right) \le \gamma(M^p_\omega(\Gamma_{x,|y-x|,|z-x|})) \text{ if } q = n \tag{1}$$

$$|f(y) - f(x)|^{n-q} Q(n,q) \le |f(y) - f(x)|^{n-q}\cdot$$
$$\cdot H_{n,q}\left(\frac{|f(z) - f(x)|}{|f(y) - f(x)|}\right) \le \gamma(M^p_\omega(\Gamma_{x,|y-x|,|z-x|})) \text{ if } n-1 < q < n \tag{2}$$

Proof. Let $\Gamma = \Gamma_{x,|y-x|,|z-x|}$. Then $f(\Gamma) = R(C_0, C_1)$ and $f(x), f(y) \in C_0$ and $\infty, f(z) \in C_1$. We see from Theorem 9 in [4] that

$$|f(y) - f(x)|^{n-q} H_{n,q}\left(\frac{|f(z) - f(x)|}{|f(y) - f(x)|}\right) \le M_q(f(\Gamma)) \le \gamma(M^p_\omega(\Gamma)). \quad \square$$

Theorem 2. *Let* $n \ge 2$, $D, D^{'}$ *be domains in* $\mathbb{R}^n$, $\omega \in BMO(D)$, $\gamma : [0,\infty) \to [0,\infty)$ *be increasing with* $\lim_{t\to 0} \gamma(t) = 0$ *and let* $f : D \to D^{'}$ *be* *a homeomorphism such that $M_n(f(\Gamma)) \leq \gamma(M_\omega^n(\Gamma))$ for every $\Gamma \in A(D)$. Then, if $\overline{B}(x,R) \subset D$ and $M = \sup\limits_{B(x,r)\subset D} \int\limits_{B(x,r)} |\omega(z) - \omega_{B(x,r)}|dz$, we have the following inequalities:*

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq (1/H_n^{-1}(\gamma(\frac{V_n e^n((M + ⨍_{B(x,|z-x|)}\omega(u)du)l_n + Me^n t_n)}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^n})))$$

if $n \geq 3$ (3)

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq (1/H_2^{-1}(\gamma(\frac{V_2 e^2(M + ⨍_{B(x,|z-x|)}\omega(u)du)l_2}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^2} + \frac{MV_2e^4}{\ln\ln(\frac{e|z-x|}{|y-x|})})))$$

if $n = 2$. (4)

Proof. We use Theorem 1 and Lemma A. □

Theorem 3. *Let $n \geq 2$, $p > 1$, $D, D^{‘}$ be domains in $\mathbb{R}^n$, $\omega \in L^1_{loc}(D)$, $\gamma : [0,\infty) \to [0,\infty)$ be increasing with $\lim\limits_{t\to 0}\gamma(t) = 0$ and let $f : D \to D^{‘}$ be a homeomorphism such that $M_n(f(\Gamma)) \leq \gamma(M_\omega^p(\Gamma))$ for every $\Gamma \in A(D)$. Then, if $\overline{B}(x,R) \subset D$ and $C_{x,R,p} = \sup\limits_{0<r<R} \int\limits_{B(x,R)} \omega(u)du/r^p < \infty$, we have the following inequalities:*

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq (1/H_n^{-1}(\gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^p}))) \text{ if } |y-x| \leq |z-x| \leq \frac{R}{2} \quad (5)$$

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \frac{\exp(\gamma(C_{x,R,p}(2+\frac{2|y-x|}{|z-x|}^p)/C(n)))}{H_n^{-1}(\gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(2e))^p}))}$$

if $\frac{|z-x|}{2} \leq |y-x| \leq |z-x| \leq \frac{R}{6}$ (6)

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \exp(\frac{\gamma(C_{x,R,p}(2+\frac{|y-x|}{|z-x|})^p)}{C(n)}) \text{ if } |z-x| \leq |y-x| \leq \frac{R}{3} \quad (7)$$

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \exp(\frac{\gamma(C_{x,R,p}(1+\frac{|y-x|}{|z-x|})^p)}{C(n)}) \text{ if } |z-x| \leq \frac{|y-x|}{2} \leq \frac{R}{6}. \quad (8)$$

Proof. The results from (7) and (8) are proved in Theorem 2 in [13]. Using Theorem 1, Lemma A and the results from [4], we see that if $|y-x| \leq |z-x| \leq R$, then $H_n(\frac{|f(z)-f(x)|}{|f(y)-f(x)|}) \leq \gamma(M_\omega^p(\Gamma_{x,|y-x|,|z-x|})) \leq \gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^p})$ and relation (5) is now proved. Let now $\frac{|z-x|}{2} \leq |y-x| \leq |z-x| \leq R$ and let $w \in S(x, 2|y-x|)$. We see from (5) that $\frac{|f(y)-f(x)|}{|f(w)-f(x)|} \leq (1/H_n^{-1}(\gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(2e))^p})))$.

Since $|z-x| \leq |w-x|$, we use (7) and we see that $\frac{|f(w)-f(x)|}{|f(z)-f(x)|} \leq$
$\leq \exp(\frac{\gamma(C_{x,R,p}(2+\frac{2|y-x|}{|z-x|})^p)}{C(n)})$ and now also (6) is proved. $\square$

Remark 1. Let $\eta : [0,\infty) \to [0,\infty)$ be defined by

$$\eta(t) = \begin{cases} (1/H_n^{-1}(\gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(\frac{e}{t}))^p}))), & \text{if } 0 \leq t < \frac{1}{2} \\ \frac{\exp(\gamma(C_{x,R,p}(2+2t)^p/C(n)))}{H_n^{-1}(\gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(2e))^p}))}, & \text{if } \frac{1}{2} \leq t < 1 \\ \exp(\frac{\gamma(C_{x,R,p}(2+t)^p)}{C(n)}), & \text{if } 1 \leq t < 2 \\ \exp(\frac{\gamma(C_{x,R,p}(1+t)^p)}{C(n)}), & \text{if } t \geq 2. \end{cases}$$

We see that η is continuous, bounded and increasing on each interval $[0,\frac{1}{2})$, $[\frac{1}{2},1)$, $[1,2)$ and $[2,\infty)$ and $\lim\limits_{t\to 0}\eta(t) = 0$ and we proved in Theorem 3 that if $\overline{B}(x,\frac{R}{6}) \subset D$, $y,z \in B(x,\frac{R}{6})$, $z \neq x$, $y \neq x$, then $\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \eta(\frac{|y-x|}{|z-x|})$. We can easy find a homeomorphism $\theta : [0,\infty) \to [0,\infty)$ such that $\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \theta(\frac{|y-x|}{|z-x|})$ if $\overline{B}(x,\frac{R}{6}) \subset D$ and $y,z \in B(x,\frac{R}{6})$, $z \neq x$, $y \neq x$, and θ depends only on p,n,γ and $C_{x,R,p}$. This relation is in connection with the well known local quasisimetry property of quasiconformal mappings and we see that if there exists $K > 0$ such that $C_{x,R,p} \leq K$ for every $x \in D$ and every $R > 0$ such that $\overline{B}(x,R) \subset D$, then it results that f is quasiconformal.

Theorem 4. *Let $n \geq 2$, $p > 1$, $n-1 < q < n$, $D, D^{'}$ be domains in $\mathbb{R}^n$, $\omega \in L^1_{loc}(D)$, $\gamma : [0,\infty) \to [0,\infty)$ be increasing with $\lim\limits_{t\to 0}\gamma(t) = 0$ and let $f : D \to D^{'}$ be a homeomorphism such that $M_q(f(\Gamma)) \leq \gamma(M^p_\omega(\Gamma))$ for every $\Gamma \in A(D)$. Then, if $\overline{B}(x,R) \subset D$ and $C_{x,R,p} = \sup\limits_{0<r<R}\int\limits_{B(x,R)}\omega(u)du/r^p < \infty$, we have the following inequalities:*

$$|f(y)-f(x)|^{n-q} - |f(z)-f(x)|^{n-q} \leq \frac{\gamma(C_{x,R,p}(2+\frac{|y-x|}{|z-x|})^p)}{C(n,q)}$$

$$\text{if } |z-x| \leq |y-x| \leq \frac{R}{3} \tag{9}$$

$$|f(y)-f(x)|^{n-q}Q(n,q) \leq |f(y)-f(x)|^{n-q}H_{n,q}(\frac{|f(z)-f(x)|}{|f(y)-f(x)|}) \leq$$

$$\leq \gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^p}) \text{ if } |y-x| \leq |z-x| \leq R \tag{10}$$

$$|f(y)-f(x)|^{n-q} - |f(z)-f(x)|^{n-q} \leq \frac{\gamma(C_{x,R,p}(2+\frac{2|y-x|}{|z-x|})^p)}{C(n,q)}$$

$$\text{if } \frac{|z-x|}{2} \leq |y-x| \leq |z-x| \leq \frac{R}{6} \tag{11}$$

It also results that

$$Q(n,q)(|f(y)-f(x)|^{n-q}-|f(z)-f(x)|^{n-q})\leq\gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^p})$$

if $|y-x|\leq|z-x|\leq R$ *and* $|f(y)-f(x)|\geq|f(z)-f(x)|$ (12)

and

$$\frac{Q(n,q)(n-q)3^{\frac{n-q-2}{2}}}{2^n}\frac{|f(y)-f(x)|^2}{|f(z)-f(x)|^{q+2-n}}\leq\gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^p})$$

if $|y-x|\leq|z-x|\leq R$ *and* $|f(z)-f(x)|\geq|f(y)-f(x)|$. (13)

Proof. Relation (9) was proved in Theorem 2 in [13]. Suppose that $|y-x|\leq|z-x|\leq R$. Using Theorem 1 and Lemma A, we find that $|f(y)-f(x)|^{n-q}Q(n,q)\leq|f(y)-f(x)|^{n-q}H_{n,q}(\frac{|f(z)-f(x)|}{|f(y)-f(x)|})\leq\gamma(M^p_\omega(\Gamma_{x,|y-x|,|z-x|}))\leq\gamma(\frac{C_{x,R,p}e^p l_p}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^p})$ and relation (10) is proved.

Let now $\frac{|z-x|}{2}\leq|y-x|\leq|z-x|\leq\frac{R}{3}$ and let $\omega\in S(x,2|y-x|)$ be such that $L(x,f,2|y-x|)=|f(w)-f(x)|$. Then $|f(y)-f(x)|\leq|f(w)-f(x)|$ and using (9), we see that $|f(y)-f(x)|^{n-q}-|f(z)-f(x)|^{n-q}\leq|f(w)-f(x)|^{n-q}-|f(z)-f(x)|^{n-q}\leq\frac{\gamma(C_{x,R,p}(2+\frac{2|y-x|}{|z-x})^p)}{C(n,q)}$ and relation (11) is now proved.

Suppose now that $|y-x|\leq|z-x|\leq R$ and $|f(y)-f(x)|\geq|f(z)-f(x)|$. We see from [4] that $H_{n,q}(r)\geq Q(n,q)(1-r^{n-q})$ if $0<r<1$, hence $Q(n,q)(|f(y)-f(x)|^{n-q}-|f(z)-f(x)|^{n-q})\leq|f(y)-f(x)|^{n-q}H_{n,q}(\frac{|f(z)-f(x)|}{|f(y)-f(x)|})$. We apply now relation (10) and we obtain relation (12).

Suppose now that $|y-x|\leq|z-x|\leq R$ and $|f(y)-f(x)|\leq|f(z)-f(x)|$. We see from [4] that $H_{n,q}(r)\geq\frac{Q(n,q)}{2^n}((r^2+2)^{\frac{n-q}{2}}-r^{n-q})$ for every $r\geq 1$. Let $b=|f(z)-f(x)|$ and $a=|f(y)-f(x)|$. Then $|f(y)-f(x)|^{n-q}H_{n,q}(\frac{|f(z)-f(x)|}{|f(y)-f(x)|})=a^{n-q}H_{n,q}(\frac{b}{a})\geq\frac{Q(n,q)}{2^n}((b^2+2a^2)^{\frac{n-q}{2}}-b^{n-q})=\frac{Q(n,q)2a^2(n-q)}{2^n(\sqrt{b^2+2a^2}+b)}c^{n-q-1}$(by the theorem of Lagrange and $b<c<\sqrt{b^2+2a^2}$) $\geq\frac{Q(n,q)(n-q)2a^2}{2^n(b^2+2a^2)^{\frac{q+2-n}{2}}}\geq\frac{Q(n,q)(n-q)}{2^n3^{\frac{q+2-n}{2}}}\frac{a^2}{b^{q+2-n}}=\frac{Q(n,q)(n-q)}{2^n3^{\frac{q+2-n}{2}}}\frac{|f(y)-f(x)|^2}{|f(z)-f(x)|^{q+2-n}}$. We apply now relation (10) and we prove relation (13). □

Theorem 5. *Let* $n\geq 2$, $D,D^{‘}$ *be domains in* $\mathbb{R}^n$, *let* $f:D\to D^{‘}$ *be a homeomorphism satisfying condition* (N) *and such that its inverse is* ACL^n *and suppose that* $K_{I,n}(f)\in BMO(D)$. *Let* $x\in D$ *and* $R>0$ *be such that* $\overline{B}(x,R)\subset D$ *and let* $M=\sup_{B(x,r)\subset D}⨍_{B(x,r)}|K_{I,n}(f)(z)-K_{I,n}(f)_{B(x,r)}|dz$. *Then we have the following inequalities:*

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|}\leq(1/H_n^{-1}(\frac{V_n e^n((M+⨍_{B(x,|z-x|)}K_{I,n}(f)(u)du)l_n+Me^n t_n)}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^n}))$$

if $n \geq 3$ (14)

$$\frac{|f(y)-f(x)|}{f(z)-f(x)|} \leq (1/H_2^{-1}(\frac{V_2e^2(M+\fint_{B(x,|z-x|)} K_{I,n}(f)(u)du)l_2}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^2}+\frac{MV_2e^4}{\ln\ln(\frac{e|z-x|}{|y-x|})}))$$

if $n = 2$. (15)

Proof. We see from Lemma B that $M_n(f(\Gamma)) \leq M^n_{K_{I,n}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we apply now Theorem 2. □

Theorem 6. *Let $n \geq 2$, $D, D^{‘}$ be domains in $\mathbb{R}^n$ and let $f : D \to D^{‘}$ be a homeomorphism satisfying condition (N) and such that its inverse is ACL^n. Then, if $\overline{B}(x,R) \subset D$ and $C_{x,R,n} = \sup\limits_{0<r<R} \int_{B(x,r)} K_{I,n}(f)(z)dz/r^n < \infty$, we have the following inequalities:*

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq (1/H_n^{-1}(\frac{C_{x,R,n}e^n l_n}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^n}))$$

if $|y-x| \leq |z-x| \leq \dfrac{R}{2}$ (16)

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \frac{\exp(C_{x,R,n}(2+\frac{|y-x|}{|z-x|})^n/C(n))}{H_n^{-1}(\frac{C_{x,R,n}e^n l_n}{(\ln\ln(2e))^n})}$$

if $\dfrac{|z-x|}{2} \leq |y-x| \leq |z-x| \leq \dfrac{R}{6}$ (17)

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \exp(\frac{C_{x,R,n}(2+\frac{|y-x|}{|z-x|})^n}{C(n)}) \text{ if } |z-x| \leq |y-x| \leq \frac{R}{3} \quad (18)$$

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \exp(\frac{C_{x,R,n}(1+\frac{|y-x|}{|z-x|})^n}{C(n)}) \text{ if } |z-x| \leq \frac{|y-x|}{2} \leq \frac{R}{6}. \quad (19)$$

Proof. We see from Lemma B that $M_n(f(\Gamma)) \leq M^n_{K_{I,n}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we apply Theorem 3. □

Theorem 7. *Let $n \geq 2$, $p > n$, $D, D^{‘}$ be domains in $\mathbb{R}^n$ and let $f : D \to D^{‘}$ be a homeomorphism satisfying condition (N) and such that its inverse is ACL^n and suppose that $C = (\int_D K_{I,n}(f)(x)^{p/(p-n)}dx)^{\frac{p-n}{p}} < \infty$. Then, if $\overline{B}(x,R) \subset D$, we have the following inequalities:*

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \exp(\frac{C}{C(n)}(\frac{V_n(|y-x|+2|z-x|)^n}{|z-x|^p})^{\frac{n}{p}})$$

if $|z-x| \leq |y-x| \leq \dfrac{R}{3}$ (20)

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq (1/H_n^{-1}(C(\frac{nV_n}{((\frac{p-1}{p-n})(|z-x|^{\frac{p-n}{p-1}}-|y-x|^{\frac{p-n}{p-1}}))^{p-1}})^{\frac{n}{p}}))$$

if $|z-x| \leq |y-x| \leq \dfrac{R}{3}$ (21)

Proof. Relation (20) is proved in Theorem 4 in [13]. We see from Lemma B that $M_n(f(\Gamma)) \leq CM_p(\Gamma)^{\frac{n}{p}}$ for every $\Gamma \in A(D)$. Using Theorem 1 and Prop. 18, page 535 in [3], we find that $H_n(\frac{|f(z)-f(x)|}{|f(y)-f(x)|}) \leq CM_p(\Gamma_{x,|y-x|,|z-x|})^{\frac{n}{p}} \leq C(\frac{nV_n}{((\frac{p-1}{p-n})(|z-x|^{\frac{p-n}{p-1}}-|y-x|^{\frac{p-n}{p-1}}))^{p-1}})^{\frac{n}{p}}$ and relation (21) is proved. □

Theorem 8. *Let $n \geq 2$, $n-1 < q < n$, $D, D^{'}$ be domains in $\mathbb{R}^n$ and let $f : D \to D^{'}$ be a homeomorphism satisfying condition (N) and such that its inverse is ACL^q. Then, if $\overline{B}(x,R) \subset D$ and*

$$C_{x,R,q} = \sup_{0<r<R} \int\limits_{B(x,R)} K_{I,q}(f)(u)du/r^q < \infty,$$

we have the following inequalities:

$$|f(y)-f(x)|^{n-q} - |f(z)-f(x)|^{n-q} \leq \frac{C_{x,R,q}(2+\frac{|y-x|}{|z-x|})^q}{C(n,q)}$$

if $|z-x| \leq |y-x| \leq \dfrac{R}{3}$ (22)

$$|f(y)-f(x)|^{n-q} - |f(z)-f(x)|^{n-q} \leq \frac{C_{x,R,q}(2+\frac{2|y-x|}{|z-x|})^q}{C(n,q)}$$

if $\dfrac{|z-x|}{2} \leq |y-x| \leq |z-x| \leq \dfrac{R}{6}$ (23)

$$|f(y)-f(x)|^{n-q}Q(n,q) \leq |f(y)-f(x)|^{n-q}H_{n,q}(\frac{|f(z)-f(x)|}{|f(y)-f(x)|}) \leq \frac{C_{x,R,q}e^q l_q}{(\ln\ln(\frac{e|z-x}{|y-x|}))^q}$$

if $|y-x| \leq |z-x| \leq R$ (24)

We also have

$$Q(n,q)(|f(y)-f(x)|^{n-q} - |f(z)-f(x)|^{n-q}) \leq \frac{C_{x,R,q}e^q l_q}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^q}$$

if $|y-x| \leq |z-x| \leq R$ *and* $|f(y)-f(x)| \geq |f(z)-f(x)|$ (25)

and

$$\frac{Q(n,q)(n-q)}{2^n 3^{\frac{q+2-n}{2}}} \frac{|f(y)-f(x)|^2}{|f(z)-f(x)|^{q+2-n}} \leq \frac{C_{x,R,q}e^q l_q}{(\ln\ln(\frac{e|z-x|}{|y-x|}))^q}$$

if $|y-x| \le |z-x| \le R$ *and* $|f(z)-f(x)| \ge |f(y)-f(x)|$. (26)

Proof. We see from Lemma B that $M_q(f(\Gamma)) \le M^q_{K_{I,q}(f)}(\Gamma)$ for every $\Gamma \in A(D)$ and we apply now Theorem 4. □

Theorem 9. *Let $n \ge 2$, $n-1 < q < n$, $D, D^{`}$ be domains in $\mathbb{R}^n$ and let $f : D \to D^{`}$ be a homeomorphism satisfying condition (N) and such that its inverse is ACL^q and suppose that $C = (\int\limits_D K_{I,q}(f)(x)^{n/(n-q)}dx)^{\frac{n-q}{n}} < \infty$. Then, if $\overline{B}(x,R) \subset D$, we have the following inequalities:*

$$|f(y)-f(x)|^{n-q} - |f(z)-f(x)|^{n-q} \le \frac{CV_n^{q/n}}{C(n,q)}(2+\frac{|y-x|}{|z-x|})^q$$

if $|z-x| \le |y-x| \le \frac{R}{3}$ (27)

$$|f(y)-f(x)|^{n-q}Q(n,q) \le |f(y)-f(x)|^{n-q}H_{n,q}(\frac{|f(z)-f(x)|}{|f(y)-f(x)|}) \le$$

$$\le C(\frac{\omega_{n-1}}{(\ln(\frac{|z-x|}{|y-x|}))^{n-1}})^{q/n} \text{ if } |y-x| \le |z-x| \le R \quad (28)$$

$$|f(y)-f(x)|^{n-q} - |f(z)-f(x)|^{n-q} \le \frac{CV_n^{q/n}}{C(n,q)}(2+\frac{2|y-x|}{|z-x|})^q$$

if $\frac{|z-x|}{2} \le |y-x| \le |z-x| \le R.$ (29)

It also results that

$$|f(y)-f(x)|^{n-q} - |f(z)-f(x)|^{n-q} \le \frac{C}{Q(n,q)}(\frac{\omega_{n-1}}{(\ln(\frac{|z-x|}{|y-x|}))^{n-1}})^{q/n}$$

if $|y-x| \le |z-x| \le R$ *and* $|f(y)-f(x)| \ge |f(z)-f(x)|$ (30)

and

$$\frac{Q(n,q)(n-q)}{2^n 3^{\frac{q+2-n}{2}}}\frac{|f(y)-f(x)|^2}{|f(z)-f(x)|^{q+2-n}} \le (\frac{\omega_{n-1}}{(\ln(\frac{|z-x|}{|y-x|}))^{n-1}})^{q/n}$$

if $|y-x| \le |z-x| \le R$ *and* $|f(z)-f(x)| \ge |f(y)-f(x)|$. (31)

Proof. Relation (27) was proved in Theorem 4 in [13]. We see from Lemma B that $M_q(f(\Gamma)) \le CM_n(\Gamma)^{q/n}$ for every $\Gamma \in A(D)$. Using Theorem 1 and taking $\gamma(t) = Ct^{q/n}$ for $t \ge 0$, we find that $|f(y)-f(x)|^{n-q}Q(n,q) \le |f(y)-f(x)|^{n-q}H_{n,q}(\frac{|f(z)-f(x)|}{|f(y)-f(x)|}) \le CM_n(\Gamma_{x,|y-x|,|z-x|})^{q/n} = C(\frac{\omega_{n-1}}{(\ln(\frac{|z-x|}{|y-x|}))^{n-1}})^{q/n}$ and relation (28) is now proved. Using relation (28) and arguing as in Theorem 4, we prove relations (30) and (31).

Let now $\frac{|z-x|}{2} \le |y-x| \le |z-x| \le R$ and let $\omega \in S(x, 2|y-x|)$ be such that $L(x,f,2|y-x|) = |f(w)-f(x)|$. Then $|f(y)-f(x)| \le |f(w)-f(x)|$

and using relation (27), we see that $|f(y)-f(x)|^{n-q}-|f(z)-f(x)|^{n-q}\leq |f(w)-f(x)|^{n-q}-|f(z)-f(x)|^{n-q}\leq \frac{CV_n^{q/n}}{C(n,q)}(2+\frac{2|y-x|}{|z-x|})^q$ and relation (29) is also proved. □

References

[1] Cabiria Andreian Cazacu, O kvasikonformnyh otobrazhenyah, *Dokl. Akad. Nauk SSSR*, **126(2)** (1959), 235-238.

[2] Cabiria Andreian Cazacu, Moduli inequalities for quasiregular mappings, *Ann. Acad. Sci. Fenn. Math.*, **2** (1976), 17-28, http://dx.doi.org/10.5186/aasfm.1976.0204.

[3] P. Caraman, Relations between p-capacity and p-module (I), *Rev. Roumaine Math. Pures Appl.*, **39**, *6* (1994), 509-553.

[4] P. Caraman, Relations between p-capacity and p-module (II), *Rev. Roumaine Math. Pures Appl.*, **39**, *6* (1994), 555-577.

[5] M. Cristea, Mappings of finite distortion. Boundary extension, *Rev. Roumaine Math. Pures Appl.*, **51**, *5-6* (2006), 607-631.

[6] M. Cristea, Mappings of finite distortion, Zoric's theorem and equicontinuity results, *Rev. Roumaine Math. Pures Appl.*, **52**, *5* (2007), 539-554.

[7] M. Cristea, Local homeomorphisms having local ACL^n inverses, *Complex Var. Elliptic Equ.*, **53**, *1* (2008), 77-99, http://dx.doi.org/10.1080/17476930701666924.

[8] M. Cristea, Open, discrete mappings having local ACL^n inverses, *Complex Var. Elliptic Equ.*, **55**, *1-3* (2010), 61-90, http://dx.doi.org/10.1080/17476930902998985.

[9] M. Cristea, Dilatations of homeomorphisms satisfying some modular inequalities, *Rev. Roumaine Math. Pures Appl.*, **56**, *4* (2011), 275-282.

[10] M. Cristea, Mappings satisfying some modular inequalities, *Rev. Roumaine Math. Pures Appl.*, **57**, *2* (2012), 117-132.

[11] M. Cristea, On generalized quasiconformal mappings, *Complex Var. Elliptic Equ.*, **59**, *2* (2014), 232-246, http://dx.doi.org/10.1080/17476933.2012.731398.

[12] M. Cristea, On generalized quasiregular mappings, *Complex Var. Elliptic Equ.*, **58**, *12* (2013), 1745-1764, http://dx.doi.org/10.1080/17476933.2012.713946.

[13] M. Cristea, Some metric relations of the homeomorphisms satisfying generalized modular inequalities (I), *Math. Rep. (Bucur.)*, **15 (65)**, *4* (2013), 387-396.

[14] I. Holopainen and P. Pankka, Mappings of finite distortion: Global homeomorphism theorem, *Ann. Acad. Sci. Fenn. Math.*, **29** (2004), 135-151.

[15] P. Koskela and J. Onninen, Mappings of finite distortion: Capacity and modulus inequalities, *J. Reine Angew. Math.*, **599** (2006), 1-26, http://dx.doi.org/10.1515/crelle.2006.076.

[16] P. Koskela, J. Onninen and K. Rajala, Mappings of finite distortion: Injectivity radius of local homeomorphisms, *Future Trends in Geometric Function Theory*, Vol. 92, Rep. Univ. Jyväskylä, Dept. Mat. Stat., Univ. of Jyväskylä (2002), 169-174.

[17] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, Mappings of finite length distortion, *J.Anal. Math.*, **93** (2004), 215-236, http://dx.doi.org/10.1007/bf02789308.

[18] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, On Q-homeomorphisms, *Ann. Acad. Sci. Fenn. Math.*, **30** (2005), 49-69.

[19] O. Martio, V. Ryazanov, U. Srebro and E. Yakubov, *Moduli in Modern Mapping Theory*, Springer Monogr. Math., Springer, New York, 2009, http://dx.doi.org/10.1007/978-0-387-85588-2.

[20] K. RAJALA, Mappings of finite distortion: Removable singularities for locally homeomorphic mappings, *Proc. Amer. Math. Soc.*, **132** (2004), 3251-3258, http://dx.doi.org/10.1090/S0002-9939-04-07459-3.

[21] K. RAJALA, Mappings of finite distortion: Removability of Cantor sets, *Ann. Acad. Sci. Fenn. Math.*, **26** (2004), 269-281.

[22] Y.G. RESHETNYAK, Space Mappings with Bounded Distortion, *Transl. Math. Monogr.*, 73, AMS, Providence, RI, 1989.

[23] S. RICKMAN, *Quasiregular Mappings*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3), 26, Springer-Verlag, 1993, http://dx.doi.org/10.1007/978-3-642-78201-5.

[24] V. RYAZANOV and E. SEVOSTYANOV, Towards the theory of ring Q-homeomorphisms, *Israel J. Math.*, **168** (2008), 101-115, http://dx.doi.org/10.1007/s11856-008-1058-2.

[25] V. RYAZANOV and E. SEVOSTYANOV, Equicontinuity of mappings quasiconformal in the mean, *Ann. Acad. Sci. Fenn. Math.*, **36** (2011), 231-244, http://dx.doi.org/10.5186/aasfm.2011.3614.

[26] R. SALIMOV, On regular homeomorphisms in the plane, *Ann. Acad. Sci. Fenn. Math.*, **35** (2010), 285-289, http://dx.doi.org/10.5186/aasfm.2010.3517.

[27] R. SALIMOV, On Q-homeomorphisms with respect to p-modulus, *Ann. Univ. Buchar. Math. Ser.*, **2 (LX)** (2011), 207-213.

[28] R. SALIMOV and E. SEVOSTYANOV, The theory of shell-based Q-mappings in geometric function theory, *Sb. Math.*, **201** (2010), 909-934, http://dx.doi.org/10.1070/sm2010v201n06abeh004096.

[29] R. SALIMOV and E. SEVOSTYANOV, ACL and differentiability of the open discrete ring mappings, *Complex Var. Elliptic Equ.*, **55**, *1-3* (2010), 49-59, http://dx.doi.org/10.1080/17476930902999009.

[30] R. SALIMOV and E. SEVOSTYANOV, On inner dilatations of the mappings with unbounded characteristic, *J. Sci. Math.*, **178**, (2011), 97-107, http://dx.doi.org/10.1007/s10958-011-0529-x.

[31] E. SEVOSTYANOV, Towards a theory of removable singularities for maps with unbounded characteristics of quasiconformity, *Izv. Math.*, **74(1)** (2010), 151-165, http://dx.doi.org/10.1070/im2010v074n01abeh002483.

[32] E. SEVOSTYANOV, Liouville theorem, Piccard and Sohotkii's theorem for ting mappings, *Ukr. Math. Bull.*, **5**, *3* (2008), 366-381.

[33] J. VÄISÄLÄ, *Lectures on n- dimensional quasiconformal mappings*, Lecture Notes in Math., 229, Springer Verlag, 1971, http://dx.doi.org/10.1007/bfb0061216.

[34] M. VUORINEN, *Conformal geometry and quasiregular mappings*, Lecture Notes in Math., 1319, Springer Verlag, 1988, http://dx.doi.org/10.1007/bfb0077904.

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