Annals of the University of Bucharest (mathematical series)
5 (LXIII) (2014), 7–20

The category of semi-simplicial analytic spaces

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To professor Cabiria Andreian Cazacu on her 85^{th} birthday

Abstract - The purpose of this note is to study the fundamental operations of Grothendieck on the categories of analytic modules defined over semi-simplicial analytic spaces (i.e. families of analytic spaces indexed by simplicial complexes, with compatible connecting morphisms). As main applications we show how the direct image functors allow one to construct the dualizing complex as well as natural Dolbeault resolutions on complex spaces with singularities.

Key words and phrases : semi-simplicial analytic space; direct image; Čech complex; Dolbeault resolution; dualizing complex.

Mathematics Subject Classification (2010) **:** 32C15, 32C22, 32C37, 32K15.

1. Introduction

A semi-simplicial analytic space is a contravariant functor from a simplicial complex (seen as a category) to the category of analytic spaces, or, equivalently, a family of analytic spaces indexed by the simpleces of a simplicial complex, together with a family of compatible connecting morphisms. Semi-simplicial (s.s.) analytic spaces and the corresponding analytic modules appeared, for instance, in Forster - Knorr [4] for the proof of Grauert's direct image theorem, in Verdier [12] for the introduction of the natural topology on the global hypercohomology groups of complexes of analytic sheaves with coherent cohomology, in Ramis, Ruget [9] for the proof of relative analytic duality, in Flenner [3] and Bănică, Putinar, Schumacher [1] for computations linked to deformation theory.

The purpose of this note is to extend the fundamental operations of Grothendieck to analytic modules over s.s.analytic spaces. While extension of the inverse image or of the tensor product are straightforward (i.e. are made componentwise), the extensions for the direct image and the direct image with proper supports are more involved. Thus, in general, the direct image associates to a module over the source s.s.analytic space a complex of modules bounded below. Similarly, the direct image with proper supports associates to a comodule a complex of comodules bounded above.

There are several interesting applications. Thus, using the direct image, one can "link together" local Dolbeault resolutions to get a Dolbeault resolution on a complex space with singularities.

In a similar manner one "links together" local dualizing complexes (given by Dolbeault resolutions with currents in local closed embeddings in manifolds) to obtain a representative for the dualizing complex on any analytic space. The first construction of the dualizing complex in analytic geometry was done by Ramis, Ruget [8] using the Cousin complex and techniques of algebraic geometry; there exists another construction due to Fouché [5] which also uses the Cousin complex and a construction based on the Godement resolution due to Golovin [6]. Let us remark that the representatives for the Dolbeault resolution and the dualizing complex constructed here have good topological properties and allow, for instance, proofs for the absolute duality theorems similar to that originally given by Serre [11]. For details see [2].

Let us mention also that the Čech complex of an analytic sheaf relative to an open covering of an analytic space appears naturally as a direct image.

2. Semi-simplicial analytic spaces

2.1. Semi-simplicial objects

For the reader's convenience we start by reminding several well-known facts.

Let $(I, \mathcal{S})$ be a simplicial complex, i.e. I is an arbitrary set and $\mathcal{S}$ is a family of non-empty finite parts of I, such that:

1. $\{i\} \in \mathcal{S}$ for all $i \in I$
2. if $\alpha' \subset \alpha \in \mathcal{S}$ then $\alpha' \in \mathcal{S}$.

If $\alpha \in \mathcal{S}$ we denote by $|\alpha| = Card(\alpha) - 1$ the length of the simplex α. Recall that $\dim((I, \mathcal{S})) = \sup\{|\alpha| \mid \alpha \in \mathcal{S}\}$.

A morphism of simplicial complexes $f : (I, \mathcal{S}) \to (J, \mathcal{T})$ is simply a mapping $f : I \to J$ such that $f(\alpha) \in \mathcal{T}$. whenever $\alpha \in \mathcal{S}$.

If $K(pt)$ is the simplicial complex over the set with one element $\{pt\}$ then denote by $a_{\mathcal{S}} : (I, \mathcal{S}) \to K(pt)$ the only morphism possible (induced by the unique mapping $I \to \{pt\}$).

Definition 2.1.

1. *Let $\mathcal{C}$ be a category. A semi-simplicial (s.s.) system of objects in $\mathcal{C}$ relative to the simplicial complex $(I, \mathcal{S})$ consists of:*

   - *a family $(X_\alpha)_{\alpha \in \mathcal{S}}$ of objects in $\mathcal{C}$*
   - *a family $(\rho_{\alpha\beta})_{\alpha \subset \beta}$ of connecting morphisms, $\rho_{\alpha\beta} : X_\beta \to X_\alpha$, such that $\rho_{\alpha\alpha} = id$ for $\alpha \in \mathcal{S}$, and $\rho_{\alpha\beta} \circ \rho_{\beta\gamma} = \rho_{\alpha\gamma}$ whenever $\alpha \subseteq \beta \subseteq \gamma$.*

2. *Let $f : (I, \mathcal{S}) \to (J, \mathcal{T})$ be a morphism of simplicial complexes and let $\mathcal{X} = ((X_\alpha)_{\alpha\in\mathcal{S}}, (\rho_{\alpha\beta})_{\alpha\subset\beta})$, $\mathcal{Y} = ((Y_\gamma)_{\gamma\in\mathcal{T}}, (\rho'_{\gamma\delta})_{\gamma\subset\delta})$ be s.s.systems of objects in $\mathcal{C}$ relative to $(I, \mathcal{S})$, respectively $(J, \mathcal{T})$. A morphism $F : \mathcal{X} \to \mathcal{Y}$ over f consists of a family of morphisms in $\mathcal{C}$, $(F_\alpha)_{\alpha\in\mathcal{S}}$, $F_\alpha : X_\alpha \to Y_{f(\alpha)}$, such that the following diagram commutes for each $\alpha \subseteq \beta$:*

$$\begin{array}{ccc} X_\beta & \xrightarrow{F_\beta} & Y_{f(\beta)} \\ \downarrow{\scriptstyle \rho_{\alpha\beta}} & & \downarrow{\scriptstyle \rho'_{f(\alpha)f(\beta)}} \\ X_\alpha & \xrightarrow{F_\alpha} & Y_{f(\alpha)} \end{array}$$

If the simplicial complex is clear from the context we shall omit mentioning it.

If $\mathcal{C}$ is the category of analytic spaces (in this paper analytic space will always mean complex analytic space) then we shall say for short s.s.analytic space instead of s.s.system of analytic spaces. Let $\mathcal{X} = ((X_\alpha)_{\alpha\in\mathcal{S}}, (\rho_{\alpha\beta})_{\alpha\subset\beta})$ be a s.s.analytic space. Here X_α is short for $(X_\alpha, \mathcal{O}_\alpha)$, where $\mathcal{O}_\alpha$ denotes the sheaf of holomorphic sections of X_α, and $\rho_{\alpha\beta}$ is short for $(\rho_{\alpha\beta}, \rho^1_{\beta\alpha})$ where $\rho_{\alpha\beta} : X_\beta \to X_\alpha$ is the topological part and $\rho^1_{\beta\alpha} : \mathcal{O}_\alpha \to \rho_{\alpha\beta *}(\mathcal{O}_\beta)$ is the section level part. If X_α is a complex manifold for all $\alpha \in \mathcal{S}$, then $\mathcal{X}$ will be called a s.s.complex manifold.

Example 2.1. An analytic space can be regarded as a s.s.analytic space relative to $K(pt)$, the simplicial complex constructed over the index set with one element.

Example 2.2. Let X be an analytic space and $\mathcal{U} = (U_i)_{i\in I}$ an open covering of X. One associates to $\mathcal{U}$ the simplicial complex $(I, \mathcal{N}(\mathcal{U}))$, where $\mathcal{N}(\mathcal{U})$ denotes the nerve of $\mathcal{U}$, and the s.s.analytic space relative to $(I, \mathcal{N}(\mathcal{U}))$, $\mathfrak{U} = ((U_\alpha)_{\alpha\in\mathcal{N}(\mathcal{U})}, (i_{\alpha\beta})_{\alpha\subset\beta})$, where U_α denotes, as usual, the intersection $\bigcap_{i\in\alpha} U_i$, and $i_{\alpha\beta} : U_\beta \to U_\alpha$ is the natural inclusion. Moreover there is a natural morphism of s.s.analytic spaces $i : \mathfrak{U} \to X$ with components $i_\alpha : U_\alpha \to X$.

Example 2.3. Let $(I, \mathcal{S})$ be a simplicial complex and $(X_i)_{i\in I}$ a family of analytic spaces. For $\alpha \in \mathcal{S}$ let $X_\alpha = \prod_{i\in\alpha} X_i$. Then $\mathcal{X} = ((X_\alpha)_{\alpha\in\mathcal{S}}, (p_{\alpha\beta})_{\alpha\subset\beta})$ is a s.s.analytic space, where $p_{\alpha\beta} : X_\beta \to X_\alpha$ is the natural projection

2.2. Embedding atlases

Let $i : X \hookrightarrow D$ be an closed embedding of the analytic space X in the complex manifold D. We call (X, i, D) an embedding triple. A morphism of embedding triples $(f, \tilde{f}) : (X_1, i_1, D_1) \to (X_2, i_2, D_2)$ is a pair of morphisms of analytic spaces such that the following diagram commutes:

$$\begin{array}{ccc} X_1 & \xrightarrow{f} & X_2 \\ \downarrow{\scriptstyle i_1} & & \downarrow{\scriptstyle i_2} \\ D_1 & \xrightarrow{\tilde{f}} & D_2 \end{array} \tag{2.1}$$

If $\tilde{f}$ is clear from the context we sometimes write f instead of $(f, \tilde{f})$.

A complex manifold D will be identified with the embedding triple (D, id, D).

Remark 2.1. A s.s.of embedding triples $(X_\alpha, k_\alpha, D_\alpha)_{\alpha\in S}$ can be seen as a triple $(\mathcal{X}, k, \mathcal{D})$ where $\mathcal{X} = (X_\alpha)_{\alpha\in\mathcal{S}}$ is a s.s.analytic space, $\mathcal{D} = (D_\alpha)_{\alpha\in\mathcal{S}}$ is a s.s.complex manifold, and $k : \mathcal{X} \to \mathcal{D}$ is a morphism of s.s. analytic spaces such that each $k_\alpha : X_\alpha \to D_\alpha$ is a closed embedding.

Definition 2.2. *Let X be an analytic space. An embedding atlas of X consists of a family of embedding triples $\mathcal{A} = (U_i, k_i, D_i)_{i\in I}$ such that*

1. *$U_i \subset X$ is an open set*
2. *$\mathcal{U} = (U_i)_{i\in I}$ is a covering of X.*

An embedding triple (U_i, k_i, D_i) of $\mathcal{A}$ will be called a chart.

The pair $(X, \mathcal{A})$ will be called locally embedded analytic space or, sometimes, a local embedding of X.

In particular, an embedding triple (X, i, D) can be seen as an embedding atlas of X with one chart.

Remark 2.2. Let X be an analytic space and $\mathcal{A} = (U_i, k_i, D_i)_{i\in I}$ an embedding atlas of X. One associates to the pair $(X, \mathcal{A})$ the s.s.of embedding triples $(\mathfrak{U}, k, \mathcal{D})$ over the simplicial complex $(I, \mathcal{N}(\mathcal{U}))$ where $\mathcal{U} = (U_i)_{i\in I}$ denotes the open covering of X, $\mathfrak{U} = ((U_\alpha)_{\alpha\in\mathcal{N}(\mathcal{U})}, (i_{\alpha\beta})_{\alpha\subset\beta})$ is the s.s.analytic space corresponding to $\mathcal{U}$ (see Example 2.2), $\mathcal{D} = ((D_\alpha)_{\alpha\in\mathcal{N}(\mathcal{U})}, (p_{\alpha\beta})_{\alpha\subset\beta})$ is the s.s.analytic space associated to the family $(D_i)_{i\in I}$ (see Example 2.3) and $k : \mathfrak{U} \to \mathcal{D}$ is the morphism deduced from the closed embeddings $k_i : U_i \to D_i$. It is easy to see that each $k_\alpha : U_\alpha \to V_\alpha$ is also a closed embedding.

Let $\mathcal{A} = (U_i, k_i, D_i)_{i\in I}$ and $\mathcal{B} = (V_j, k_j, D_j)_{j\in J}$ be embedding atlases of the analytic space X, respectively Y. A morphism $F : (X, \mathcal{A}) \to (Y, \mathcal{B})$ of locally embedded analytic spaces consists of the following data:

- a morphism of analytic spaces $f : X \to Y$
- a mapping $\tau : I \to J$ such that $f(U_i) \subset V_{\tau(i)}$
- a family $(\tilde{f}_i)_{i\in I}$ of morphisms, $\tilde{f}_i : D_i \to D_{\tau(i)}$, such that $(f|U_i, \tilde{f}_i) : (U_i, k_i, D_i) \to (V_{\tau(i)}, k_{\tau(i)}, D_{\tau(i)})$ is a morphism of embedding triples.

2.3. Modules and comodules over a s.s.analytic space

The notions of $\mathfrak{X}$-modules and $\mathfrak{X}$-comodules defined below were introduced in [9] under the name "modules à liaisons covariantes", respectively "modules à liaisons contravariantes".

Throughout this paragraph $\mathfrak{X} = ((X_\alpha, \mathcal{O}_\alpha)_{\alpha\in\mathcal{S}}, (\rho_{\alpha\beta}, \rho^1_{\beta\alpha})_{\alpha\subset\beta})$ will denote a s.s.analytic space relative to the simplicial complex $(I, \mathcal{S})$.

Definition 2.3.

1. *An $\mathfrak{X}$-module consists of*
   - *a family $(\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}$ where $\mathcal{F}_\alpha$ is an $\mathcal{O}_\alpha$-module on X_α*
   - *a family of connecting morphisms $(\varphi_{\beta\alpha})_{\alpha\subset\beta}$, where $\varphi_{\beta\alpha} : \mathcal{F}_\alpha \to \rho_{\alpha\beta *}(\mathcal{F}_\beta)$, is a morphism of $\mathcal{O}_\alpha$-modules such that $\varphi_{\alpha\alpha} = id$ for all $\alpha \in \mathcal{S}$,and $\rho_{\beta\gamma *}(\varphi_{\gamma\beta}) \circ \varphi_{\beta\alpha} = \varphi_{\gamma\alpha}$., whenever $\alpha \subseteq \beta \subseteq \gamma$.*
2. *If $\mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}, (\varphi_{\beta\alpha})_{\alpha\subset\beta})$, $\mathcal{G} = ((\mathcal{G}_\alpha)_{\alpha\in\mathcal{S}}, (\psi_{\beta\alpha})_{\alpha\subset\beta})$ are $\mathfrak{X}$-modules, then a morphism of $\mathfrak{X}$-modules $u : \mathcal{F} \to \mathcal{G}$ consists of a family $(u_\alpha)_{\alpha\in\mathcal{S}}$, where $u_\alpha : \mathcal{F}_\alpha \to \mathcal{G}_\alpha$ is a morphism of $\mathcal{O}_\alpha$-modules, such that for $\alpha \subseteq \beta$ $\rho_{\alpha\beta *}(u_\beta) \circ \varphi_{\beta\alpha} = \psi_{\beta\alpha} \circ u_\alpha$.*

We denote by $Mod(\mathfrak{X})$ the abelian category of $\mathfrak{X}$-modules and by $C(\mathfrak{X})$ the category of complexes with terms in $Mod(\mathfrak{X})$.

Example 2.4. $((\mathcal{O}_\alpha)_{\alpha\in\mathcal{S}}, (\rho^1_{\beta\alpha})_{\alpha\subset\beta})$ is obviously an $\mathfrak{X}$-module that we denote $\mathcal{O}_\mathcal{X}$.

Example 2.5. In the context of Example 2.2 let $\mathcal{F} \in Mod(\mathcal{O}_X)$. Then $(\mathcal{F}|U_\alpha)_{\alpha\in\mathcal{S}}$ with the obvious connecting morphisms is an $\mathfrak{U}$-module that we denote $\mathcal{F}|\mathfrak{U}$.

Example 2.6. $((k_{\alpha *}(\mathcal{O}_\alpha))_{\alpha\in\mathcal{N}(\mathcal{U})}, (k_{\alpha *}(i^1_{\beta\alpha}))_{\alpha\subset\beta})$ is a $\mathcal{D}$-module (see the notation in Remark 2.2).

Definition 2.4.

1. *An $\mathfrak{X}$-comodule consists of:*
   - *a family $(\mathcal{M}_\alpha)_{\alpha\in\mathcal{S}}$ where $\mathcal{M}_\alpha$ is an $\mathcal{O}_\alpha$-module*
   - *a family of connecting morphisms $(\theta_{\alpha\beta})_{\alpha\subset\beta}$, where $\theta_{\alpha\beta} : \rho_{\alpha\beta !}(\mathcal{F}_\beta) \to \mathcal{F}_\alpha$ is a morphism of $\mathcal{O}_\alpha$-modules such that $\theta_{\alpha\alpha} = id$ for all $\alpha \in \mathcal{S}$,and, if $\alpha \subseteq \beta \subseteq \gamma$ then $\theta_{\alpha\beta} \circ \rho_{\beta\gamma !}(\theta_{\beta\gamma}) = \theta_{\alpha\gamma}$.*
2. *If $\mathcal{M} = ((\mathcal{M}_\alpha)_{\alpha\in\mathcal{S}}, (\theta_{\alpha\beta})_{\alpha\subset\beta})$, $\mathcal{N} = ((\mathcal{N}_\alpha)_{\alpha\in\mathcal{S}}, (\widetilde{\theta}_{\alpha\beta})_{\alpha\subset\beta})$ are $\mathcal{X}$-comodules, then a morphism of $\mathfrak{X}$-comodules $v : \mathcal{M} \to \mathcal{N}$ consists of a family of morphisms $(v_\alpha)_{\alpha\in\mathcal{S}}$, where $v_\alpha : \mathcal{M}_\alpha \to \mathcal{N}_\alpha$ is a morphism of $\mathcal{O}_\alpha$-modules, such that for $\alpha \subseteq \beta$ $\widetilde{\theta}_{\alpha\beta} \circ \rho_{\alpha\beta !}(v_\beta) = v_\alpha \circ \theta_{\alpha\beta}$.*

We denote by $coMod(\mathcal{X})$ the abelian category of $\mathcal{X}$-comodules and by $coC(\mathcal{X})$ the category of complexes with terms in $coMod(\mathcal{X})$.

Example 2.7. In the context of Example 2.2 let $\mathcal{F} \in Mod(\mathcal{O}_X)$. Then $(\mathcal{F}|U_\alpha)_{\alpha\in\mathcal{S}}$ together with the connecting morphisms given by the extensions with 0 $\theta_{\alpha\beta} : i_{\alpha\beta!}(\mathcal{F}|U_\beta) \to \mathcal{F}|U_\alpha$ is an $\mathfrak{U}$-comodule that we denote $\mathcal{F}|\mathfrak{U}$.

3. The Grothendieck operations on the categories of modules and comodules

Since in this note we want to avoid using derived categories we will not treat the exceptional inverse image $f^!$.

Let $f : (I, \mathcal{S}) \to (J, \mathcal{T})$ be a morphism of simplicial complexes, let $\mathcal{X} = ((X_\alpha)_{\alpha\in\mathcal{S}}, (\rho_{\alpha\beta})_{\alpha\subset\beta})$, $\mathcal{Y} = ((Y_\gamma)_{\gamma\in\mathcal{T}}, (\rho'_{\gamma\delta})_{\gamma\subset\delta})$ be s.s.analytic spaces relative to $(I, \mathcal{S})$, respectively $(J, \mathcal{T})$, and $F : \mathcal{X} \to \mathcal{Y}$ a morphism over f.

We start with the functors that can be extended componentwise.

3.1. The tensor product

The tensor product on analytic modules induces a bifunctor $\otimes : Mod(\mathcal{X}) \times Mod(\mathcal{X}) \to Mod(\mathcal{X})$, namely if one considers $\mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}, (\varphi_{\beta\alpha})_{\alpha\subset\beta})$, $\mathcal{G} = ((\mathcal{G}_\alpha)_{\alpha\in\mathcal{S}}, (\psi_{\beta\alpha})_{\alpha\subset\beta}) \in Mod(\mathcal{X})$ then $((\mathcal{F}_\alpha \otimes \mathcal{G}_\alpha)_{\alpha\in\mathcal{S}}, (\varphi_{\beta\alpha} \otimes \psi_{\beta\alpha})_{\alpha\subset\beta})$ is an $\mathcal{X}$-module.

3.2. The Hom functors

Beside the Hom functors of $Mod(\mathcal{X})$ and $coMod(\mathcal{X})$ one can define a natural bifunctor

$$\mathcal{H}om : Mod(\mathcal{X}) \times coMod(\mathcal{X}) \to coMod(\mathcal{X})$$

as follows: if $\mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}, (\varphi_{\beta\alpha})_{\alpha\subset\beta})$ is an $\mathcal{X}$-module and $\mathcal{M} = ((\mathcal{M}_\alpha)_{\alpha\in\mathcal{S}}, (\theta_{\alpha\beta})_{\alpha\subset\beta})$ is an $\mathcal{X}$-comodule then $\mathcal{H}om(\mathcal{F}, \mathcal{M}) = ((\mathcal{H}om(\mathcal{F}_\alpha, \mathcal{M}_\alpha))_{\alpha\in\mathcal{S}}, (\theta_{\alpha\beta} \circ \bullet \circ \varphi_{\beta\alpha})_{\alpha\subset\beta})$ is an $\mathcal{X}$-comodule.

3.3. The inverse image

Let $\mathcal{F} = ((\mathcal{F}_\gamma)_{\gamma\in\mathcal{T}}) \in Mod(\mathcal{Y})$. Then $F^*(\mathcal{F}) = (F^*(\mathcal{F})_{\alpha\in\mathcal{S}})$ with $F^*(\mathcal{F})_\alpha = F^*_\alpha(\mathcal{F}_{f(\alpha)})$ and the obvious connecting morphisms is an $\mathcal{X}$-module.

Remark that in the context of Examples 2.2 and 2.5 $\mathcal{F}|\mathfrak{U} = (\mathcal{F}|U_\alpha)_{\alpha\in\mathcal{S}}$ coincides with $i^*(\mathcal{F})$.

3.4. Alternate $\mathcal{X}$-modules and $\mathcal{X}$-comodules

In order to define direct images of $\mathcal{X}$-modules and $\mathcal{X}$-comodules we shall construct alternate versions of them. For this, let $(I, \mathcal{S})$ be a simplicial complex and fix a total order on I. We use the following notations:

- if $\alpha \in \mathcal{S}$ and $j \in [0, |\alpha|]$, then $v(\alpha; j) =$ the j-th vertex of α (with respect to the order on I, the counting starting from 0), and $\sigma(\alpha; j) = \alpha \setminus \{v(\alpha; j)\}$
- if $\alpha \in \mathcal{S}$, $|\alpha| \geq 1$ and $j, k \in [0, |\alpha|]$, $j \neq k$, then $\sigma(\alpha; j, k) = \alpha \setminus \{v(\alpha; j), v(\alpha; k)\}$. Obviously, if $j < k$ then $\sigma(\alpha; j, k) = \sigma(\sigma(\alpha; k); j) = \sigma(\sigma(\alpha; j); k-1)$

Thus if $\alpha = \{i_0, ..., i_n\}$ and $i_0 < i_1 < ... < i_n$, one checks immediately that $|\alpha| = n$, $v(\alpha; j) = v_j$, $\sigma(\alpha; j) = \{i_0, ..., \hat{i}_j, ..., i_n\}$, $\sigma(\alpha; j, k) = \{i_0, ..., \hat{i}_j, ..., \hat{i}_k, ..., i_n\}$

Let $\mathfrak{X} = ((X_\alpha)_{\alpha \in \mathcal{S}}, (\rho_{\alpha\beta})_{\alpha \subset \beta})$ be a s.s.analytic space. To simplify notation we use the subscript (α, j) to refer to the mappings along the edge $[\alpha, \sigma(\alpha; j)]$ of the simplicial complex $(I, \mathcal{S})$. Thus we write $\rho_{(\alpha,j)}$ instead of $\rho_{\sigma(\alpha;j)\alpha} : X_\alpha \to X_{\sigma(\alpha;j)}$. Similarly, if $\mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha \in \mathcal{S}}, (\varphi_{\beta\alpha})_{\alpha \subset \beta})$ is an $\mathfrak{X}$-module we write $\varphi_{(\alpha,j)}$ instead of $\varphi_{\alpha\sigma(\alpha;j)} : \mathcal{F}_{\sigma(\alpha;j)} \to \rho_{(\alpha;j)*}(\mathcal{F}_\alpha)$; if $\mathcal{M} = ((\mathcal{M}_\alpha)_{\alpha \in \mathcal{S}}, (\theta_{\alpha\beta})_{\alpha \subset \beta})$ is an $\mathfrak{X}$-comodule we write $\theta_{(\alpha,j)}$ instead of $\theta_{\sigma(\alpha;j)\alpha} : \rho_{(\alpha;j)!}(\mathcal{M}_\alpha) \to \mathcal{M}_{\sigma(\alpha;j)}$.

Remark 3.1. The family of commuting morphisms $(\rho_{\alpha\beta})_{\alpha \subset \beta}$ can be "reconstructed" (by finite compositions) from the subfamily $(\rho_{(\alpha;j)})_{(\alpha;j)}$. If $\alpha \in \mathcal{S}$, $|\alpha| \geq 1$, $j, k \in [0, |\alpha|]$, with, for instance, $j < k$, then the following rectangular diagram commutes:

$$\begin{array}{ccc} X_\alpha & \xrightarrow{\rho_{(\alpha;j)}} & X_{\sigma(\alpha;j)} \\ \downarrow{\scriptstyle \rho_{(\alpha;k)}} & & \downarrow{\scriptstyle \rho_{(\sigma(\alpha;j),k-1)}} \\ X_{\sigma(\alpha;k)} & \xrightarrow{\rho_{(\sigma(\alpha;k),j)}} & X_{\sigma(\alpha;j,k)} \end{array} \qquad (D(\alpha; j, k))$$

Conversely, any family of morphisms $(\rho_{(\alpha;j)})_{(\alpha;j)}$ such that the diagrams $D(\alpha; j, k)$ commute, generates a family of commuting morphisms for the family of analytic spaces $(X_\alpha)_{\alpha \in \mathcal{S}}$. Similarly, the connecting morphisms of the $\mathfrak{X}$-module $\mathcal{F}$ are uniquely determined by the subfamily $(\varphi_{(\alpha;j)})_{(\alpha;j)}$ and the obvious rectangular diagrams commute:

$$\begin{array}{ccc} \rho_{(\alpha;j,k)*}\mathcal{F}_\alpha & \xleftarrow{\rho_{(\sigma(\alpha;j),k-1)*}(\varphi_{(\alpha;j)})} & \rho_{(\sigma(\alpha;j),k-1)*}(\mathcal{F}_{\sigma(\alpha;j)}) \\ \uparrow{\scriptstyle \rho_{(\sigma(\alpha;k),j)*}(\varphi_{(\alpha;k)})} & & \uparrow{\scriptstyle \varphi_{(\sigma(\alpha;j),k-1)}} \\ \rho_{(\sigma(\alpha;k),j)*}(\mathcal{F}_{\sigma(\alpha;k)}) & \xleftarrow{\varphi_{(\sigma(\alpha;k),j)}} & \mathcal{F}_{\sigma(\alpha;j,k)} \end{array} \qquad (D(\mathcal{F}; \alpha; j, k))$$

Ditto for the $\mathfrak{X}$-comodule $\mathcal{M}$ — its connecting morphisms are uniquely determined by the subfamily $(\theta_{(\alpha;j)})_{(\alpha;j)}$ and the corresponding rectangular diagrams $D(\mathcal{M}; \alpha; j, k)$ commute.

Definition 3.1.

1. *An alternate $\mathcal{X}$-module consists of a family $(\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}$, where each $\mathcal{F}_\alpha$ is an $\mathcal{O}_\alpha$-module, together with the family of connecting morphisms $(\varphi_{(\alpha;j)})_{(\alpha;j)}$, $\varphi_{(\alpha;j)} : \mathcal{F}_{(\alpha;j)} \to \rho_{(\alpha;j)*}(\mathcal{F}_\alpha)$ such that the diagrams $D(\mathcal{F};\alpha;j,k)$ anti-commute.*
2. *Let $\mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}\ ,(\varphi_{(\alpha;j)})_{(\alpha;j)})$, $\mathcal{G} = ((\mathcal{G}_\alpha)_{\alpha\in\mathcal{S}}\ ,(\psi_{(\alpha;j)})_{(\alpha;j)})$ be alternate $\mathcal{X}$-modules. A morphism of alternate $\mathcal{X}$-modules $u : \mathcal{F} \to \mathcal{G}$ consists of a family $(u_\alpha)_{\alpha\in\mathcal{S}}$, $u_\alpha : \mathcal{F}_\alpha \to \mathcal{G}_\alpha$ morphism of $\mathcal{O}_\alpha$-modules, such that for each pair $(\alpha;j)$ the diagram commutes:*

$$\begin{array}{ccc} \mathcal{F}_{(\alpha;j)} & \xrightarrow{u_{(\alpha;j)}} & \mathcal{G}_{(\alpha;j)} \\ \downarrow{\scriptstyle \varphi_{(\alpha;j)}} & & \downarrow{\scriptstyle \psi_{(\alpha;j)}} \\ \rho_{(\alpha;j)*}\mathcal{F}_\alpha & \xrightarrow{u_\alpha} & \rho_{(\alpha;j)*}\mathcal{G}_\alpha \end{array} \qquad (D(\mathcal{F},\mathcal{G};\alpha;j))$$

   One denotes by $aMod(\mathcal{X})$ the category of alternate $\mathcal{X}$-modules.
3. *With the notations at point 2, an anti-morphism of alternate $\mathcal{X}$-modules is a family of morphisms $u = (u_\alpha)_{\alpha\in\mathcal{S}}$ such that the diagrams $D(\mathcal{F},\mathcal{G};\alpha;j)$ anti-commute. A complex of alternate $\mathcal{X}$-modules with anti-morphism differentials will be called an alternate complex of alternate $\mathcal{X}$-modules. One denotes by $aC(\mathcal{X})$ the category of alternate complexes of alternate $\mathcal{X}$-modules.*
4. *In exactly the same way as above one defines the notions of alternate $\mathcal{X}$-comodule, morphism and anti-morphism of alternate $\mathcal{X}$-comodules and alternate complex of alternate $\mathcal{X}$-comodules. One denotes by $acoMod(\mathcal{X})$ the category of alternate $\mathcal{X}$-comodules and by $acoC(\mathcal{X})$ the category of alternate complexes of alternate $\mathcal{X}$-comodules.*

To the edge (α,j) of the simplicial complex $(I,\mathcal{S})$ we associate the alternating coeficient $\varepsilon(\alpha,j) = (-1)^j$. Note that if $\mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}\ ,(\varphi_{(\alpha;j)})_{(\alpha;j)})$ is an $\mathcal{X}$-module then $alt(\mathcal{F}) = ((\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}\ ,(\varepsilon(\alpha,j)\varphi_{(\alpha;j)})_{(\alpha;j)})$ is an alternate $\mathcal{X}$-module. One checks easily that $alt : Mod(\mathcal{X}) \to aMod(\mathcal{X})$ is an isomorphism of categories with an obvious inverse that we denote by alt^{-1}. The functor alt extends to an isomorphism of categories $C(\mathcal{X}) \to aC(\mathcal{X})$. Indeed, if $\mathcal{F}^\bullet \in C(\mathcal{X})$, $\mathcal{F}^\bullet = ((\mathcal{F}^\bullet_\alpha)_{\alpha\in\mathcal{S}}\ ,(\varphi^\bullet_{(\alpha;j)})_{(\alpha;j)})$ then the terms of $alt(\mathcal{F}^\bullet)$ are obtained from the terms of $\mathcal{F}^\bullet$ via the functor alt, while the differentials of each complex $\mathcal{F}^\bullet_\alpha$ are multiplied by $(-1)^{|\alpha|}$.

Similarly there is an isomorphism of categories $alt : coMod(\mathcal{X}) \to acoMod(\mathcal{X})$ which associates to the $\mathcal{X}$-comodule $\mathcal{M} = ((\mathcal{M}_\alpha)_{\alpha\in\mathcal{S}}\ ,(\theta_{(\alpha;j)})_{(\alpha;j)})$ the alternate $\mathcal{X}$-comodule $alt(\mathcal{M}) = ((\mathcal{M}_\alpha)_{\alpha\in\mathcal{S}}\ ,(\varepsilon(\alpha,j)\theta_{(\alpha;j)})_{(\alpha;j)})$; as above, this isomorphism extends to an isomorphisms $alt : coC(\mathcal{X}) \to acoC(\mathcal{X})$.

Remark 3.2. The notions of alternate $\mathcal{X}$-module, alternate $\mathcal{X}$-comodule and alternate complex of $\mathcal{X}$-modules or $\mathcal{X}$-comodules do not depend on the total order on I. The *alt* functors do. However for two total orders on I there is a (non-unique) functorial isomorphism between the two corresponding *alt* functors.

3.5. Direct image

Consider the following setting.

- $f : (I, \mathcal{S}) \to (J, \mathcal{T})$ a morphism of simplicial complexes
- total orders on I and J such that $f : I \to J$ is increasing
- $\mathcal{X} = ((X_\alpha)_{\alpha\in\mathcal{S}}, (\rho_{\alpha\beta})_{\alpha\subset\beta})$, $\mathcal{Y} = ((Y_\gamma)_{\gamma\in\mathcal{T}}, (\rho'_{\gamma\delta})_{\gamma\subset\delta})$ s.s.analytic spaces relative to $(I, \mathcal{S})$, respectively $(J, \mathcal{T})$
- $F : \mathcal{X}\to\mathcal{Y}$ a morphism of s.s.analytic spaces over$f$, that is one has $F = (F_\alpha, F^*_\alpha)_{\alpha\in\mathcal{S}}$ with $F_\alpha : X_\alpha \to Y_{f(\alpha)}$ morphism of analytic spaces such that for $\alpha \subseteq \beta$ the following diagram commutes

$$\begin{array}{ccc} X_\beta & \xrightarrow{F_\beta} & Y_{f(\beta)} \\ \downarrow{\scriptstyle \rho_{\alpha\beta}} & & \downarrow{\scriptstyle \rho'_{f(\alpha)f(\beta)}} \\ X_\alpha & \xrightarrow{F_\alpha} & Y_{f(\alpha)} \end{array}$$

Remark that $\mathcal{S}$ is the disjoint union of the sets $(\mathcal{S}_\gamma)_{\gamma\in\mathcal{T}}$, with $\mathcal{S}_\gamma = \{\alpha \in \mathcal{S} | f(\alpha) = \gamma\}$. Let $\mathcal{F} \in aMod(\mathcal{X})$, $\mathcal{F} = ((\mathcal{F}_\alpha)_{\alpha\in\mathcal{S}}, (\varphi_{(\alpha;j)})_{(\alpha;j)})$. For $\gamma \in \mathcal{T}$ $((F_{\alpha*}(\mathcal{F}_\alpha))_\alpha, (F_{\sigma(\alpha;j)*}(\varphi_{(\alpha,j)}))_{(\alpha;j)})_{f(\alpha)=f(\sigma(\alpha;j))=\gamma}$ is a multicomplex of Y_γ-modules (recall that multicomplex means anti-commuting rectangles) and consider the simple complex associated to this multicomplex:

$$... \to \prod_{\substack{f(\alpha)=\gamma \\ |\alpha|=i}} F_{\alpha*}(\mathcal{F}_\alpha) \to \prod_{\substack{f(\alpha)=\gamma \\ |\alpha|=i+1}} F_{\alpha*}(\mathcal{F}_\alpha) \to ... \qquad (C^\bullet(\gamma))$$

Here the product with $i = |\gamma|$ is considered in degree 0. The connecting morphisms of $\mathcal{F}$ induce anti-morphisms $C^\bullet(\sigma(\gamma; j)) \to \rho_{(\gamma;j)*}(C^\bullet(\gamma))$ and one checks that $(C^\bullet(\gamma))_{\gamma\in\mathcal{T}}$ is an alternated complex of alternated $\mathcal{Y}-$modules.

If we start with $\mathcal{F}$ an alternated complex of alternated $\mathcal{X}$-modules instead of an alternated $\mathcal{X}$-module then $C^\bullet(\gamma)$ is a double complex.

Definition 3.2.

1. *If $\mathcal{F} \in aMod(\mathcal{X})$ then $F_*(\mathcal{F})$ is the alternated complex of alternated $\mathcal{Y}$-modules with $F_*(\mathcal{F})_\gamma = C^\bullet(\gamma)$ and connecting morphisms induced by those of $\mathcal{F}$.*

2. *If $\mathcal{F} \in aC(\mathcal{X})$ then $F_*(\mathcal{F})$ is the alternated complex of alternated $\mathcal{Y}$-modules with $F_*(\mathcal{F})_\gamma$ =the simple complex associated to the double complex $C^\bullet(\gamma)$ (with the product for $i = |\gamma|$ considered in degree 0) and connecting morphisms induced by those of $\mathcal{F}$.*

3. *If $\mathcal{F} \in Mod(\mathcal{X})$ (respectively $\mathcal{F} \in C(\mathcal{X})$) then*

$$F_*(\mathcal{F}) = alt^{-1}(F_*(alt(\mathcal{F}))).$$

One checks easily that the definition of the direct image is compatible with the natural inclusion functors $aMod(\mathcal{X}) \to aC(\mathcal{X})$ and $Mod(\mathcal{X}) \to C(\mathcal{X})$.

Example 3.1. The components $(F^*_\alpha)_{\alpha\in\mathcal{S}}$ of the morphism $F : \mathcal{X}\to\mathcal{Y}$ determine a morphism of $\mathcal{Y}$-modules $F^* : \mathcal{O}_\mathcal{Y} \to F_*(\mathcal{O}_\mathcal{X})$.

Example 3.2. If $f : (I, \mathcal{S}) \to (J, \mathcal{T})$ is bijective (in particular if f is the identity of $(I, \mathcal{S})$) then $F_*(\mathcal{F})_\gamma = F_{\alpha *}(\mathcal{F}_\alpha)$ where $\alpha = f^{-1}(\gamma)$. In particular if $F : X \to Y$ is a morphism of analytic spaces and $\mathcal{F} \in Mod(X)$ then the usual direct image $F_*(\mathcal{F})$ coincides with the direct image of $\mathcal{F}$ as module over X seen as s.s. analytic space relative to $K(pt)$ (see Example 2.1).

Since computing the direct images comes down to taking cartesian products over subsets of type $\mathcal{S}_\gamma$ in a multiple complex, by associativity of the cartesian product and of the usual direct image one has:

Lemma 3.1. *Let $f : (I_1, \mathcal{S}_1) \to (I_2, \mathcal{S}_2)$, $g : (I_2, \mathcal{S}_2) \to (I_3, \mathcal{S}_3)$ be morphisms of simplicial complexes and assume that we have fixed total orders on I_1, I_2, I_3. Let $F : \mathcal{X}\to\mathcal{Y}$, $G : \mathcal{Y}\to\mathcal{Z}$ be morphisms of s.s.analytic spaces overf, respectively g, where $\mathcal{X}$, respectively $\mathcal{Y}$, repectively $\mathcal{Z}$ is a s.s.analytic space relative to $(I_1, \mathcal{S}_1)$, respectively to $(I_2, \mathcal{S}_2)$, respectively relative to $(I_3, \mathcal{S}_3)$ Then if $\mathcal{F} \in aMod(\mathcal{X})$ or $\mathcal{F} \in aC(\mathcal{X})$ or $\mathcal{F} \in Mod(\mathcal{X})$ or $\mathcal{F} \in C(\mathcal{X})$ one has $(G \circ F)_*(\mathcal{F}) = G_*F_*(\mathcal{F})$.*

Let $\mathcal{F} \in Mod(\mathcal{X})$ and $\mathcal{G} \in Mod(\mathcal{Y})$. One checks that there is an adjunction isomorphism:

$$Hom(F^*(\mathcal{G}), \mathcal{F}) \widetilde{\longrightarrow} Hom(\mathcal{G}, F_*(\mathcal{F})). \tag{3.1}$$

3.6. Direct image with proper supports

Consider the same setting as in paragraph 3.5.

In the case of comodules one defines an analogue of the direct image with proper supports. Let $\mathcal{M} \in acoMod(\mathcal{X})$, $\mathcal{M} = ((\mathcal{M}_\alpha)_{\alpha\in\mathcal{S}}\ , (\theta_{(\alpha;j)})_{(\alpha;j)})$. For $\gamma \in \mathcal{T}$ $((F_{\alpha!}(\mathcal{M}_\alpha))_\alpha, (F_{\sigma(\alpha;j)!}(\theta_{(\alpha,j)}))_{(\alpha;j)})_{f(\alpha)=f(\sigma(\alpha;j))=\gamma}$ is a multicomplex

of Y_γ-modules and consider the following simple complex associated to this multicomplex:

$$... \to \bigoplus_{\substack{f(\alpha)=\gamma \\ |\alpha|=i}} F_{\alpha!}(\mathcal{G}_\alpha^\bullet) \to ... \to \bigoplus_{\substack{f(\alpha)=\gamma \\ |\alpha|=|\gamma|}} F_{\alpha!}(\mathcal{G}_\alpha^\bullet) \to 0 \qquad (C_!^\bullet(\gamma))$$

where the sum for $i = |\gamma|$ is considered in degree 0. The connecting morphisms of $\mathcal{M}$ induce anti-morphisms $\rho_{(\gamma;j)!}(C_!^\bullet(\gamma)) \to C_!^\bullet(\sigma(\gamma;j))$ and one checks that $(C_!^\bullet(\gamma))_{\gamma\in\mathcal{T}}$ is an alternated complex of alternated $\mathcal{Y}$-comodules.

If we start with $\mathcal{M}$ an alternated complex of alternated $\mathcal{X}$-comodules instead of an alternated $\mathcal{X}$-comodule then $C_!^\bullet(\gamma)$ is a double complex.

Definition 3.3.

1. *If $\mathcal{M} \in acoMod(\mathcal{X})$ then $F_!(\mathcal{M})$ is the alternated complex of alternated $\mathcal{Y}$-comodules with $F_!(\mathcal{M})_\gamma = C_!^\bullet(\gamma)$ and connecting morphisms induced by those of $\mathcal{M}$.*
2. *If $\mathcal{M} \in acoC(\mathcal{X})$ then $F_!(\mathcal{M})$ is the alternated complex of alternated $\mathcal{Y}$-comodules with $F_!(\mathcal{M})_\gamma$ =the simple complex associated to the double complex $C_!^\bullet(\gamma)$ (with the sum for $i = |\gamma|$ considered in degree 0) and connecting morphisms induced by those of $\mathcal{M}$.*
3. *If $\mathcal{M} \in coMod(\mathcal{X})$ (respectively $\mathcal{M} \in coC(\mathcal{X})$) then*

$$F_!(\mathcal{M}) = alt^{-1}(F_!(alt(\mathcal{M}))).$$

One checks easily that the definition of the direct image is compatible with the natural inclusion functors $acoMod(\mathcal{X}) \to acoC(\mathcal{X})$ and $coMod(\mathcal{X}) \to coC(\mathcal{X})$.

Example 3.3. If $f : (I, \mathcal{S}) \to (J, \mathcal{T})$ is bijective (in particular if f is the identity of $(I, \mathcal{S})$) then $F_!(\mathcal{M})_\gamma = F_{\alpha!}(\mathcal{M}_\alpha)$ where $\alpha = f^{-1}(\gamma)$.

As for the direct images for $\mathcal{X}$-modules, computing direct images with proper supports comes down to taking direct sums over subsets of type $\mathcal{S}_\gamma$ in a multiple complex. By associativity of the direct sum and of the usual direct image with proper supports one has:

Lemma 3.2. *Let $f : (I_1, \mathcal{S}_1) \to (I_2, \mathcal{S}_2)$, $g : (I_2, \mathcal{S}_2) \to (I_3, \mathcal{S}_3)$ be morphisms of simplicial complexes and assume that we have fixed total orders on I_1, I_2, I_3. Let $F : \mathcal{X} \to \mathcal{Y}$, $G : \mathcal{Y} \to \mathcal{Z}$ be morphisms of s.s.analytic spaces overf, respectively g, where $\mathcal{X}$, respectively $\mathcal{Y}$, repectively $\mathcal{Z}$ is a s.s.analytic space relative to $(I_1, \mathcal{S}_1)$, respectively to $(I_2, \mathcal{S}_2)$, respectively relative to $(I_3, \mathcal{S}_3)$. Then if $\mathcal{F} \in acoMod(\mathcal{X})$ or $\mathcal{F} \in acoC(\mathcal{X})$ or $\mathcal{F} \in coMod(\mathcal{X})$ or $\mathcal{F} \in coC(\mathcal{X})$ one has $(G \circ F)_!(\mathcal{F}) = G_!F_!(\mathcal{F})$.*

4. Applications and examples

4.1 The Čech complex

Let X be an analytic space, $\mathcal{U} = (U_i)_{i\in I}$ an open covering of X and $\mathcal{F} \in Mod(\mathcal{O}_X)$. Consider $\mathfrak{U} = ((U_\alpha)_{\alpha\in\mathcal{N}(\mathcal{U})})$ the s.s.analytic space corresponding to the covering $\mathcal{U}$ (see Example 2.2) and $i : \mathfrak{U} \to X$ the morphism given by the inclusions. Then one checks easily that $i_*(i^*(\mathcal{F})) = i_*(\mathcal{F}|\mathfrak{U})$ is the Čech complex of $\mathcal{F}$ with respect to the covering $\mathcal{U}$, $\mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$. Moreover, the natural morphism $\mathcal{F} \to \mathcal{C}^\bullet(\mathcal{U}, \mathcal{F})$ is exactly the adjunction morphism $\mathcal{F} \to i_*(i^*(\mathcal{F}))$. Remark that according to [10], chap 1, §4, Lemme 1, this morphism is a quasi-isomorphism and, consequently, an isomorphism in the derived category.

If instead of the morphism $i : \mathfrak{U} \to X$ above one considers the morphism $a : \mathfrak{U} \to \{pt\}$, where $\{pt\}$ is the reduced analytic space with one element, then $a_*(i^*(\mathcal{F})) = a_*(\mathcal{F}|\mathfrak{U})$ is the Čech complex of $\mathcal{F}$ with respect to the covering $\mathcal{U}$, $C^\bullet(\mathcal{U}, \mathcal{F})$.

In the context of Example 2.7 let $\mathcal{G} \in Mod(\mathcal{O}_X)$. Then $\mathcal{G}|\mathfrak{U} = (\mathcal{G}|U_\alpha)_{\alpha\in\mathcal{N}(\mathcal{U})}$ can be seen as an $\mathfrak{U}$-comodule one checks easily that $i_!(\mathcal{G}|\mathfrak{U})$ is the semi-simplicial resolution of $\mathcal{G}$ with respect to the covering $\mathcal{U}$:

$$... \to \bigoplus_{|\alpha|=1} \mathcal{G}_{U_\alpha} \to \bigoplus_{|\alpha|=0} \mathcal{G}_{U_\alpha} \to 0.$$

Thus the natural morphisms $i_!(\mathcal{G}|\mathcal{U}) \to \mathcal{G}$ is a quasi-isomorphism and hence is an isomorphism in the derived category.

4.2 The Dolbeault resolution on an analytic space

Let X be an n-dimensional complex manifold and consider the Dolbeault-Grothendieck resolution of $\mathcal{O}_X$

$$0 \longrightarrow \mathcal{E}_X^{0,0} \xrightarrow{\overline{\partial}} ... \xrightarrow{\overline{\partial}} \mathcal{E}_X^{0,n} \longrightarrow 0.$$

For $\mathcal{F} \in Mod(\mathcal{O}_X)$ we denote by $\mathcal{D}olb(X; \mathcal{F})$ the complex:

$$0 \longrightarrow \mathcal{E}_X^{0,0} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow ... \longrightarrow \mathcal{E}_X^{0,n} \otimes_{\mathcal{O}_X} \mathcal{F} \longrightarrow 0.$$

Since the stalks of the sheaves $\mathcal{E}_X^{p,q}$ are $\mathcal{O}_X$-flat (see [7]), $\mathcal{D}olb(X; \mathcal{F})$ is a resolution of $\mathcal{F}$.

Consider now (X, i, D) an embedding triple (see paragraph 2.2). Then one defines

$$\mathcal{D}olb(i; \mathcal{O}_X) = i^{-1}\mathcal{D}olb(X; i_*(\mathcal{O}_X)).$$

Finally let $(X, \mathcal{A})$ be an analytic space with an embedding atlas, where $\mathcal{A} = (U_i, k_i, D_i)_{i\in I}$ (see paragraph 2.2), let $\mathcal{U} = (U_i)_{i\in I}$ be the open covering

of X given by the atlas $\mathcal{A}$, $\mathfrak{U} = ((U_\alpha)_{\alpha\in\mathcal{N}(\mathcal{U})})$ the s.s.analytic space corresponding to $\mathcal{U}$ and $i : \mathfrak{U} \to X$ the morphism given by the inclusion mappings (see Example 2.2). Functoriality of the pull-back of differential forms implies that $(\mathcal{D}olb(k_\alpha; \mathcal{O}_X|U_\alpha))_{\alpha\in\mathcal{N}(\mathcal{U})} = (k_\alpha^{-1}\mathcal{D}olb(D_\alpha; k_{\alpha *}i_\alpha^{-1}(\mathcal{O}_X)))_{\alpha\in\mathcal{N}(\mathcal{U})}$ is a complex of $\mathfrak{U}$-modules that we denote by $\mathcal{D}olb(k; i^*(\mathcal{O}_X))$. By definition

$$\mathcal{D}olb(\mathcal{A}; \mathcal{O}_X) = i_*(\mathcal{D}olb(k; i^*(\mathcal{O}_X)))$$

is the Dolbeault resolution of $\mathcal{O}_X$ on $(X, \mathcal{A})$. One checks that it is indeed a resolution. Moreover, if X is countable at infinity and the covering $\mathcal{U}$ is locally finite, then the terms of $\mathcal{D}olb(\mathcal{A}; \mathcal{O}_X)$ are soft sheaves endowed with a canonical Frechet-Schwartz topology.

4.3 The dualizing complex on an analytic space

Let X be an n-dimensional complex manifold and consider the Dolbeault-Grothendieck resolution for Ω_X^n:

$$0 \longrightarrow \mathcal{K}_X^{n,0} \xrightarrow{\overline{\partial}} ... \xrightarrow{\overline{\partial}} \mathcal{K}_X^{n,n} \longrightarrow 0.$$

For $\mathcal{F} \in Mod(\mathcal{O}_X)$ we denote by $\mathcal{KD}olb(X; \mathcal{F})$ the complex:

$$0 \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_X^{n,0}) \longrightarrow ... \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F}, \mathcal{K}_X^{n,n}) \longrightarrow 0$$

where the term containing $\mathcal{K}_X^{n,n}$ is considered in degree 0. Since the stalks of the sheaves $\mathcal{K}_X^{p,q}$are $\mathcal{O}_X$-injective (see [7]), and if $\mathcal{F}$ is a coherent sheaf then $\mathcal{KD}olb(X; \mathcal{F})$ is a representative for $R\mathcal{H}om(\mathcal{F}, \Omega_X^n)[n]$.

Consider now (X, i, D) an embedding triple (see paragraph 2.2). Then one defines

$$\mathcal{KD}olb(i; \mathcal{O}_X) = i^{-1}\mathcal{KD}olb(X; i_*(\mathcal{O}_X)).$$

Let as above $(X, \mathcal{A})$ be an analytic space with an embedding atlas, where $\mathcal{A} = (U_i, k_i, D_i)_{i\in I}$ (see paragraph 2.2), let $\mathcal{U} = (U_i)_{i\in I}$ be the open covering of X given by the atlas $\mathcal{A}$, $\mathfrak{U} = ((U_\alpha)_{\alpha\in\mathcal{N}(\mathcal{U})}$ the s.s.analytic space corresponding to $\mathcal{U}$ and $i : \mathfrak{U} \to X$ the morphism given by the inclusion mappings (see Example 2.2). Functoriality of the pull-back of differential forms and the duality differential forms - currents implies that $(\mathcal{KD}olb(k_\alpha; \mathcal{O}_X|U_\alpha))_{\alpha\in\mathcal{N}(\mathcal{U})} = (k_\alpha^{-1}\mathcal{KD}olb(D_\alpha; k_{\alpha *}i_\alpha^{-1}(\mathcal{O}_X)))_{\alpha\in\mathcal{N}(\mathcal{U})}$ is a complex of $\mathfrak{U}$-comodules that we denote by $\mathcal{KD}olb(k; i^*(\mathcal{O}_X))$. By definition

$$\mathcal{KD}olb(\mathcal{A}; \mathcal{O}_X) = i_!(\mathcal{KD}olb(k; i^*(\mathcal{O}_X))).$$

Moreover, if X is countable at infinity and the covering $\mathcal{U}$ is locally finite, then the terms of $\mathcal{KD}olb(\mathcal{A}; \mathcal{O}_X)$ are soft sheaves with $\mathcal{O}_X$-injective stalks. $\mathcal{KD}olb(\mathcal{A}; \mathcal{O}_X)$ is a representative for the dualizing complex on X.

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