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On the location of the zeros of the derivative of Dirichlet L-functions

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Dedicated to Professor Cabiria Andreian Cazacu on the occasion of her 85th birthday

Abstract - A connection between the Riemann Hypothesis and the distribution of the zeros of the derivative ζ' of the Riemann Zeta function ζ has been revealed 80 years ago, but it remained dormant until recently, when a lot of studies began to be devoted to the distribution of the zeros of the derivatives of ζ. Although the first ideas were based on geometric grounds, the recent studies have more a number-theoretical flavour. We revive in this paper the initial geometrical ideas and study the problem in the larger context of the family of Dirichlet L-functions, to which the Riemann Zeta function belongs.

Key words and phrases : Dirichlet character, Dirichlet L-function, non trivial zero, fundamental domain.

Mathematics Subject Classification (2010) **:** 30C35, 11M26.

1. Introduction

The idea of using the pre-image of the real axis *(die Methode der reellen Züge)* in the study of the location of the zeros of ζ' is present in Speiser's paper [13] who was inspired by the thesis of A. Utzinger, *Über die reellen Züge der Riemannschen Zetafunction, Zürich 1934*. However, the lack of modern computational tools did not allow him to fully benefit of the power of this method. In particular, it was impossible at that time to find the fundamental domains of the Riemann Zeta function, which are essential in the visualization of the conformal mapping realized by this function. This has been accomplished in [3]-[5] and [7]-[9]. Moreover, in [6] we succeeded to do the same thing for arbitrary Dirichlet L-functions.

Let us present the concepts we need in order to tackle this problem. A *Dirichlet character* modulo q, with q a positive integer, is a totally multiplicative periodic function χ of period q defined on $\mathbb{Z}$ (see [11], [6]). A *Dirichlet L-series* is a series of the form

$$L(s\,;\chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s}. \tag{1.1}$$

It is known (see [11]) that every Dirichlet L-series can be extended to an analytic function in the whole plane, except possibly at $s = 1$, where it can have a simple pole. Such a function is called *Dirichlet L-function*. The Riemann Zeta function is the particular Dirichlet L-function obtained for $q = 1$. If $d|q$ and χ^* is a Dirichlet character modulo d, then $\chi(n) = \chi^*(n)$ if $(n, q) = 1$ and $\chi(n) = 0$ otherwise, defines a character modulo q for which

$$L(s\,;\chi) = L(s\,;\chi^*) \prod_{p|q,p>1} \left(1 - \frac{\chi(p)}{p^s}\right), \tag{1.2}$$

where p are prime numbers. We say that χ^* *induces* χ. A Dirichlet character χ is called *primitive* if it is not induced by any other character. The formula (1.2) shows that if χ^* induces χ, then $L(s\,;\chi)$ and $L(s\,;\chi^*)$ have the same zeros, except for the zeros of the product, which are all on the imaginary axis. We consider them as trivial zeros of $L(s\,;\chi)$ and therefore $L(s\,;\chi)$ and $L(s\,;\chi^*)$ have the same non trivial zeros. We would like to do a similar classification of the zeros of the derivative $L'(s\,;\chi)$. This was straightforward for the derivative of the Riemann Zeta function for which the real zeros were considered naturally as trivial, while the others were considered non trivial. A criterion of triviality for the zeros of $L'(s\,;\chi)$ could come from the way these zeros are *generated* by the zeros of $L(s\,;\chi)$. Indeed, we have seen in [6] that for any bounded region D of the plane, we can find $r > 0$ such that the intersection of D with the pre-image of a circle γ_r centred at the origin and of radius r is a set of disjoint closed curves containing each one a unique zero of $L(s\,;\chi)$ situated in D. When r increases, these curves expand and for some value r_0 of r two of them touch each other at a point v_0. This point is a branch point of the branched Riemann surface generated by $L(s\,;\chi)$ and therefore a zero of $L'(s\,;\chi)$. By letting r increase past r_0 the two curves fuse into a unique closed curve containing both zeros and for a greater value of r some other zeros of $L'(s\,;\chi)$ are obtained and so on. For the Riemann Zeta function, the trivial zeros of ζ have generated in this way the trivial zeros of ζ' and the non trivial zeros of ζ have generated the non trivial zeros of ζ'. The situation becomes more complicated for arbitrary Dirichlet L-functions, since trivial zeros on the imaginary axis and non trivial ones can generate together zeros of its derivative. Which ones of these zeros will be called trivial and which ones non trivial? In order to device a proper criterion, we need to remember (see [6] and [8]) that all the zeros of $L(s\,;\chi)$ and $L'(s\,;\chi)$ are simple zeros. Moreover, there are unbounded strips S_k, $k \neq 0$ bounded each one by a couple of unbounded curves which are mapped bijectively by $L(s\,;\chi)$ onto the interval $(1, +\infty)$ of the real axis such that every S_k contains $j_k \geq 1$ non trivial zeros located on the critical axis $\operatorname{Re} s = 1/2$ and a number $m_k \geq 0$ of trivial zeros located on the imaginary axis. There is also a strip S_0 bounded by the same type of curves which contains all the trivial real zeros of $L(s\,;\chi)$. The zeros of $L'(s\,;\chi)$ are obtained by expanding the components of the pre-image by $L(s\,;\chi)$ of γ_r. If both components touching at a $v_{k,j}$ contain non trivial zeros of $L(s\,;\chi)$, we will say that $v_{k,j}$ is a non trivial zero of $L'(s\,;\chi)$, otherwise we will call it trivial. This definition agrees with the known definition of the (non)triviality of the zeros of the Riemann Zeta function. It is an easy exercise to check that S_k contains j_{k-1} non trivial zeros and m_k trivial zeros of $L'(s\,;\chi)$, where j_k and m_k are the non trivial and respectively trivial zeros of $L(s\,;\chi)$ from S_k. These numbers agree with those known for Dirichlet L-functions defined by primitive characters, in which case $m = 0$. They are obtained by building a complete binary tree having as leaves the zeros of $L(s\,;\chi)$ from S_k and as internal nodes the zeros of $L'(s\,;\chi)$ from S_k. It is known that if such a tree has $j_k + m_k$ leaves, it will have $j_k + m_k - 1$ internal nodes. From these $j_k - 1$ correspond to the non trivial zeros of $L'(s\,;\chi)$, as it can be seen by building separately the tree generated by the non trivial zeros of $L(s\,;\chi)$, i.e. by simply ignoring the existence of trivial zeros of $L(s\,;\chi)$. We call *progenitor* of a leaf an internal node from which one can descend to that leaf. Figure 1 below illustrates the strips S_k as well as the components of the pre-images of circles γ_r for different values of r in the case of a Dirichlet L-function defined by a complex Dirichlet character χ modulo 14, which is imprimitive and thus $L(s\,;\chi)$ has some imaginary trivial zeros. The way the zeros of $L(s\,;\chi)$ generate the zeros of $L'(s\,;\chi)$ is shown in the adjacent binary tree. We can distinguish trivial and non trivial zeros of $L(s\,;\chi)$ as well as of $L'(s\,;\chi)$.

A lot of studies (see [10]-[14]) have been devoted lately to the distribution of the zeros of the derivatives of ζ and most of them make reference to Speiser's paper without expanding on the method he used. We take in this paper a deeper look into the geometry of the pre-image of the real axis in a more general setting, namely for an arbitrary Dirichlet L-function, in order to draw conclusions related to the distribution of the zeros of its derivative.

2. The pre-image of the real axis by $L(s\,;\chi)$

The alternative notation $L(q,j,s)$ is used for a Dirichlet L-function defined by the j-th Dirichlet character modulo q when it is important to specify both of these parameters. The location of the zeros of such a function becomes obvious when coloring differently the pre–images of the negative and of the positive real half axes, namely the zeros appear at the junction of the two colors. By the Big Picard Theorem, if $x_0 \in \mathbb{R}$ is a non lacunary value of $L(q,j,s)$, then there is a countable number of points $s_n \in \mathbb{C}$ such that $L(q,j,s_n) = x_0$. The continuation along $\mathbb{R}$ from every s_n (see [2]) produces a countable number of unbounded curves, called *components* of the pre-image of $\mathbb{R}$ by $L(q,j,s)$. Sometimes, in the continuation process, branch points of the respective branched Riemann surface can be met. These are the points where $L'(q,j,s) = 0$. It is known (see [1], page 133) that in the neighborhood of such a point s_0 we have:

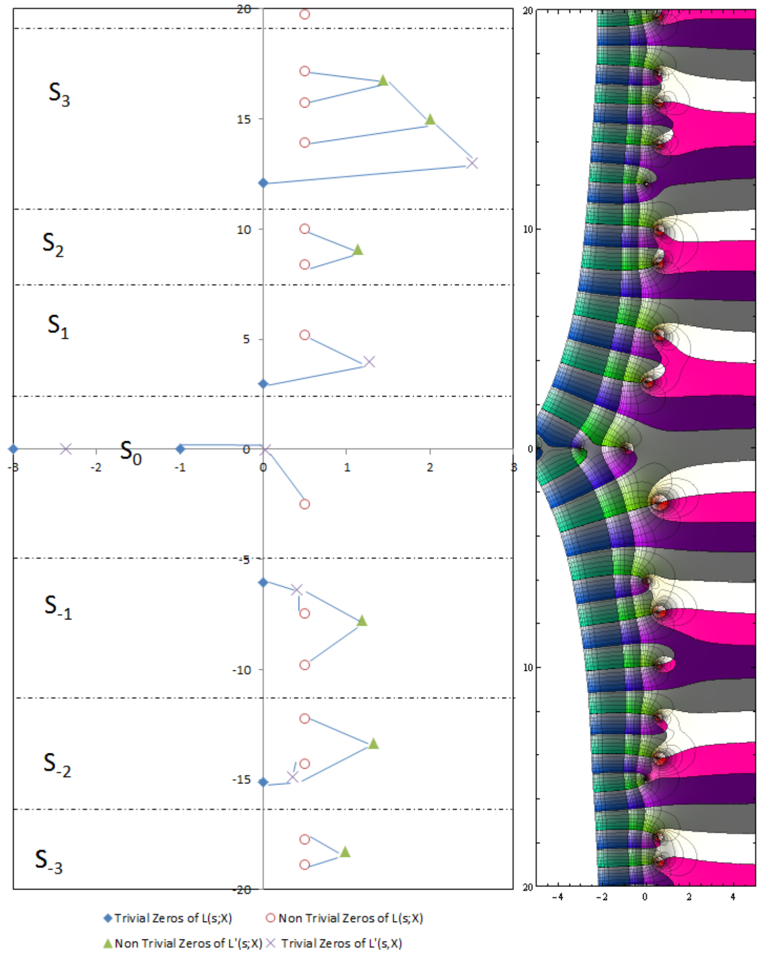

Figure 1. Trivial and non trivial zeros of $L(s\,;\chi)$ and $L'(s\,;\chi)$.

$$L(q,j,s) = L(q,j,s_0) + (s-s_0)^2 g(s), \tag{2.1}$$

where $g(s)$ is an analytic function and $g(s_0) \neq 0$. Indeed, it is known (see [6]) that the zeros of $L'(q,j,s)$ are all simple, hence two components of the pre-image of $\mathbb{R}$ by $L(q,j,s)$ touch each other at s_0 and are orthogonal.

It has been shown in [6] that for any Dirichlet L-function there are four kinds of components of the pre-image of $\mathbb{R}$. Namely, there are countable many components Γ'_k which are mapped bijectively by the function onto the interval $(1, +\infty)$. They do not intersect each other and form infinite strips S_k. Every strip S_k contains a unique component $\Gamma_{k,0}$ of the pre-image of $\mathbb{R}$ which is mapped bijectively by the function onto the interval $(-\infty, 1)$ and a finite number $j_k - 1$ of components $\Gamma_{k,j}$, $j \neq 0$ which are mapped bijectively by the function onto the whole real axis. The curves $\Gamma_{k,j}$ do not intersect each other and they do not intersect any curve Γ'_k. Every curve $\Gamma_{k,j}$ contains a unique zero $s_{k,j}$ of the L-function, which is non trivial if the function is generated by a primitive character and can be a trivial zero on the imaginary axis if the function is generated by an imprimitive one. Finally, there are those components containing trivial real zeros, which belong all to the strip S_0 and which are mapped by the function onto the whole real axis. Adjacent components touch each other orthogonally on the real axis if the character is real. If the character is not real, then they do not touch each other and every strip between two adjacent components contains a unique branch point of the function. We can reach these branch points, as well as those belonging to the strips S_k by arcs starting at points $u_{k,j}$ for which $L(u_{k,j};\chi) = 1$ or by unbounded curves on which $\lim_{\sigma->+\infty} L(\sigma + it\,;\chi) = 1$ and such that the images of these arcs and unbounded curves are the segments connecting $z = 1$ with the image by $L(s\,;\chi)$ of the respective branch points. The following is in line with Speiser's results.

Theorem 2.1. *All the non trivial zeros of $L'(s\,;\chi)$ have the real part greater than $1/2$. The trivial zeros of $L'(s\,;\chi)$ with imaginary progenitors have the real part greather than 0.*

Proof. Let $v_{k,j}$ be such a zero of $L'(s\,;\chi)$ and suppose that it is a progenitor of the non trivial zero $s_{k,j}$. Due to the color alternating rule (see [5], [6]), $v_{k,j}$ is situated outside the parabola-like curve $\Gamma_{k,j}$. It is obviously enough to study the case when $j > 0$, i.e. when $\Gamma_{k,j}$ is above $\Gamma_{k,0}$. Then $L(v_{k,j};\chi)$ belongs to the upper half plane in Figure 2, hence $0 < \alpha < \pi$, where $\alpha = \arg L(v_{k,j};\chi)$. In this paper $\arg z \in [0, 2\pi)$ is the angle made by the ray from the origin through z with the positive real half axis. The pre-image of the ray determined by $L(v_{k,j};\chi)$ contains two curves which are orthogonal at $v_{k,j}$. Indeed, it is known (see [6]) that the zeros of $L'(s\,;\chi)$ are simple, hence $v_{k,j}$ is a ramification point of order two. The angles at $v_{k,j}$ are doubled by $L(s\,;\chi)$, hence the four arcs meeting at $v_{k,j}$ make the angles $\alpha/2, \pi/2 + \alpha/2, \pi + \alpha/2$ and $3\pi/2 + \alpha/2$ with a horizontal line. The angle of $\pi/2 + \alpha/2$ made by the second arc (which ends at $s_{k,j}$) is less than the angle β made by the tangent to it at any other point and the horizontal line. If $\operatorname{Re} v_{k,j} \leq \operatorname{Re} s_{k,j}$, then there must be a point on this arc for which $\beta = \pi/2$, therefore $\pi/2 + \alpha/2 < \pi/2$, i.e. $\alpha < 0$, which is absurd and this contradiction proves the first part of

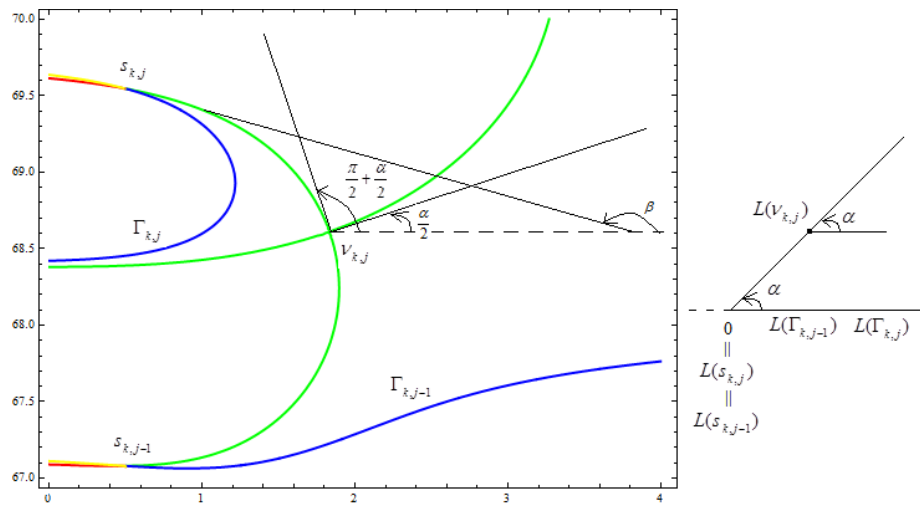

Figure 2. $\operatorname{Re} v_{k,j} > \operatorname{Re} s_{k,j}$.

the theorem. The second part is obtained taking $\operatorname{Re} s_{k,j} = 0$. Figure 2 below illustrates such a situation for $\zeta(s)$ in the strip S_7 and the non trivial zero $s_{7,1} = 1/2 + 69.54i$ of ζ as well as the corresponding zero $v_{7,1} = 1.83 + 68.61i$ of ζ'. $\square$

3. The second derivative of $L(s\,;\chi)$

Let us take now the pre-image by $L'(s\,;\chi)$ of the real axis. Some of the components are this time curves Υ'_k which bound strips Σ_k and are mapped bijectively by $L'(s\,;\chi)$ onto the interval $(-\infty, 0)$. Every strip Σ_k contains a unique component $\Upsilon_{k,0}$ of the pre-image of the real axis, which is mapped bijectively by $L'(s\,;\chi)$ onto the interval $(0, +\infty)$ and a number of components which are mapped bijectively by $L'(s\,;\chi)$ onto $\mathbb{R}$. Let us take also the pre-image by $L'(s\,;\chi)$ of a circle γ_r centered at the origin and of radius r. Since $\lim_{\sigma->+\infty} L'(\sigma + it\,;\chi) = 0$, for every $\epsilon > 0$, there is σ_t such that $\sigma > \sigma_t$ implies that $|L'(\sigma + it\,;\chi)| < \epsilon$. Therefore, for $r < \epsilon$ some components of the pre-image of γ_r must intersect the unbounded sets $\{s = \sigma + it\ |\sigma > \sigma_t\}$ for every $t \in \mathbb{R}$. In fact, we can show that there is just one such component. Indeed, by Big Picard Theorem, the pre-image of the point $z = -r$ is a countable set of points accumulating to ∞. Each one of these points belongs to a unique component of the pre-image of the negative real half axis. In other words every curve Υ'_k contains one of these points and so does every curve $\Upsilon_{k,j}$. Starting from such a point on Υ'_k and performing continuation along γ_r counterclockwise we reach the point on Υ'_{k+1} and if the continuation is clockwise, we reach the point on Υ'_{k-1}. It is obvious that in this way we visit all the points belonging to the pre-image of $z = -r$ which are situated on curves Υ'_k and the unbounded curve we obtain in this way is the unique (due to the monodromy theorem) unbounded component of the pre-image of γ_r. For an arbitrary bounded region of the plane, we can take r small enough such that the bounded components of the pre-image by $L'(s\,;\chi)$ of γ_r included in that region are disjoint closed curves containing each one a zero of $L'(s\,;\chi)$. Indeed, when $r->0$ the respective components contract each one to a point, which is a zero of $L'(s\,;\chi)$.

Theorem 3.1. *The number of zeros of $L''(s\,;\chi)$ and of $L'(s\,;\chi)$ from a given strip Σ_k is the same.*

Proof. Suppose that Σ_k contains m zeros of $L'(s\,;\chi)$. Considering them as the leaves of a binary tree whose internal nodes are obtained as the touching points of the bounded components of the pre-image of γ_r, we obtain a complete binary tree. It is known that the number of internal nods of this tree must be $m-1$. These are not all the zeros of $L''(s\,;\chi)$. One more zero is obtained as the touching point of one of these components with the unbounded component of the pre-image of γ_r. Every zero of $L''(s\,;\chi)$ in Σ_k can be obtained in this way and therefore the number of zeros of $L''(s\,;\chi)$ in Σ_k is exactly m. □

Some of the zeros of $L'(s\,;\chi)$ from Σ_k can be trivial zeros, as defined in Section 1. We apply that definition recursively for the zeros of any derivative of $L(s\,;\chi)$. Therefore, the non trivial zeros of $L''(s\,;\chi)$ are obtained as internal nodes of the complete binary tree built on the non trivial zeros of $L'(s\,;\chi)$ as leaves. The other zeros of $L''(s\,;\chi)$ from Σ_k are trivial, even if they are progenitors of some non trivial zeros of $L'(s\,;\chi)$.

Theorem 3.2. *To every zero $v_{k,j}$ of $L'(s\,;\chi)$ from Σ_k corresponds a zero $w_{k,j}$ of $L''(s\,;\chi)$ such that $\operatorname{Re} w_{k,j} > \operatorname{Re} v_{k,j}$ and $\operatorname{Im} w_{k,j} \approx \operatorname{Im} v_{k,j}$.*

Proof. For a zero $w_{k,j}$ of $L''(s\,;\chi)$, the pre-image by $L'(s\,;\chi)$ of the ray determined by $L'(w_{k,j})$ has two components orthogonal in $w_{k,j}$. If $\arg L'(w_{k,j};\chi) = \alpha$, then as in Theorem 2.1 we can deal just with the case $0 < \alpha < \pi$. The four arcs starting at $w_{k,j}$ form with a horizontal line respectively the angles $\alpha/2, \pi/2+\alpha/2, \pi+\alpha/2$ and $3\pi/2+\alpha/2$. The second arc ends in $v_{k,j}$ and the tangent to it makes with a horizontal line an angle $\beta > \pi/2+\alpha/2$. For $\operatorname{Re} w_{k,j} \leq \operatorname{Re} v_{k,j}$, there would be a point on this arc for which $\beta = \pi/2$, hence $\pi/2 > \pi/2+\alpha/2$, i.e. $\alpha < 0$, which is a contradiction, proving the first part of the theorem. For the second, we need to explain first the meaning of the sign $\approx$ used there. We have seen that $\lim_{\sigma->+\infty} L'(\sigma+it\,;\chi) = 0$ implies the existence of an unbounded component of the pre-image by $L'(s\,;\chi)$ of every circle γ_r centered at the origin and of radius r. We can take r small enough such that this component and the bounded components from a given S_k are all disjoint. Letting r increase, the bounded components expand, remaining in S_k. Indeed, due to the color alternating rule, they cannot cut Γ'_k or Γ'_{k+1}. The unbounded component moves indefinitely to the left with the increasing r, since for $\sigma + it \in \Upsilon'_k$ we have that $\lim_{\sigma->-\infty} L'(\sigma + it\,;\chi) = -\infty$. There must be a value of r for which the bounded component containing $v_{k,j}$ meets the unbounded component of the pre-image of γ_r at a point $w_{k,j}$. Since the last does not differ too much in Σ_k of a vertical line, this explains the fact that the imaginary parts of $v_{k,j}$ and $w_{k,j}$ do not differ too much. If we continue to increase r the respective curves will fuse into a unique unbounded curve leaving both $v_{k,j}$ and $w_{k,j}$ at the right. □

Let us notice that there is a hint of the correspondence between $s_{k,j}, v_{k,j}$ and $w_{k,j}$ for the Riemann Zeta function in [14], Figure 1, and Table II, except that the idea of a mysterious *bouncing effect* (see [14], p. 680) is hiding its true nature. Moreover, the existence of the non trivial zero $-0.36+3.59i$ of ζ'', as Spira notices, contradicts either the generality of this correspondence, or Speiser's results. Indeed, either this zero does not correspond to a non trivial zero of ζ', contradicting the generality of the respective correspondence, or there is a non trivial zero of ζ' at the left of the critical line and therefore Speiser's result is not true, admitting the Riemann Hypothesis. However, this correspondence concerns the non trivial zeros of ζ, ζ' and ζ'' and for a meaningful discussion on this topic, we should first clarify the nature of the zero $-0.36 + 3.59i$. We anticipate by saying that it is a trivial zero of ζ'' and the whole problem can be now looked upon from a different perspective. It is also worth to compare at this moment the effectiveness of the number-theoretical method used by Spira and that of the geometric function theory consistently used by us in [3]-[9]. The data regarding the non trivial zeros of ζ and ζ' from [14], Figure 1 are obtained in [4] and [8] by taking the pre-image by ζ of the real axis, as well as of some circles γ_r centered at the origin. The first pre-image partitions the complex plane into strips S_k in which the second pre-image operates locating the zeros of ζ'. In every strip S_k the number of these zeros is one unit less than those of ζ. The reason is the way they appear as the internal nodes of a complete binary tree whose leaves are the zeros of ζ contained by the respective strip. This difference explains the gaps noticed by Spira and labelled bouncing. Moreover, starting now with the zeros of ζ' from a given strip S_k, the color matching rule established in [7] allows us to substitute Σ_k with S_k and obtain in the same way the zeros of ζ'' situated in S_k, except that this time there is an unbounded component of the pre-image of γ_k, creating a zero of ζ'', such that the number of zeros of ζ'' in S_k is the same as that of ζ'. Moreover, it appears that these zeros are at the intersection of the unbounded component of the pre-image of γ_r with the bounded ones, which explains the fact noticed by Spira that $\operatorname{Im} w_{k,j}$ is approximately the same with $\operatorname{Im} v_{k,j}$. Having this in

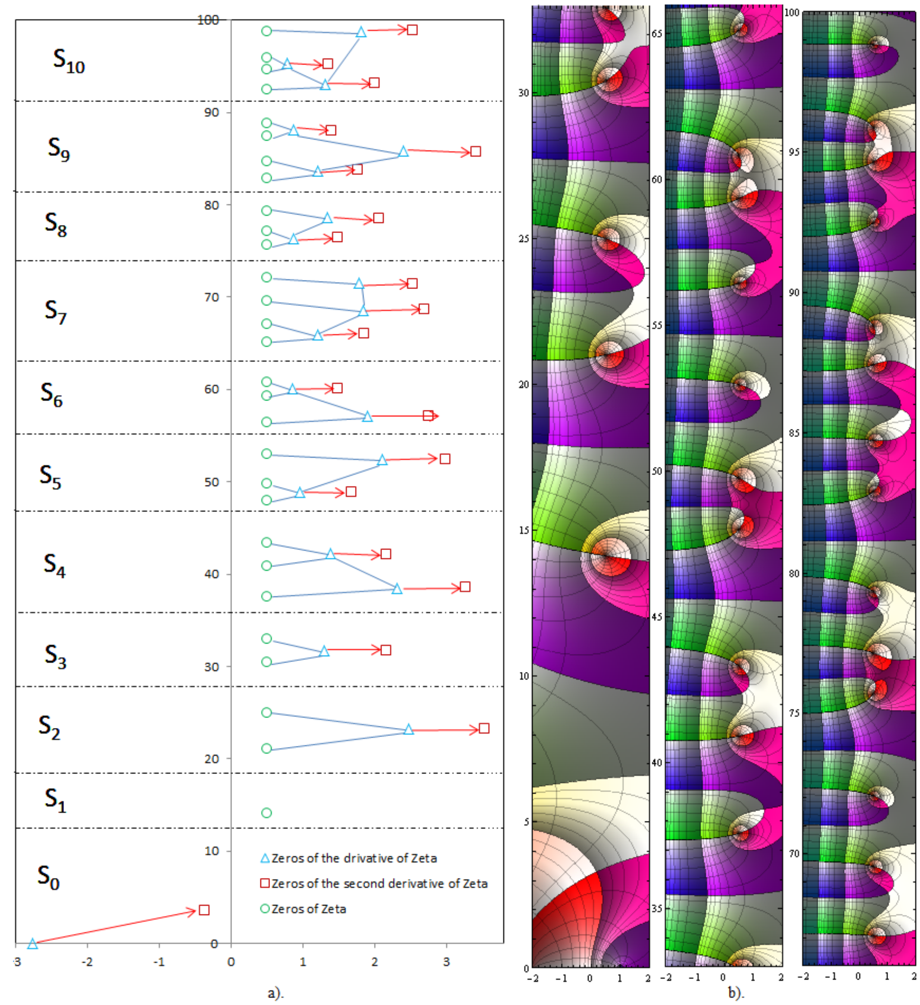

Figure 3. The correspondence of the zeros of ζ, ζ' and ζ''.

mind, we can complete, as shown in Figure 3a below, Spira's Figure 1 ([14], p. 681), by indicating the strips S_k, the corresponding complete binary trees of the zeros of ζ and ζ', as well as the correspondence $v_{k,j} -> w_{k,j}$.

The Figure 3 above shows the non trivial zeros of ζ, ζ' and ζ'' as calculated by Spira, but also as deduced by our geometric method. We state as a theorem the surprising result regarding the zero $-0.36 + 3.59i$.

Theorem 3.3. *The function ζ'' has two non real trivial zeros symmetric with respect to the real axis.*

Proof. Indeed, a trivial zero of ζ'' is obtained when two components of a pre-image by $\zeta'(s)$ of a ray cross each other and at least one of them ends in

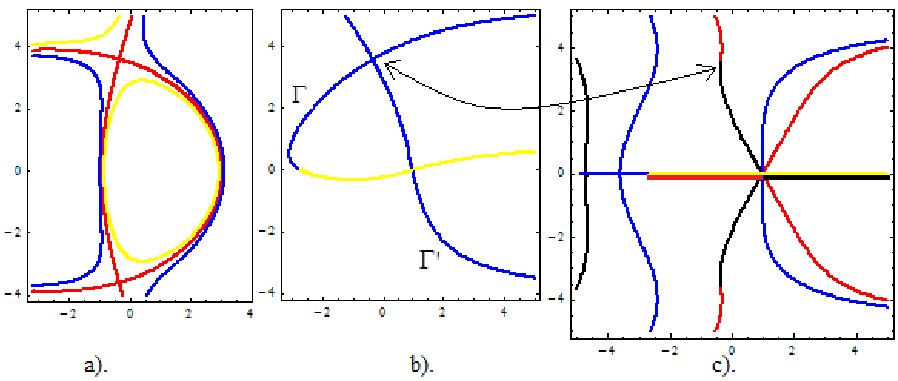

Figure 4. Non real trivial zeros of ζ''.

a trivial zero of ζ'. This is exactly the situation with the zero $-0.36+3.59i$ of ζ''. Indeed, the pre-image of the ray passing through $\zeta'(-0.36+3.59i)$ contains two curves passing through the pole $s=1$ and crossing each other once more at $-0.36+3.59i$ (see Figure 4b). One of the components Γ of this pre-image passes through the first trivial zero of ζ'. Let us notice that $\lim_{\sigma->+\infty}\zeta'(\sigma+it)=0$ when $\sigma+it\in\Gamma$. The other component Γ' stretches for σ from -∞ to $+\infty$ and $\lim_{\sigma->+\infty}\zeta'(\sigma+it)=0$ and $\lim_{\sigma->-\infty}\zeta'(\sigma+it)=\infty$ when $\sigma+it\in\Gamma'$. We notice also that the same zero $-0.36+3.59i$ is obtained as a touching point of two components of the pre-image by $\zeta'(s)$ of the circle γ_r of radius $r=|\zeta'(-0.36+3.59i)|$ and centered at the origin, one bounded and the other one unbounded (see Figure 4a). We varied slightly the value of r to show the way these components approach each other, touch each other and then fuse to a unique unbounded component. Figure 4c reveals the existence of the trivial zero $-0.36-3.59i$ of ζ''. It also illustrates the fact that $s=1$ is a double pole for ζ' (blue and yellow) and a triple pole for ζ'' (red and black). □

In our knowledge, the question whether the zeros of $L^{(n)}(s\,;\chi)$ have the real part as big as we want for n big enough, was not answered yet. Spira's affirmation that *it seems probable that the zeros of* $\zeta^{(k)}(s)$ *will have real parts tending to* ∞ *with* k has no solid support in his paper and we did not find a similar affirmation elsewhere.

All we can say based on Theorem 3.2 is that if σ_n+it_n is a zero of $L^{(n)}(s\,;\chi)$, then there is a zero $\sigma_{n+1}+it_{n+1}$ of $L^{(n+1)}(s\,;\chi)$ such that $\sigma_{n+1}>\sigma_n$.

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