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Weak stability of the solutions of a fluid-rigid body problem

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Abstract - We consider a concept of weak solution for a boundary value problem modelling the interactive motion of a coupled system consisting in a rigid body immersed in a viscous fluid. The time variation of the fluid's domain (due to the motion of the rigid body) is not known apriori, so we deal with a free boundary value problem. Our main theorem asserts that the weak limit of any weakly convergent sequence of solutions still is a solution. The proof uses only the basic theory of Navier-Stokes equations, simplifying the approach in [2]

Key words and phrases : fluid-structure interactions, weak solutions, stability.

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1. Introduction

We consider a coupled system of nonlinear partial and ordinary differential equations modelling the motion of a rigid body inside a fluid flow. The governing equations for the fluid flow are the classical Navier-Stokes system, whereas the motion of the solid is governed by the balance equations for linear and angular momentum.

Let $A \subset \mathbb{R}^3$ be an open bounded set representing the domain occupied by both the fluid and the rigid body. For the sake of simplicity, we assume the rigid body to be a moving ball of radius 1 and the fluid to be homogeneous of density one. If we denote by $\Omega(t)$ (resp. $B(t)$) the domain occupied by the fluid (resp. by the solid) at the instant t, then the full system of equations modelling the motion of the fluid and of the rigid body is

$$\vec{u}' - \nu\Delta\vec{u} + \operatorname{div}(\vec{u}\otimes\vec{u}) + \nabla p = 0, \qquad \vec{x}\in\Omega(t), t\in[0,T], \tag{1.1}$$
$$\operatorname{div}\vec{u} = 0, \qquad \vec{x}\in\Omega(t), t\in[0,T], \tag{1.2}$$
$$\vec{u} = 0, \qquad \vec{x}\in\partial A,\ t\in[0,T], \tag{1.3}$$
$$\vec{u} = \vec{h}'(t) - \vec{\omega}(t)\times\vec{n}, \qquad \vec{x}\in\partial B(t),\ t\in[0,T], \tag{1.4}$$
$$M\vec{h}''(t) = -\int_{\partial B(t)} \sigma\vec{n}d\Gamma, \qquad t\in[0,T], \tag{1.5}$$
$$J\vec{\omega}'(t) = \int_{\partial B(t)} \vec{n}\times\sigma\vec{n}d\Gamma, \qquad t\in[0,T], \tag{1.6}$$
$$\vec{u}(x,0) = \vec{u}^0(x), \qquad x\in\Omega(0), \tag{1.7}$$
$$\vec{h}(0) = \vec{h}^0\in\mathbb{R}^3,\ \vec{h}'(0) = \vec{h}^1\in\mathbb{R}^3,\ \vec{\omega}(0) = \vec{\omega}^0\in\mathbb{R}^3. \tag{1.8}$$

In the above system the unknowns are $\vec{u}(\vec{x},t)$ (the Eulerian velocity field of the fluid), $\vec{h}(t)$ (the position of the center of the rigid ball) and $\vec{\omega}(t)$ (the angular velocity of the rigid body).

We have denoted by ∂A the boundary of A, by $\partial B(t)$ the boundary of the rigid ball at instant t and by $\vec{n}(\vec{x},t)$ the unit normal to $\partial B(t)$ at the point $\vec{x}\in\partial B(t)$, directed to the interior of the ball. Further, we have denoted by M (respectively by J) the mass (respectively the moment of inertia) of the rigid ball and by

$$\sigma_{ij}(x,t) = -p(x,t)\delta_{ij} + \nu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \tag{1.9}$$

the stress tensor in the fluid.

The main difficulties in the analysis of (1.1)–(1.8) are:

1. The Navier-Stokes equations are valid in a non-cylindrical domain, so that classical Galerkin type methods cannot be applied.

2. The form of this domain (in the $(\vec{x},t)$ space) depends on the solution, i.e., this is a free boundary problem.

Various concepts of weak solutions of (1.1)–(1.8) were studied in [1], [2] and [3]. In [2] the authors proved a stability result related to ours, by using the notion of renormalized solution introduced in [5]. The aim of this work is to give an elementary proof of the fact that any weak limit of a sequence of solutions of (1.1)–(1.8) still is a solution. We mention that we don't use the notion of renormalized solution.

The plan of this paper is as follows: In Section 2 we introduce some notation and state the main result. The main stability result is proved in Section 3.

2. Notation and main result

We first introduce the function spaces necessary to the definition of weak solutions. Let $A \subset \mathbb{R}^3$ be an open bounded set such that

$$A^\circ = \{\vec{x} \in A \,|\, d(\vec{x}, \partial A) > 1\} \neq \emptyset. \tag{2.1}$$

The following spaces are usually associated to the analysis of Navier-Stokes equations (see [6, ch.1]):

$$\begin{aligned} \mathcal{V}(A) &= \{\vec{v} \in \mathcal{D}(\ A; \mathbb{R}^3) \,|\, \operatorname{div} \vec{v} = 0\}, \\ H(\ A) &= \text{the closure of } \mathcal{V}(\ A) \text{ in } [L^2(\ A)]^3, \\ V(\ A) &= \text{the closure of } \mathcal{V}(\ A) \text{ in } [H^1(\ A)]^3. \end{aligned}$$

For $\vec{h} \in A^\circ$, we denote

$$B_{\vec{h}} = \left\{ x \in \mathbb{R}^3 | \ |\vec{x} - \vec{h}| < 1 \right\}.$$

The introduction of the following spaces is motivated by the presence of the rigid body

$$\begin{aligned} \mathcal{W}_{\vec{h}}(A) &= \left\{ (\vec{v}, \vec{\ell}, \vec{k}) \in \mathcal{V}(A) \times \mathbb{R}^3 \times \mathbb{R}^3 \ | \ \vec{v}|_{B_{\vec{h}}}(\vec{y}) = \vec{\ell} + \vec{k} \times (\vec{y} - \vec{h}) \right\}, \\ \mathbb{H}_{\vec{h}}(A) &= \text{the closure of } \mathcal{W}(A) \ \text{ in } L^2(A)^3 \times \mathbb{R}^3 \times \mathbb{R}^3, \\ \mathbb{V}_{\vec{h}}(A) &= \text{the closure of } \mathcal{W}(A) \ \text{ in } H^1(A)^3 \times \mathbb{R}^3 \times \mathbb{R}^3, \end{aligned}$$

On these spaces we consider the inner products defined by

$$\left\langle \left(\vec{v}_1, \vec{\ell}_1, \vec{k}_1\right), \left(\vec{v}_2, \vec{\ell}_2, \vec{k}_2\right) \right\rangle_{\mathbb{H}_{\vec{h}}(A)} = \int_{A \setminus B_{\vec{h}}} \vec{v}_1 \cdot \vec{v}_2 \, dy + M \vec{\ell}_1 \cdot \vec{\ell}_2 + J \vec{k}_1 \cdot \vec{k}_2,$$

$$\left\langle \left(\vec{v}_1, \vec{\ell}_1, \vec{k}_1\right), \left(\vec{v}_2, \vec{\ell}_2, \vec{k}_2\right) \right\rangle_{\mathbb{V}_{\vec{h}}(A)} = \int_{A \setminus B_{\vec{h}}} \vec{v}_1 \cdot \vec{v}_2 \, dy + M \vec{\ell}_1 \cdot \vec{\ell}_2 + J \vec{k}_1 \cdot \vec{k}_2 + $$
$$+ 2\nu \int_A \nabla^s \vec{v}_1 : \nabla^s \vec{v}_2 \, dy,$$

where

$$\left(\nabla^s \vec{U}\right)_{ij} = \frac{1}{2} \left(\frac{\partial \vec{U}_i}{\partial x_j} + \frac{\partial \vec{U}_j}{\partial x_i} \right).$$

Remark 2.1. Since $\vec{v}_1$ and $\vec{v}_2$ are rigid movements in $B_{\vec{h}}$, an easy computation shows that the scalar product in $\mathbb{H}_{\vec{h}}(A)$ can be written equivalently as follows:

$$\left\langle \left(\vec{v}_1, \vec{\ell}_1, \vec{k}_1\right), \left(\vec{v}_2, \vec{\ell}_2, \vec{k}_2\right) \right\rangle_{\mathbb{H}(A)} = \int_A \rho_{\vec{h}} \vec{v}_1 \cdot \vec{v}_2 \, dy,$$

where ρ is equal to the liquid's density in $A \setminus B_{\vec{h}}$, and to the solid's density in $B_{\vec{h}}$, so this scalar product is equivalent to the scalar product in $H(A)$.

Moreover, if $\vec{h}(t)$ is a function from $[0,T]$ in A° then we define the following spaces:

$$\begin{aligned}
L^p\left(0,T;\mathbb{H}_{\vec{h}(\cdot)}\right) &= \left\{f\in L^p\left(0,T;\mathbb{H}(A)\right)\ \mid f(t)\in\mathbb{H}_{\vec{h}(t)} \text{ for a.e. } t\in[0,T]\right\},\\
L^p\left(0,T;\mathbb{V}_{\vec{h}(\cdot)}\right) &= \left\{f\in L^p\left(0,T;\mathbb{V}(A)\right)\ \mid f(t)\in\mathbb{V}_{\vec{h}(t)} \text{ for a.e. } t\in[0,T]\right\},\\
\mathcal{C}^k\left([0,T];\mathcal{W}_{\vec{h}(\cdot)}\right) &= \left\{f\in \mathcal{C}^k\left([0,T],\mathcal{W}(A)\right)\ \mid f(t)\in\mathcal{W}_{\vec{h}(t)}\ \forall t\in[0,T]\right\},\\
\mathcal{C}^k\left([0,T];\mathbb{V}_{\vec{h}(\cdot)}\right) &= \left\{f\in \mathcal{C}^k\left([0,T],\mathbb{V}(A)\right)\ \mid f(t)\in\mathbb{V}_{\vec{h}(t)}\ \forall t\in[0,T]\right\},
\end{aligned}$$

where $1\le p\le\infty$ and $0\le k\le\infty$. Let $\rho_s>0$ be a constant denoting the density of the rigid. We recall that the fluid is supposed homogen0us of density eaual to 1. If $\vec{h}:[0,T]\to A^0$ we wefine the function $\rho_{\vec{h}(t)}:A\to\mathbb{R}$ determining the mass distribution ar instant t by

$$\rho_{\vec{h}(t)}(\vec{x})=\begin{cases}1 & if \quad \vec{x}\in\Omega(t)\\ \rho_s & if \quad \vec{x}\in B(t)\end{cases}$$

Our definition of weak solutions will be motivated by the following proposition, which can be proved by the use of simple integration by parts and of a transport formula (see for instance [1] for the details):

Proposition 2.1. *If $(\vec{u},\vec{h},\vec{\omega})$ is a classical solution of* (1.1)–(1.8)*, then the identity:*

$$\begin{aligned}
&-\int_0^T\int_A \rho_{\vec{h}(t)}\vec{u}\cdot\vec{v}' dy + 2\nu\int_0^T\int_A \nabla^s\vec{u}:\nabla^s\vec{v}dydt\\
&-\int_0^T\int_A\left[(\vec{u}\otimes\vec{u}):\nabla^s\vec{v}\right]dydt=\int_A\vec{u}^0\cdot\vec{v}(0)dy
\end{aligned}\tag{2.2}$$

holds true for any $(\vec{v},\vec{\ell},\vec{k})\in\mathcal{C}^1([0,T],\mathbb{V}_{\vec{h}(\cdot)})$ such that $\vec{v}(\cdot,T)=\vec{\ell}(T)=\vec{k}(T)=0$.

We can now define finite energy solutions of (1.1)-(1.8).

Definition 2.1. *A triplet $(\vec{u},\vec{h},\vec{\omega})$ is called a weak solution of* (1.1)-(1.8) *if:*

$$\vec{h}\in W^{1,\infty}(0,T;\mathbb{R}^3)\cap\mathcal{C}^0([0,T];A^\circ),\tag{2.3}$$

$$(\vec{u},\vec{h}',\vec{\omega})\in L^2\left(0,T;\mathbb{V}_{\vec{h}(\cdot)}\right)\cap L^\infty\left(0,T;\mathbb{H}_{\vec{h}(\cdot)}\right)\ \textit{and}\tag{2.4}$$

$$(\vec{u},\vec{h}',\vec{\omega})\ \textit{satisfies}\ (2.2)\ \textit{for any}\ (\vec{v},\vec{\ell},\vec{k})\in\mathcal{C}^1([0,T];\mathbb{V}_{\vec{h}(\cdot)})\tag{2.5}$$

$$\textit{such that}\ \vec{v}(\cdot,T)=\vec{\ell}(T)=\vec{k}(T)=0$$

Remark 2.2. By Proposition 2.1 we know that any classical solution is a weak solution. On the other hand, one can also easily check that any smooth weak solution is a classical one.

The existence of weak solutions in the sense defined above was proved in [1]. The proof given in the reference above cannot be adapted for the case of several rigid bodies. The main result of this paper, stated below, can be easily adapted for the case of several rigid bodies.

Theorem 2.1. *Suppose that* $(\vec{u}_n, \vec{h}_n, \vec{\omega}_n)$ *is a sequence of weak solutions of* (1.1)-(1.8) *such that*

$$\vec{u}_n \to \vec{u} \text{ in } L^2(0,T;V(A)) \text{ weak and in } L^\infty(0,T,H(A)) \text{ weak star.} \tag{2.6}$$

Then

$$\vec{h}_n \to \vec{h} \text{ in } W^{1,\infty}(0,T;\mathbb{R}^3) \text{ weak star,} \tag{2.7}$$

$$\vec{\omega}_n \to \vec{\omega} \text{ in } L^\infty(0,T;\mathbb{R}^3) \text{ weak star.} \tag{2.8}$$

and $(\vec{u}, \vec{h}, \vec{\omega})$ *is a weak solution of* (1.1)-(1.8).

3. Proof of the main result

Lemma 3.1. *Suppose that* $(\vec{u}_n, \vec{h}_n, \vec{\omega}_n)$ *satisfy the assumptions of Theorem* 2.1. *Then* $(\vec{h}_n) \in \mathcal{C}^0([0,T];A^\circ)$ *and relations* (2.7), (2.8) *hold true. Moreover* $\vec{h}_n \to \vec{h}$ *in* $\mathcal{C}^0([0,T];\mathbb{R}^3)$ *strongly and* $\rho_{\vec{h}_n(\cdot)} \to \rho_{\vec{h}(\cdot)}$ *in* $L^p(]0,T[\times A)$ *strongly, for all* $p \in [1,\infty)$.

Proof. Relations (2.7), (2.8) follow by direct integration from (2.6) and the relation

$$\vec{u}_n(\vec{x},t) = \vec{h}_n'(t) + \omega_n(t) \times (\vec{x} - \vec{h}_n(t)),\ \vec{x} \in B_{\vec{h}_n(t)}.$$

Moreover assumption (2.7) and the Arzela-Ascoli theorem imply that $\vec{h}_n \to \vec{h}$ in $\mathcal{C}^0([0,T];\mathbb{R}^3)$ strongly. In order to show that $\rho_{\vec{h}_n(\cdot)} \to \rho_{\vec{h}(\cdot)}$ in $L^p(]0,T[\times\widetilde{A})$ strongly it suffices to remark that there exist a constant $C > 0$ such that

$$\int_0^T\int_{\widetilde{A}} |\rho_{\vec{h}_n(t)} - \rho_{\vec{h}(t)}|^p dydt \le C \int_0^T |\vec{h}_n(t) - \vec{h}(t)| dt.$$

Here above $\mu(\cdot)$ is the usual Lebesgue measure in $\mathbb{R}^3$. □

If $\epsilon \ge 0$ we denote

$$Q^\epsilon_{\vec{h}} = \left\{(x,t) \in A \times (0,T) |\ |x - \vec{h}(t)| > 1 + \epsilon\right\},\ Q_{\vec{h}} = Q^0_{\vec{h}}.$$

Moreover suppose that $D \subset A$ and $\alpha, \beta > 0$ are such that $D \times (\alpha, \beta) \subset Q_{\vec{h}}$. Consider the function spaces

$$M(D) = \left\{\vec{v} \in L^2(\overline{D}, \mathbb{R}^3) \mid \operatorname{div} \vec{v} = 0 \text{ in } D\right\},$$

$$Z(D) = H(D) \cap M(D),\ F(D) = H^2(D)^3 \cap H^1_0(D)^3 \cap M(D).$$

We denote by $P(D)$ the orthogonal projector from $M(D)$ in $Z(D)$. We will need the following compactness result

Lemma 3.2. *Suppose that D, α, β satisfy the assumtions above and denote by $(\vec{w}_n)$ (respectively by $(\vec{w})$ the sequence of restrictions of $\vec{u}_n$ to $D \times (\alpha, \beta)$ (respectively the restriction of $\vec{u}$ to $D \times (\alpha, \beta)$). Then (up to the extraction of a subsequence) $P(D)\vec{w}_n \to P(D)\vec{w}$ strongly in $L^2(D \times (\alpha, \beta))$.*

Proof. Since $[\alpha, \beta] \times D \subseteq Q$, if we choose the test function $\vec{v}$ in (2.2) to be equal to $a(t)\varphi$, with $\varphi \in F(D)$ and $a \in \mathcal{D}(\alpha, \beta)$ we obtain that

$$\frac{d}{dt}\int_D P(D)\vec{w}_n \cdot \vec{\varphi} = \langle f_n, \vec{\varphi}\rangle_{[F(D)]', F(D)}, \tag{3.1}$$

in $\mathcal{D}'(\alpha, \beta)$ where

$$\langle f_n, \vec{\varphi}\rangle_{[F(D)]', F(D)} = -2\nu \int_D \nabla^s \vec{w}_n : \nabla^s \vec{\varphi} dy +$$

$$+\int_D \vec{w}_n \cdot [(\vec{w}_n \cdot \nabla)\vec{\varphi}]dy - \int_D \vec{w}_n \cdot [(\vec{h}'_n \cdot \nabla)\vec{\varphi}]dy = 0, \qquad \forall \vec{\varphi} \in F(D). \tag{3.2}$$

Since $(\vec{w}_n)$ is uniformly bounded in $L_2(0, T; H^1(D))$ relation (3.2) combined with classical estimates (see for instance [4, p. 72–74]) imply that

$$||f_n||_{L^2(0,T;[F(D)]')} \leq M_2, \quad \forall n \geq 1, \tag{3.3}$$

where $M_2 > 0$ is a constant. From (3.1) and (3.3) we obtain

$$||\frac{\partial}{\partial t} P(D)\vec{w}_n||_{L^2(0,T;[F(D)]')} \leq M_2, \quad \forall n \geq 1. \tag{3.4}$$

Using the fact that $(\vec{w}_n)$ is uniformly bounded in $L_2(0, T; H^1(D))$, the compact inclusions $Z(D) \cap H^1(D)^3 \subseteq Z(D) \subseteq [F(D)]'$, relation (3.4) and a version of Aubin's lemma (see for instance [3]) we obtain that the sequence $(P(D)\vec{w}_n)$ is relatively compact in $L^2((\alpha, \beta) \times D)$. $\square$

Proof of Theorem 2.1. We will follow an idea from [3]. Let $\epsilon > 0$ and ψ be a smooth function with $\nabla^s \psi = 0$ for $(\vec{x}, t) \notin Q^\epsilon_{\vec{h}}$. Due to Lemma 3.1 we have that $\psi \in \mathcal{C}^k\left([0, T]; \mathbb{V}_{\vec{h}_n(\cdot)}\right)$ for n large enough. Since all the linear terms in (2.2) obviously pass to the limit it suffices to show that

$$\int_0^T \int_A \rho_{\vec{h}_n(t)} \vec{u} \cdot \vec{\psi}' dy \to \int_0^T \int_A \rho_{\vec{h}(t)} \vec{u} \cdot \vec{\psi}' dy, \tag{3.5}$$

$$\int_0^T \int_A (\vec{u}_n \otimes \vec{u}_n) : \nabla^s \psi dx dt \to \int_0^T \int_A (\vec{u} \otimes \vec{u}) : \nabla^s \psi dx dt. \tag{3.6}$$

Relation (3.5) folows directly from Lemma 3.1 so we concentrate on (3.6). Denote by $\vec{\gamma}$ a function such that $\nabla^s \vec{\gamma} \equiv 0$ and $\vec{\gamma}(x, t) = \vec{\psi}(x, t)$ if $(\vec{x}, t) \notin Q^\epsilon_{\vec{h}}$. Then

$$\int_0^T \int_A (\vec{u}_n \otimes \vec{u}_n) : \nabla^s \vec{\psi} dx dt = \int_0^T \int_A (\vec{u}_n \otimes \vec{u}_n) : \nabla^s \vec{\eta} dx dt$$

where $\vec{\eta} = \vec{\psi} - \vec{\gamma}$, $\vec{\eta}(x,t) = \vec{\gamma}(x,t)$ for $x \in \partial A$ and $\vec{\eta}(\vec{x},t) = 0$, for $(\vec{x},t) \notin Q^{\epsilon}_{\vec{h}}$. Moreover, by Lemma 7.1 in [4, ch.1, p.103], for any $\delta > 0$ there exists a function $g \in L_2(0,T;H^1(\Omega))$, div $\vec{g} = 0$, such that $\vec{g}(x,t) = 0$ for $(\vec{x},t) \notin Q^{\epsilon}_{\vec{h}}$, $\vec{g}(x,t) = \gamma(x,t)$ for $x \in \partial A$ and

$$\left| \int_0^T \int_A (\vec{u}_n \otimes \vec{u}_n) : \nabla^s \vec{g} dx dt \right| \leq \delta \|\vec{u}_n\|^2_{L^2(0,T;H^1(A))}.$$

Since $(\vec{u}_n)$ is uniformly bounded in $L^2(0,T;H^1_0(A))$, the inequality above shows that it is enough to pass to the limit for test functions $\vec{\eta}$ which are equal to zero outside $Q^{\epsilon}_{\vec{h}}$.

Consider an arbitrary cylinder $E = D \times [\alpha, \beta]$, $\alpha, \beta \in (0,T)$, included in $Q^{\epsilon}_{\vec{h}}$. We suppose that D has a smooth boundary. and we take the test function $\vec{v}$ equal to zero outside E. Let $P(D)$ be the orthogonal projector from $M(D)$ in $Z(D)$. According to a result in [6, p.16], any function $\vec{w} \in M(D)$ can be written

$$\vec{w} = P(D)\vec{w} + \nabla \zeta^D,$$

where ζ^D is harmonic in D. Due to Lemma 3.2 we have only to prove that

$$\int_{[0,T]\times A} (\nabla \zeta^D_n \otimes \nabla \zeta^D_n) : \nabla^s \vec{\eta} dx dt = \int_{[0,T]\times A} (\nabla \zeta^D \otimes \nabla \zeta^D) : \nabla^s \vec{\eta} dx dt \quad (3.7)$$

But, since $\vec{\eta}(x) = 0$ for $x \in \partial D$,

$$\int_D (\nabla \zeta^D \otimes \nabla \zeta^D) : \nabla^s \vec{\eta} dx = -\int_D div(\nabla \zeta^D \otimes \nabla \zeta^D) \cdot \vec{\eta} dx =$$

$$= -\int_D (\Delta \zeta^D \nabla \zeta^D + \frac{1}{2} \nabla |\nabla \zeta^D|^2) \cdot \vec{\eta} dx = \int_D \frac{1}{2} |\nabla \zeta_D|^2) \text{div } \vec{\eta} dx = 0.$$

we conclude that

$$\int_E (\nabla \zeta^D_n \otimes \nabla \zeta^D_n) : \nabla^s(\eta) dx dt = \int_E (\nabla \zeta^D \otimes \nabla \zeta^D) : \nabla^s(\eta) dx dt = 0, \ \forall n \geq 1,$$

so (3.7) holds true. Since E is arbitrary we obtain the conclusion of the Theorem. □

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