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Existence, uniqueness and homogenization for ferroelectric materials

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Abstract - We discuss existence, uniqueness, and homogenization results for a class of rate-independent, nonlinear ferroelectric models. We show that these models can be formulated in an energetic framework which is based on the elastic and the electric displacements as reversible variables and on interior, irreversible variables like the remanent polarization. The PDE system defining the model is restated in terms of a stability condition and an energy balance law using an energy-storage functional and a dissipation functional.
We provide quite general conditions on the constitutive laws which guarantee the existence of a solution. Under more restrictive assumptions we are also able to establish uniqueness results.
By the method of *weak and strong two-scale convergence* via *periodic unfolding*, we show that the solutions of the problem with periodicity converge to the energetic solution of the homogenized problem associated with the corresponding Γ-limits of the functionals. The main difficulties are the nonlinearity of the model, as well as the general form considered for the stored energy, which is neither convex nor quadratic.

Key words and phrases : rate independence, ferroelectric materials, polarization, energetic formulation, dissipation potential.

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1. Introduction

Ceramic materials and single crystals showing ferroelectric behavior are being used in many applications in electronics and optics, in areas such as dielectric ceramics for capacitor applications, ferroelectric thin films for nonvolatile memories, piezoelectric materials for medical ultrasound imaging and actuators, and electro-optic materials for data storage and displays.

While for homogenization in the case of linear ferroelectric (piezoelectric) models there is a rich literature, for the nonlinear case we found only few references (see for instance [5, 12, 26]). The nonlinear model considered here was proposed in [20]. This model (recalled in Section 2) captures the hysteretic behavior of ferroelectrics, by keeping in the mean time the general perspective for treating multi-axial behavior and complex geometries. It is based on the rate-independent, three-dimensional models used in the engineering literature; see [14, 10, 8, 9, 7, 15, 11, 24]. These models work in the framework of small deformations and the quasistatic approximation for the elastic and electrostatic equilibria. However, certain internal variables q, like the remanent polarization, are history dependent by an activation threshold, thus leading to a rate-independent evolution process.

Using as primary reversible variables the elastic displacement $u : \Omega \to \mathbb{R}^d$ and the electric displacement $D : \mathbb{R}^d \to \mathbb{R}^d$, the process can be written in an energetic formulation which is based on the stored-energy functional

$$\begin{aligned}\mathcal{E}(t,u,D,q) = \int_\Omega \big(W(x,\varepsilon(u),D,q) + \alpha(x,\nabla q)\big)\,\mathrm{d}x \\ + \int_{\mathbb{R}^d\setminus\Omega} \frac{1}{2\epsilon_0}|D|^2\,\mathrm{d}x - \langle \ell(t),(u,D)\rangle\end{aligned}$$

and a dissipation potential of the form

$$\mathcal{R}(\dot q(t)) = \int_\Omega R(x,\dot q(t,x))\,\mathrm{d}x.$$

The nonlocal term $\alpha(x,\nabla q)$ in $\mathcal{E}$ usually takes the form $\frac{\kappa}{2}|\nabla q|^2$, with $\kappa > 0$. This term penalizes rapid changes of the internal variable by introducing a length scale which determines the minimal width of the interfaces between domains of different polarization.

The energetic formulation was originally developed for shape-memory alloys but ever since it has been shown to apply for many different rate-independent material models such as finite-strain elastoplasticity, damage, brittle fracture, delamination and vortex pinning in superconductors; see [16] for a survey.

The theory is based on a purely static stability condition (S) and on the energy balance (E), which have to hold for all $t \in [0,T]$:

$$\text{(S):}\quad \mathcal{E}(t,u(t),D(t),q(t)) \le \mathcal{E}(t,\widehat u,\widehat D,\widehat q) + \mathcal{R}(\widehat q - q(t)) \quad \text{for all } \widehat u,\widehat D,\widehat q\,;$$

$$\begin{aligned}\text{(E):}\quad \mathcal{E}(t,u(t),D(t),q(t)) + \int_0^t \mathcal{R}(\dot q(s)))\,\mathrm{d}s = \mathcal{E}(0,u(0),D(0),q(0)) \\ - \int_0^t \langle \dot\ell(s),(u(s),D(s))\rangle\,\mathrm{d}s.\end{aligned}$$

The major advantage of this formulation is that it does involve neither derivatives of the constitutive functions W and R, nor derivatives of the solution (u,D,q) that are not controlled by the energy functional $\mathcal{E}$. The dissipation integral $\int_0^t \mathcal{R}(\dot q(s)))\,\mathrm{d}s$ can be reformulated as a total variation, hence time derivatives of the solutions do not occur.

In [20] we proved the existence of solutions for (S) & (E) in suitable function spaces and uniqueness under more restrictive conditions.

After introducing the length scale parameter ε, we assume that W, α, R depend periodically on x with a period proportional with ε and we have to deal with the following functionals.

$$\begin{aligned}\mathcal{E}_\varepsilon(t,u,D,q) \;=\; & \int_\Omega \big(W(\tfrac{x}{\varepsilon}, \boldsymbol{e}(u), D, q) + \alpha(\tfrac{x}{\varepsilon}, \nabla q)\big)\,\mathrm{d}x \\ & + \int_{\mathbb{R}^d\setminus\Omega} \frac{1}{2\epsilon_0}|D|^2\,\mathrm{d}x - \langle \ell(t), (u,D)\rangle,\end{aligned}$$

$$\mathcal{R}_\varepsilon(\dot q) = \int_\Omega R\left(\tfrac{x}{\varepsilon}, \dot q(x)\right)\mathrm{d}x.$$

Our purpose is to show that the solutions of the energetic problem associated with $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$ converge to some limits which are the solutions of a suitable homogenized problem. Because of the non-smoothness and the hysteretic behavior of the evolution of the systems it will not be possible to find a homogenized limit equation in the classical sense. This would mean to find limit functionals defined on the same domain Ω. Instead we will need the so-called two-scale homogenization that decomposes solutions into macroscopic and microscopic behavior. The choice of the two-scale homogenization is particularly appropriate for numerical simulations.

We are able to prove our homogenization result by using some technical tools already established in [21]. Some of these tools are related to the classical notion of two-scale convergence introduced by Nguetseng [23] in 1989 and developed by Allaire [2] in 1992. This concept is now used in various applications in continuum mechanics (see for instance [1, 3, 13, 22, 29]).

One of the main difficulties is to show that weak two-scale limits of stable states are still stable. For this we will use a sufficient condition from [17], that asks for the existence of a *joint recovery sequence* satisfying the variational inequality (5.1) below. In [21] this was proved by exploiting the quadratic nature of the energies. Here this is no longer possible, since the free energy has a quite general form. Thus our homogenization result may be viewed as a generalization of that from [21]. Instead of using the "quadratic trick" we will take advantage of some results from Section 4 (Theorem 4.1 and Proposition 4.4). The lack of convexity of W is another difficulty that we have to face, but here the still valid weakly lower semicontinuity of the energy functionals saved us. In [26] homogenization for piezoelectric composites with periodic microstructure is also performed. Only the internal energy functional is homogenized, but not the dissipational one. This functional does not depend on the time and on the internal variables, and it is assumed to be convex.

Our paper is organized as follows. In Section 2 we recall the ferroelectric model from [20] and the existence result obtained there. In Section 3 we formulate our ε-problem as well as a suitable homogenized problem. In

Section 4 we summarize the relevant material on *weak and strong two-scale convergence, periodic folding operator and unfolding operator* and some related results (for the proofs we refer the reader to [21] and [28]). In Section 5 our homogenization result is stated and proved.

2. Modeling and existence result for ferroelectric materials

For the convenience of the reader, in this section we recall the model proposed in [20].

The basic quantities in the theory are the elastic displacement field $u : \Omega \to \mathbb{R}^d$ and the electric displacement field $D : \mathbb{R}^d \to \mathbb{R}^d$. Here, the electric displacement is also defined outside the body, as the interior polarization of a ferroelectric material generates an electric field E and a displacement D in all of $\mathbb{R}^d$ via the static Maxwell equation in $\mathbb{R}^d$. Commonly, the polarization P is used for modeling, and it is defined via

$$D = \epsilon_0 E + P,$$

where ϵ_0 the dielectric constant (or permetivity) in the medium surrounding the body Ω. In contrast to D and E, the polarization P is defined only inside the body Ω, and set equal to 0 outside. Nonetheless, our formulation stays with D, since it leads to a simple and consistent thermomechanical model.

In addition, we use internal variables $q : \Omega \to \mathbb{R}^{d_q}$ which, for instance, may be taken as a remanent strain $\boldsymbol{\varepsilon}_{\mathrm{rem}}$ or a remanent polarization P_{rem}.

The stored-energy functional has the following form:

$$\begin{aligned}\mathcal{E}(t,u,D,q) = \int_\Omega \big(W(x,\boldsymbol{\varepsilon}(u),D,q) + \alpha(x,\nabla q)\big)\,\mathrm{d}x \\ + \int_{\mathbb{R}^d\setminus\Omega} \frac{1}{2\epsilon_0}|D|^2\,\mathrm{d}x - \langle \ell(t),(u,D)\rangle,\end{aligned} \tag{2.1}$$

where W is the Helmholtz free energy and $\boldsymbol{\varepsilon}(u)$ is the infinitesimal strain tensor given by

$$\boldsymbol{\varepsilon}(u) = \frac{1}{2}(\nabla u + \nabla u^{\mathsf{T}}) \in \mathbb{R}^{d\times d}_{\mathrm{sym}} := \{\boldsymbol{\varepsilon} \in \mathbb{R}^{d\times d} \mid \boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}^{\mathsf{T}}\}. \tag{2.2}$$

The external loading $\ell(t)$ depends on the process time t and it is usually given by

$$\begin{aligned}\langle \ell(t),(u,D)\rangle = \int_{\mathbb{R}^d} E_{\mathrm{ext}}(t,x)\cdot D(x)\,\mathrm{d}x + \int_\Omega f_{\mathrm{vol}}(t,x)\cdot u(x)\,\mathrm{d}x \\ + \int_{\Gamma_{\mathrm{Neu}}} f_{\mathrm{surf}}(t,x)\cdot u(x)\,\mathrm{d}a(x),\end{aligned}$$

where E_{ext}, f_{vol} and f_{surf} are applied, external fields.

For the dissipation potential $\mathcal{R}$ we take the very simple form

$$\mathcal{R}(\dot{q}) = \int_\Omega R(x, \dot{q}(x))\,\mathrm{d}x, \tag{2.3}$$

where $R(x,\cdot) : \mathbb{R}^{d_q} \to [0,\infty)$ is a convex function which is positively homogeneous of degree 1. Note that the dissipation potential acts on the rate $\dot{q} = \frac{\partial}{\partial t} q$ of the internal variable only.

To formulate the rate-independent evolution law we use the thermomechanically conjugated forces

$$\sigma = \frac{\partial}{\partial \boldsymbol{\varepsilon}} W \in \mathbb{R}^{d\times d}_{\mathrm{sym}}, \quad E = \begin{cases} \frac{\partial}{\partial D} W & \text{on } \Omega, \\ \frac{1}{\epsilon_0} D & \text{on } \mathbb{R}^d \backslash \Omega \end{cases}, \qquad X_q \in \mathbb{R}^{d_q},$$

where $X_q = \frac{\partial}{\partial q} W - \mathrm{div}(\mathrm{D}\alpha(x, \nabla q))$ and σ is the stress tensor and E the electric field. The elastic equilibrium equation and the Maxwell equations read

$$- \operatorname{div} \sigma + f_{\mathrm{vol}}(t,\cdot) = 0 \quad \text{in } \Omega, \tag{2.4a}$$

$$\operatorname{div} D = 0 \text{ and } \operatorname{curl}(E - E_{\mathrm{ext}}(t,\cdot)) = 0 \quad \text{in } \mathbb{R}^d, \tag{2.4b}$$

where $\operatorname{curl} E$ is defined as $\nabla E - (\nabla E)^{\mathsf{T}}$ for general dimensions. Thus, these equations are static and respond instantaneously to changes of the loadings $f_{\mathrm{vol}}(t,\cdot)$ and $E_{\mathrm{ext}}(t,\cdot)$.

The evolution of q follows a force balance which uses the multi-valued dissipational force

$$\partial R(x,\dot{q}) = \{\, X \in \mathbb{R}^{d_q} \mid R(x,V) \geq R(x,\dot{q}) + X\cdot(V - \dot{q}) \text{ for all } V \in \mathbb{R}^{d_q} \,\},$$

which is in fact the subdifferential of the convex function $R(x,\cdot)$. The force balance takes the simple form

$$0 \in \partial R(x,\dot{q}) + X_q. \tag{2.5}$$

We now want to consider these equations in appropriate functions spaces. The space $\mathcal{F}$ of the variable (u, D) is defined by

$$\mathcal{F} = \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d) \times \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d),$$
$$\text{where} \begin{cases} \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d) := \{\, \psi \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{R}^d) \mid \operatorname{div}\psi = 0 \,\}, \\ \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega;\mathbb{R}^d) := \{ v \in \mathrm{H}^1(\Omega,\mathbb{R}^d) \mid v|_{\Gamma_{\mathrm{Dir}}} \equiv 0 \}. \end{cases}$$

The space $\mathcal{Q}$ contains the internal state functions q and is taken to be $\mathrm{H}^1(\Omega;\mathbb{R}^{d_q})$. We define the state space

$$\mathcal{Y} = \mathcal{F} \times \mathcal{Q}.$$

The definition of the space $\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d)$ already includes Gauß' law, which is part of the Maxwell equations. Using the well-known fact (cf. [27, Thm. 1.4]) that the total space $\mathrm{L}^2(\mathbb{R}^d;\mathbb{R}^d)$ decomposes in an orthogonal way into the two closed subspaces $\mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d)$ and

$$\mathrm{L}^2_{\mathrm{curl}}(\mathbb{R}^d) = \{\psi \in \mathrm{L}^2(\mathbb{R}^d;\mathbb{R}^d) \mid \mathrm{curl}\,\psi = 0\}$$

we obtain the following result.

Proposition 2.1. *Assume that W is twice continuously differentiable and satisfies suitable growth conditions. Let $\mathrm{D}_D\mathcal{E}(t,u,D,q)[\widehat{D}]$ denote the Gâteaux derivative of $\mathcal{E}$ in the direction $\widehat{D}$. Then*

$$\begin{aligned} &\mathrm{D}_D\mathcal{E}(t,u,D,q)[\widehat{D}] = 0 \text{ for every } \widehat{D} \in \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d) \\ &\iff \mathrm{curl}\left(\tfrac{\partial}{\partial D}\widetilde{W} - E_{\mathrm{ext}}(t,\cdot)\right) = 0 \text{ in } \mathbb{R}^d, \end{aligned}$$

where $\widetilde{W} = W$ for $x \in \Omega$ and $\widetilde{W} = \frac{1}{2\epsilon_0}|D|^2$ else.

Thus, we implement the Maxwell equations (2.4b) simply by choosing a suitable functions space and the condition $\mathrm{D}_D\mathcal{E}(t,u,D,q) = 0$. Similarly, the elastic equilibrium (2.4a) is obtained by $\mathrm{D}_u\mathcal{E}(t,u,D,q) = 0$.

The dissipative force balance can also be rewritten in functional form, and so the full problem may be written as

$$\begin{cases} \mathrm{D}_u\mathcal{E}(t,u(t),D(t),q(t)) = 0, \\ \mathrm{D}_D\mathcal{E}(t,u(t),D(t),q(t)) = 0, \\ 0 \in \partial\mathcal{R}(\dot{q}(t)) + \mathrm{D}_q\mathcal{E}(t,u(t),D(t),q(t)). \end{cases} \tag{2.6}$$

In fact, our theory is not based on the force balance (2.6). Instead, following [19, 18], we use a weaker formulation which is based on energies only. Under suitable smoothness and convexity assumptions, the energetic formulation is equivalent to (2.6). We call (u,D,q) an ***energetic solution*** of the problem associated with $\mathcal{E}$ and $\mathcal{R}$, if for all $t \in [0,T]$ the *stability condition* (S) and the *energy balance* (E) hold:

$$\begin{aligned} &\text{(S):}\quad \mathcal{E}(t,u(t),D(t),q(t)) \le \mathcal{E}(t,\widehat{u},\widehat{D},\widehat{q}) + \mathcal{R}(\widehat{q}-q(t)) \quad \text{for all } \widehat{u},\widehat{D},\widehat{q}; \\ &\text{(E):}\quad \mathcal{E}(t,u(t),D(t),q(t)) + \int_0^t \mathcal{R}(\dot{q}(s)))\,\mathrm{d}s = \mathcal{E}(0,u(0),D(0),q(0)) \\ &\qquad\qquad - \int_0^t \langle \dot{\ell}(s),(u(s),D(s))\rangle\,\mathrm{d}s. \end{aligned} \tag{2.7}$$

For every $t \in [0,T]$ let us define the *set of stable states*

$$\mathcal{S}(t) = \left\{ (u,D,Q) \in \mathcal{Y} \;\middle|\; \begin{array}{r} \mathcal{E}(t,u,D,q) \le \mathcal{E}(t,\widehat{u},\widehat{D},\widehat{q}) + \mathcal{R}(\widehat{q}-q) \\ \text{for all } (\widehat{u},\widehat{D},\widehat{q}) \in \mathcal{Y} \end{array} \right\}.$$

Thus (S) reads as $(u(t), D(t), q(t)) \in \mathcal{S}(t)$.

Suppose Ω and Γ_{Dir} comply with the assumptions from Korn's inequality, that is, $\Omega \subset \mathbb{R}^d$ is a nonempty connected open bounded set with Lipschitz boundary Γ, and Γ_{Dir} is a measurable subset of Γ, such that $\int_{\Gamma_{\mathrm{Dir}}} 1\,\mathrm{d}a > 0$.

We now impose suitable conditions on the functions W, α, R, in order to get our existence result. The function $R : \Omega \times \mathbb{R}^{d_q} \to [0, \infty)$ satisfies the conditions

$$\begin{cases} R \in \mathrm{C}^0(\overline{\Omega} \times \mathbb{R}^{d_q}), \\ c_R|V| \le R(x, V) \le C_R|V| \quad \text{for all } x \in \Omega, V \in \mathbb{R}^{d_q} \end{cases} \tag{A1}$$

for some fixed constants $c_R, C_R > 0$,

$$R(x, \cdot) : \mathbb{R}^{d_q} \to [0, \infty) \text{ is 1-homogeneous and convex for every } x \in \Omega. \tag{A2}$$

The functions W and α have to fulfill the following conditions:

$$\begin{cases} W : \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \times \mathbb{R}^{d_q} \to [0, \infty] \text{ is a Caratheodory function,} \\ \alpha : \Omega \times \mathbb{R}^{d_q \times d} \to [0, \infty] \text{ is a Caratheodory function.} \end{cases} \tag{A3}$$

For W (and similarly for α) this means that the function $W(\cdot, \boldsymbol{\varepsilon}, D, q)$ is measurable on Ω for each $(\boldsymbol{\varepsilon}, D, q)$, and that the mapping $W(x, \cdot, \cdot, \cdot)$ is continuous on $\mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \times \mathbb{R}^{d_q}$ for a.e. $x \in \Omega$. Further, we need coercivity and convexity assumptions:

$$\begin{gathered} W(x, \boldsymbol{\varepsilon}, D, q) + \alpha(x, V) \ge c(|\boldsymbol{\varepsilon}|^2 + |D|^2 + |q|^2 + |V|^2) - C \\ \text{for every } (x, \boldsymbol{\varepsilon}, D, q, V) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \times \mathbb{R}^{d_q} \times \mathbb{R}^{d_q \times d}, \end{gathered} \tag{A4}$$

for some fixed constants $c, C > 0$,

$$\begin{cases} W(x, \cdot, \cdot, q) : \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \to [0, \infty] \text{ is convex for every } (x, q) \in \Omega \times \mathbb{R}^{d_q}, \\ \alpha(x, \cdot) : \mathbb{R}^{d_q \times d} \to [0, \infty] \text{ is convex for every } x \in \Omega. \end{cases} \tag{A5}$$

The fact that convexity in the variable q is not needed is the basis for the ability to model the ferroelectric effect, since the choices of W presented in the applied works are certainly not convex in q.

For the external loading $\ell(t)$ we assume that

$$\ell \in \mathrm{C}^1\big([0, T], (\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d))^* \times \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d)^*\big). \tag{A6}$$

In [20] we showed that under the above reasonable continuity and convexity assumptions the energetic formulation (S) & (E) has solutions for suitable initial data. The uniqueness was established under much stronger conditions on the constitutive functions W and α.

Theorem 2.1 (Existence theorem) *Assume* (A1)–(A6) *hold. Then for each stable initial data* (u_0, D_0, q_0) $(\in \mathcal{S}(0))$ *the energetic problem* (S) & (E) *from* (2.7) *has a solution* $(u, D, q) : [0, T] \to \mathcal{Y}$*, with*

$$(u(0), D(0), Q(0)) = (u_0, D_0, q_0).$$

Moreover, we have $(u, D, q) \in \mathrm{L}^\infty([0, T]; \mathcal{Y})$.

3. The homogenization problem

3.1. ε problem

Let $d \in \mathbb{N}$ be the space dimension. The periodicity in $\mathbb{R}^d$ is expressed by a d-dimensional periodicity lattice

$$\Lambda = \left\{ \lambda = \sum_{j=1}^{d} k_j b_j \;:\; k = (k_1, k_2, ..., k_d) \in \mathbb{Z}^d \right\},$$

where $\{b_1, ..., b_d\}$ is an arbitrary vector basis in $\mathbb{R}^d$. The associated unit cell is

$$Y = \left\{ x = \sum_{1}^{d} \gamma_j b_j \;\middle|\; \gamma_j \in [0, 1) \right\} \subset \mathbb{R}^d,$$

and so $\mathbb{R}^d$ is the disjoint union of all translated cells $\lambda + Y$, with $\lambda \in \Lambda$. For simplicity, we may assume further on that $\Lambda = \mathbb{Z}^d$, hence that $Y = [0, 1)^d$.

We introduce a length scale parameter $\varepsilon > 0$ and we assume that material properties are periodic with respect to the microscopic lattice $\varepsilon\Lambda$. In addition to the conditions (A1)–(A6) from Section 2, we assume the functions W, α, R to be Y-periodic in the first argument. For every $\varepsilon > 0$ let us consider the following energy and respectively dissipation functionals.

$$\begin{aligned} \mathcal{E}_\varepsilon(t, u, D, q) = \int_\Omega \big(W(\tfrac{x}{\varepsilon}, \boldsymbol{e}(u), D, q) + \alpha(\tfrac{x}{\varepsilon}, \nabla q) \big) \,\mathrm{d}x \\ + \int_{\mathbb{R}^d \setminus \Omega} \frac{1}{2\epsilon_0} |D|^2 \,\mathrm{d}x - \langle \ell(t), (u, D) \rangle, \end{aligned} \tag{3.1}$$

$$\mathcal{R}_\varepsilon(\dot{q}) = \int_\Omega R\left(\tfrac{x}{\varepsilon}, \dot{q}(x) \right) \mathrm{d}x. \tag{3.2}$$

We call $(u_\varepsilon, D_\varepsilon, q_\varepsilon)$ an ***energetic solution*** of the problem associated with $\mathcal{E}_\varepsilon$ and $\mathcal{R}_\varepsilon$, if for every $t \in [0, T]$ the *stability condition* (S^ε) and the *energy balance* (E^ε) hold:

$$\begin{aligned} &(\mathrm{S}^\varepsilon)\text{: } \mathcal{E}_\varepsilon(t, u_\varepsilon(t), D_\varepsilon(t), q_\varepsilon(t)) \le \mathcal{E}_\varepsilon(t, \widehat{u}, \widehat{D}, \widehat{q}) + \mathcal{R}_\varepsilon(\widehat{q} - q_\varepsilon(t)) \quad \text{for all } \widehat{u}, \widehat{D}, \widehat{q}; \\ &(\mathrm{E}^\varepsilon)\text{: } \mathcal{E}_\varepsilon(t, u_\varepsilon(t), D_\varepsilon(t), q_\varepsilon(t)) + \textstyle\int_0^t \mathcal{R}_\varepsilon(\dot{q}_\varepsilon(s))) \,\mathrm{d}s \\ &\qquad = \mathcal{E}_\varepsilon(0, u_\varepsilon(0), D_\varepsilon(0), q_\varepsilon(0)) - \textstyle\int_0^t \langle \dot{\ell}(s), (u_\varepsilon(s), D_\varepsilon(s)) \rangle \,\mathrm{d}s. \end{aligned} \tag{3.3}$$

For every $t \in [0,T]$ let us define the *set of ε-stable states*

$$\mathcal{S}_\varepsilon(t) = \{(u,D,Q) \in \mathcal{Y} \,|\, \mathcal{E}_\varepsilon(t,u,D,q) \le \mathcal{E}_\varepsilon(t,\widehat{u},\widehat{D},\widehat{q}) + \mathcal{R}_\varepsilon(\widehat{q}-q) \\ \text{for all } (\widehat{u},\widehat{D},\widehat{q}) \in \mathcal{Y}\}.$$

Thus (S^ε) reads as $(u_\varepsilon(t), D_\varepsilon(t), q_\varepsilon(t)) \in \mathcal{S}_\varepsilon(t)$.

Applying the same reasoning as for Theorem 2.1 leads to the following existence result.

Proposition 3.1. *For all $\varepsilon > 0$ and stable $(u_\varepsilon^0, D_\varepsilon^0, q_\varepsilon^0) \in \mathcal{S}_\varepsilon(0)$, the energetic problem (S^ε) & (E^ε) has a solution $(u_\varepsilon, D_\varepsilon, q_\varepsilon) : [0,T] \to \mathcal{Y}$, with*

$$(u_\varepsilon(0), D_\varepsilon(0), q_\varepsilon(0)) = (u_\varepsilon^0, D_\varepsilon^0, q_\varepsilon^0).$$

Moreover, we have $(u_\varepsilon, D_\varepsilon, q_\varepsilon) \in \mathrm{L}^\infty([0,T];\mathcal{Y})$.

3.2. The homogenized problem

In this section we formulate the problem $(\mathbf{S})$ & $(\mathbf{E})$. Our final aim is to prove that $(\mathbf{S})$ & $(\mathbf{E})$ is the two-scale homogenized problem for (S^ε) & (E^ε). We first introduce some useful functions spaces.

Definition 3.1. *Let $\mathrm{C}^\infty_{\mathrm{per}}(Y)$ be the subspace of $\mathrm{C}^\infty(\mathbb{R}^d)$ consisting of Y-periodic functions. Let $\mathrm{H}^1_{\mathrm{per}}(Y)$ denote the closure of $\mathrm{C}^\infty_{\mathrm{per}}(Y)$ for the H^1-norm. Let us also consider the Banach space*

$$\mathrm{H}^1_{\mathrm{av}}(Y) := \left\{ u \in \mathrm{H}^1_{\mathrm{per}}(Y) \,\middle|\, \int_Y u(y)\mathrm{d}y = 0 \right\}.$$

The Banach space $\mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(Y))^d$ is considered according to the following definition.

Definition 3.2. *For any Banach space X, measurable set $T \subset \mathbb{R}^d$, and $p \in [1,\infty]$, let $\mathrm{L}^p(T;X)$ denote the Banach space of (equivalence classes of) strongly measurable maps $u : T \to X$, such that $\|u(\cdot)\|_X \in \mathrm{L}^p(T)$.*

Let us consider

$$\begin{aligned}
&\mathbf{H} := \mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(Y))^d,\\
&\mathrm{L}^2_{\mathrm{Div}}(\mathbb{R}^d \times Y) := \left\{ \psi \in \mathrm{L}^2(\mathbb{R}^d \times Y)^d \,\middle|\, \mathrm{div}_y\,\psi = 0\,, \int_Y \mathrm{div}_x\,\psi(x,y)\,\mathrm{d}y = 0 \right\},\\
&\mathbf{Q} := \mathrm{H}^1(\Omega)^{d_q} \times \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(Y))^{d_q},\\
&\mathbf{Z} := \mathbf{H} \times \mathrm{L}^2_{\mathrm{Div}}(\mathbb{R}^d \times Y) \times \mathbf{Q}.
\end{aligned}$$

For all $U = (u_0, U_1) \in \mathbf{H}, \mathbb{D} \in \mathrm{L}^2_{\mathrm{Div}}(\mathbb{R}^d \times Y)$, and $Q = (q_0, Q_1) \in \mathbf{Q}$, let us define

$$\mathbf{E}(t, U, \mathbb{D}, Q) = \int_\Omega \int_Y \big(W(y, \hat{\boldsymbol{e}}(U), \mathbb{D}, q_0) + \alpha(y, \nabla_x q_0 + \nabla_y Q_1)\big)\, \mathrm{d}y\, \mathrm{d}x \\ + \int_{\mathbb{R}^d \setminus \Omega} \int_Y \frac{1}{2\epsilon_0} |\mathbb{D}|^2 \, \mathrm{d}y\, \mathrm{d}x - \langle \ell(t), (u_0, \overline{\mathbb{D}}) \rangle,$$

where $\overline{\mathbb{D}}(x) := \int_Y \mathbb{D}(x, y)\, \mathrm{d}y$ for every $x \in \mathbb{R}^d$ and $\hat{\boldsymbol{e}}(U) := \boldsymbol{e}_x(u_0) + \boldsymbol{e}_y(U_1)$, that is, $\hat{\boldsymbol{e}}(U)(x, y) = \boldsymbol{e}_x(u_0(\cdot))(x) + \boldsymbol{e}_y(U_1(x, \cdot))(y)$. We also consider the functional

$$\mathbf{R}(\dot{Q}) = \int_\Omega \int_Y R(y, \dot{q}_0(x))\, \mathrm{d}y\, \mathrm{d}x.$$

The energetic formulation for the two-scale homogenized problem $(\mathbf{S})\,\&\,(\mathbf{E})$ reads: *for all $t \in [0, T]$, the stability condition* $(\mathbf{S})$ *and the energy balance* $(\mathbf{E})$ *hold, where*

$$(\mathbf{S}):\ \mathbf{E}(t, U(t), \mathbb{D}(t), Q(t)) \le \mathbf{E}(t, \widetilde{U}, \widetilde{\mathbb{D}}, \widetilde{Q}) + \mathbf{R}(\widetilde{Q} - Q(t)) \\ \text{for every } (\widetilde{U}, \widetilde{\mathbb{D}}, \widetilde{Q}) \in \mathbf{Z},$$

$$(\mathbf{E}):\ \mathbf{E}(t, U(t), \mathbb{D}(t), Q(t)) + \int_0^t \mathbf{R}(\dot{Q}(s))\, \mathrm{d}s = \mathbf{E}(0, U(0), \mathbb{D}(0), Q(0)) \\ - \int_0^t \langle \dot{\ell}(s), (u_0(s), \overline{\mathbb{D}}(s)) \rangle \, \mathrm{d}s.$$

4. Needed results

In this section we summarize the relevant material from [21].

Throughout this section, the domain Ω will be a bounded open subset of $\mathbb{R}^d$. Let $\mathcal{D}(\Omega; \mathrm{C}^\infty_{\mathrm{per}}(Y))$ denote the space of all measurable functions u on $\Omega \times \mathbb{R}^d$, with $u(x, \cdot) \in \mathrm{C}^\infty_{\mathrm{per}}(Y)$ for every $x \in \Omega$, and such that the map $\Omega \ni x \mapsto u(x, \cdot) \in \mathrm{C}^\infty_{\mathrm{per}}(Y)$ is indefinitely differentiable and with compact support, that is, $u \in \mathrm{C}^\infty_{\mathrm{c}}(\Omega; \mathrm{C}^\infty_{\mathrm{per}}(Y))$.

From now on we will assume that $p \in (1, \infty)$.

Definition 4.1. *Let $(v_\varepsilon)_\varepsilon$ be a sequence in $\mathrm{L}^p(\Omega)$. One says that $(v_\varepsilon)_\varepsilon$ two-scale converges to $V = V(x, y)$ in $\mathrm{L}^p(\Omega \times Y)$, if for any function $\psi = \psi(x, y)$ in $\mathcal{D}(\Omega; \mathrm{C}^\infty_{\mathrm{per}}(Y))$, one has*

$$\lim_{\varepsilon \to 0} \int_\Omega v_\varepsilon(x) \psi\left(x, \frac{x}{\varepsilon}\right) \, \mathrm{d}x = \int_\Omega \int_Y V(x, y) \psi(x, y) \, \mathrm{d}y \, \mathrm{d}x. \tag{4.1}$$

We then write $v_\varepsilon \overset{\mathrm{ts}}{\rightharpoonup} V$, where "ts" stands for "two-scale" convergence.

We next recall the *periodic unfolding operator* $\mathcal{T}_\varepsilon$ and *periodic folding operator* $\mathcal{F}_\varepsilon$. For the semi-open unit cell $Y = [0, 1)^d$ we have $\cup_{\lambda \in \mathbb{Z}^d} (\lambda + Y) =$

$\mathbb{R}^d$, and all translated cells $\lambda + Y$ ($\lambda \in \mathbb{Z}^d$) are mutually disjoint. Let Ω_ε denote the smallest open set of the form $\mathrm{int}\left(\bigcup_{\lambda\in J} \varepsilon(\lambda+Y)\right)$ (with $J \subset \mathbb{Z}^d$) containing Ω. An explicit formula reads as follows

$$\begin{aligned}\Omega_\varepsilon &= \mathrm{int}\Big(\bigcup\nolimits_{\lambda\in\mathbb{Z}^d\cap\left(\frac{\Omega}{\varepsilon}-Y\right)} \varepsilon(\lambda+Y)\Big)\\ &= \mathrm{int}\{\varepsilon(\lambda+y)\,|\,\lambda\in\mathbb{Z}^d, y\in Y,\ \varepsilon(\lambda+\mathrm{int}(Y))\cap\Omega\neq\emptyset\}.\end{aligned}$$

This construction is such that $(\Omega_\varepsilon)_\varepsilon = \Omega_\varepsilon$. It can be shown that $\lim_{\varepsilon\to 0}|\Omega_\varepsilon \setminus \Omega| = |\partial\Omega|$. The Ω_ε-type sets are strongly related to the behavior of the folding and unfolding operators, defined in a very natural way via extending functions outside of Ω by 0:

$$\begin{aligned}\mathrm{L}^p_{\mathrm{ex}}(\Omega) &= \{v\in\mathrm{L}^p(\mathbb{R}^d)\,|\,v\equiv 0 \text{ a.e. on } \mathbb{R}^d\backslash\Omega\} \subset \mathrm{L}^p(\mathbb{R}^d),\\ \mathrm{L}^p_{\mathrm{ex}}(\Omega\times Y) &= \{U\in\mathrm{L}^p(\mathbb{R}^d\times Y)\,|\,U\equiv 0 \text{ a.e. on } (\mathbb{R}^d\backslash\Omega)\times Y\}\\ &\quad\subset \mathrm{L}^p(\mathbb{R}^d\times Y).\end{aligned}$$

This will implement one of the many choices of the "interpolations" one has to do at the boundary $\partial\Omega$. On the full space $\mathbb{R}^d$ we define the *periodic unfolding operator* $\mathcal{T}_\varepsilon$ via

$$\mathcal{T}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d)\to\mathrm{L}^p(\mathbb{R}^d\times Y);\quad \mathcal{T}_\varepsilon v(x,y) = v\left(\varepsilon\left[\frac{x}{\varepsilon}\right]+\varepsilon y\right). \tag{4.2}$$

The construction is such that $\mathcal{T}_\varepsilon v$ is piecewise constant in x, namely on each rescaled unit cell $\varepsilon(\lambda+Y)$ with $\lambda\in\mathbb{Z}^d$. The idea is now to define the periodic folding operator $\mathcal{F}_\varepsilon$ such that it is a left inverse of $\mathcal{T}_\varepsilon$. To this end we define the projection

$$\mathcal{P}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d\times Y)\to\mathrm{L}^p(\mathbb{R}^d\times Y);\quad \mathcal{P}_\varepsilon U(x,y) = \frac{1}{\varepsilon^d}\int_{\varepsilon\left(\left[\frac{x}{\varepsilon}\right]+Y\right)} U(\xi,y)\,\mathrm{d}\xi \tag{4.3}$$

Again, $\mathcal{P}_\varepsilon U(\cdot,y)$ is constant on rescaled unit cells. With this we define the *periodic folding operator* via

$$\mathcal{F}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d\times Y)\to\mathrm{L}^p(\mathbb{R}^d);\quad \mathcal{F}_\varepsilon U(x) = \mathcal{P}_\varepsilon U\left(x,\left\{\tfrac{x}{\varepsilon}\right\}\right). \tag{4.4}$$

Since $\mathcal{P}_\varepsilon U(\cdot,y)$ is constant on each $\varepsilon(\lambda+Y)$, we may write $\mathcal{F}_\varepsilon U(x) = \mathcal{P}_\varepsilon U\left(\varepsilon\left[\frac{x}{\varepsilon}\right],\left\{\frac{x}{\varepsilon}\right\}\right)$. It is useful to note that $\mathcal{T}_\varepsilon,\mathcal{P}_\varepsilon,\mathcal{F}_\varepsilon$ may be defined between L^p spaces for every $p\in(1,\infty)$.

Proposition 4.1. *The operators $\mathcal{T}_\varepsilon,\mathcal{P}_\varepsilon,\mathcal{F}_\varepsilon$ have the following properties:*

(i) *$\mathcal{T}_\varepsilon$ is an isometry and $\mathcal{T}_\varepsilon(\mathrm{L}^p_{\mathrm{ex}}(\Omega))\subset\mathrm{L}^p_{\mathrm{ex}}(\Omega_\varepsilon\times Y)$.*

(ii) *$\|\mathcal{P}_\varepsilon\|\le 1$ and $\mathcal{P}_\varepsilon(\mathrm{L}^p_{\mathrm{ex}}(\Omega\times Y))\subset\mathrm{L}^p_{\mathrm{ex}}(\Omega_\varepsilon\times Y)$.*

(iii) *$\mathcal{F}_\varepsilon(\mathrm{L}^p_{\mathrm{ex}}(\Omega\times Y))\subset\mathrm{L}^p_{\mathrm{ex}}(\Omega_\varepsilon)$.*

(iv) *We have* $\mathcal{T}_\varepsilon \mathcal{F}_\varepsilon = \mathcal{P}_\varepsilon$, $\mathcal{F}_\varepsilon \mathcal{T}_\varepsilon = \mathrm{id}_{\mathrm{L}^p(\mathbb{R}^d)}$, *and* $\mathcal{P}_\varepsilon \mathcal{T}_\varepsilon = \mathcal{T}_\varepsilon$.

(v) *The adjoint of* $\mathcal{T}_\varepsilon : \mathrm{L}^p(\mathbb{R}^d) \to \mathrm{L}^p(\mathbb{R}^d \times Y)$ *is the operator*

$\mathcal{F}_\varepsilon : \mathrm{L}^q(\mathbb{R}^d \times Y) \to \mathrm{L}^q(\mathbb{R}^d)$, *where* $\frac{1}{p} + \frac{1}{q} = 1$.

Proposition 4.2. (relation with the classical two-scale convergence). *Assume* Ω *has Lipschitz boundary. Let* $V \in \mathrm{L}^p(\Omega \times Y)$ *and a bounded sequence* $(v_\varepsilon)_\varepsilon$ *in* $\mathrm{L}^p(\Omega)$. *Then*

$$v_\varepsilon \overset{\mathrm{ts}}{\rightharpoonup} V \iff \mathcal{T}_\varepsilon v_\varepsilon|_{\Omega \times Y} \rightharpoonup V \text{ (weakly) in } \mathrm{L}^p(\Omega \times Y).$$

We next introduce the notions of *weak/strong two-scale convergence* as follows:

Definition 4.2. *Let* $V \in \mathrm{L}^p(\Omega \times Y)$. *A bounded sequence* $(v_\varepsilon)_\varepsilon$ *in* $\mathrm{L}^p(\Omega)$

(w2): *weakly two-scale converges to* V *(we write* $v_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} V$*), if*

$$\mathcal{T}_\varepsilon v_\varepsilon \rightharpoonup V \text{ (weakly) in } \mathrm{L}^p(\mathbb{R}^d \times Y).$$

(s2): *strongly two-scale converges to* V *(we write* $v_\varepsilon \overset{\mathrm{s2}}{\rightharpoonup} V$*), if*

$$\mathcal{T}_\varepsilon v_\varepsilon \to V \text{ (strongly) in } \mathrm{L}^p(\mathbb{R}^d \times Y).$$

Clearly, the above weak two-scale convergence is stronger than the classical. Our next proposition shows how these two notions are related. The difference between $\overset{\mathrm{ts}}{\rightharpoonup}$ and $\overset{\mathrm{w2}}{\rightharpoonup}$ disappears, if we a priori impose the boundedness of the sequence.

Proposition 4.3. *Let* $(u_\varepsilon)_\varepsilon$ *be a bounded family in* $\mathrm{L}^p(\Omega)$ *with* $p \in (1, \infty)$. *Then, the following statements are equivalent:*

$$\begin{aligned}
&\text{(i) } u_\varepsilon \overset{\mathrm{ts}}{\rightharpoonup} U \text{ in } \mathrm{L}^p(\Omega \times Y),\\
&\text{(ii) } \mathcal{T}_\varepsilon u_\varepsilon\big|_{\Omega \times Y} \rightharpoonup U \text{ in } \mathrm{L}^p(\Omega \times Y),\\
&\text{(iii) } u_\varepsilon \overset{\mathrm{w2}}{\rightharpoonup} U \text{ in } \mathrm{L}^p(\Omega \times Y).
\end{aligned}$$

Notation
For any function $v \in \mathrm{L}^p(\Omega)$ we will denote by Ev the function defined on $\Omega \times Y$ via $Ev(x, y) = v(x)$ for all $(x, y) \in \Omega \times Y$.

Remark 4.1. For every $v \in \mathrm{L}^p_{\mathrm{ex}}(\Omega)$, we have

$$\mathcal{T}_\varepsilon v \to Ev \text{ strongly in } \mathrm{L}^p(\mathbb{R}^d \times Y) \text{ as } \varepsilon \to 0.$$

The next theorem gives further information on the two-scale convergence of bounded sequences in $\mathrm{H}^1(\Omega)$.

Theorem 4.1. *Assume $|\partial\Omega| = 0$. Let $(v_\varepsilon)_\varepsilon$ be a sequence in $\mathrm{H}^1(\Omega)$, with $v_\varepsilon \rightharpoonup v_0$ weakly in $\mathrm{H}^1(\Omega)$. Then*

$$v_\varepsilon \xrightarrow{\text{s2}} Ev_0,$$

and there is a subsequence $(v_{\varepsilon'})_{\varepsilon'}$ and a function $V_1 = V_1(x,y)$ in $\mathrm{L}^2(\Omega; \mathrm{H}^1_{\mathrm{av}}(Y))$, such that

$$\nabla v_{\varepsilon'} \xrightarrow{\text{w2}} E\nabla_x v_0 + \nabla_y V_1.$$

Proposition 4.4. *For every $U = (u_0, U_1) \in \mathbf{H}$, there exists a family $(u_\varepsilon)_\varepsilon$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega)^d$, such that*

$$u_\varepsilon \xrightarrow{\text{s2}} u_0, \quad \nabla u_\varepsilon \xrightarrow{\text{s2}} \nabla u_0 + \nabla_y U_1.$$

Theorem 4.2.

(a) *Assume $|\partial\Omega| = 0$. Then for any bounded sequence $(v_\varepsilon)_\varepsilon$ in $\mathrm{L}^p(\Omega)$, there exists a subsequence $(v_{\varepsilon'})_{\varepsilon'}$ and a function $V \in \mathrm{L}^p(\Omega \times Y)$, such that $v_{\varepsilon'} \xrightarrow{\text{w2}} V$.*

(b) *Any function $V \in \mathrm{L}^p(\Omega \times Y)$ is attained as a strong two-scale limit.*

The above results are easier to obtain in the case $\Omega = \mathbb{R}^d$ (see for instance [30]).

5. Convergence results

In this section we state and prove the homogenization result. Throughout we assume for $W + \alpha$ the following growth condition:

$$\begin{aligned} &W(x,\boldsymbol{\varepsilon},D,q) + \alpha(x,V) \le \widetilde{c}(1 + |\boldsymbol{\varepsilon}|^2 + |D|^2 + |q|^2 + |V|^2) \\ &\text{for every } (x,\boldsymbol{\varepsilon},D,q,V) \in \Omega \times \mathbb{R}^{d\times d}_{\mathrm{sym}} \times \mathbb{R}^d \times \mathbb{R}^{d_q} \times \mathbb{R}^{d_q \times d}, \end{aligned} \tag{A7}$$

for some fixed constant $\widetilde{c} > 0$.
We need the following helpful results that are proved in [28].

Proposition 5.1. *For every $\mathbb{D} \in \mathrm{L}^2_{\mathrm{Div}}(\mathbb{R}^d \times Y)$, there exists a sequence $(D_\varepsilon)_\varepsilon \subset \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d)$, such that*

$$D_\varepsilon \xrightarrow{\text{s2}} \mathbb{D}.$$

We next introduce the notions of *two-scale cross-convergence* and *strong two-scale cross-convergence*.

Definition 5.1. *Let $(U,\mathbb{D},Q)\in\mathbf{Z}$, with $U=(u_0,U_1)$ and $Q=(q_0,Q_1)$. A sequence $(u_\varepsilon,D_\varepsilon,q_\varepsilon)_\varepsilon$ in $\mathcal{Y}$ is called*

(w2c): *weakly two-scale cross-convergent to $(U,\mathbb{D},Q)$, if*

$$u_\varepsilon \overset{\text{w2}}{\rightharpoonup} Eu_0, \qquad \nabla u_\varepsilon \overset{\text{w2}}{\rightharpoonup} \nabla u_0+\nabla_y U_1, \qquad D_\varepsilon \overset{\text{w2}}{\rightharpoonup} \mathbb{D},$$

$$q_\varepsilon \overset{\text{w2}}{\rightharpoonup} Eq_0, \qquad \nabla q_\varepsilon \overset{\text{w2}}{\rightharpoonup} \nabla q_0+\nabla_y Q_1.$$

We write this as $(u_\varepsilon,D_\varepsilon,q_\varepsilon)\overset{\text{w2c}}{\rightharpoonup}(U,\mathbb{D},Q)$.

(s2c): *strongly two-scale cross-convergent to $(U,\mathbb{D},Q)$, if*

$$u_\varepsilon \overset{\text{s2}}{\rightharpoonup} Eu_0, \qquad \nabla u_\varepsilon \overset{\text{s2}}{\rightharpoonup} \nabla u_0+\nabla_y U_1, \qquad D_\varepsilon \overset{\text{s2}}{\rightharpoonup} \mathbb{D},$$

$$q_\varepsilon \overset{\text{s2}}{\rightharpoonup} Eq_0, \qquad \nabla q_\varepsilon \overset{\text{s2}}{\rightharpoonup} \nabla q_0+\nabla_y Q_1.$$

We write this as $(u_\varepsilon,D_\varepsilon,q_\varepsilon)\overset{\text{s2c}}{\rightharpoonup}(U,\mathbb{D},Q)$.

Now Propositions 4.4 and 5.1 together read as:

Corollary 5.1. *For every $(U,\mathbb{D},Q)\in\mathbf{Z}$, there exists a sequence $(u_\varepsilon,D_\varepsilon,q_\varepsilon)_\varepsilon$ in $\mathcal{Y}$, such that*

$$(u_\varepsilon,D_\varepsilon,q_\varepsilon)\overset{\text{s2c}}{\rightharpoonup}(U,\mathbb{D},Q)$$

As a consequence of Lemmas 2.11 and 2.12(a) from [21] we obtain:

Proposition 5.2. *Let $(q_\varepsilon)_\varepsilon$ a bounded sequence in $\mathrm{H}^1(\Omega)^{d_q}$ and $Q=(q_0,Q_1)\in\mathbf{Q}$.*

(i) *If $q_\varepsilon \overset{\text{w2}}{\rightharpoonup} Eq_0$, then $\liminf_{\varepsilon\to0}\mathcal{R}_\varepsilon(q_\varepsilon)\geq\mathbf{R}(Q)$.*

(ii) *If $q_\varepsilon \overset{\text{s2}}{\rightharpoonup} Eq_0$, then $\lim_{\varepsilon\to0}\mathcal{R}_\varepsilon(q_\varepsilon)=\mathbf{R}(Q)$.*

Hence the convergence of $\mathcal{R}_\varepsilon$ to the limit $\mathbf{R}$ may be viewed as a two-scale Γ-convergence.

Proposition 5.3. *Let $(u_\varepsilon,D_\varepsilon,q_\varepsilon)_\varepsilon$ a bounded sequence in $\mathcal{Y}$ and $(U,\mathbb{D},Q)\in\mathbf{Z}$.*

(i) *If $(u_\varepsilon,D_\varepsilon,q_\varepsilon)\overset{\text{w2c}}{\rightharpoonup}(U,\mathbb{D},Q)$ then $\liminf_{\varepsilon\to0}\mathcal{E}_\varepsilon(t,u_\varepsilon,D_\varepsilon,q_\varepsilon)\geq\mathbf{E}(t,U,\mathbb{D},Q)$.*

(ii) *If $(u_\varepsilon,D_\varepsilon,q_\varepsilon)\overset{\text{s2c}}{\rightharpoonup}(U,\mathbb{D},Q)$ then $\lim_{\varepsilon\to0}\mathcal{E}_\varepsilon(t,u_\varepsilon,D_\varepsilon,q_\varepsilon)=\mathbf{E}(t,U,\mathbb{D},Q)$.*

Hence $\mathcal{E}_\varepsilon$ is Γ-two-scale cross-convergent to $\mathbf{E}$.

The following result can be found in a more abstract setting in [17].

Proposition 5.4. *Assume that for arbitrary stable sequence $(u_\varepsilon, D_\varepsilon, q_\varepsilon)_\varepsilon$ (that is, $(u_\varepsilon, D_\varepsilon, q_\varepsilon) \in \mathcal{S}_\varepsilon(t)$) weakly two-scale cross-convergent to some $(U, \mathbb{D}, Q) \in \mathbf{Z}$, and for every test state $(\widetilde{U}, \widetilde{\mathbb{D}}, \widetilde{Q}) \in \mathbf{Z}$, there is a sequence of test functions $(\widetilde{u}_\varepsilon, \widetilde{D}_\varepsilon, \widetilde{q}_\varepsilon)_\varepsilon \subset \mathcal{Y}$, such that*

$$\begin{aligned} \limsup_{\varepsilon\to 0} \big[\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{D}_\varepsilon, \widetilde{q}_\varepsilon) + \mathcal{R}_\varepsilon(\widetilde{q}_\varepsilon - q_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, D_\varepsilon, q_\varepsilon)\big] \\ \le \mathbf{E}(t, \widetilde{U}, \widetilde{\mathbb{D}}, \widetilde{Q}) + \mathbf{R}(\widetilde{Q} - Q) - \mathbf{E}(t, U, \mathbb{D}, Q). \end{aligned} \tag{5.1}$$

Then $(U, \mathbb{D}, Q)$ is stable (satisfies $(\mathbf{S})$).

We can now formulate our homogenization result which shows that $(\mathbf{S})\,\&\,(\mathbf{E})$ is the two-scale homogenized problem for $(\mathrm{S}^\varepsilon)\,\&\,(\mathrm{E}^\varepsilon)$.

Theorem 5.1. *Let $(u_\varepsilon, D_\varepsilon, q_\varepsilon) : [0, T] \to \mathcal{Y}$ be a solution for $(\mathrm{S}^\varepsilon)\,\&\,(\mathrm{E}^\varepsilon)$. Assume*

$$\mathcal{E}_\varepsilon(0, u_\varepsilon(0), D_\varepsilon(0), q_\varepsilon(0)) \to \mathbf{E}(0, U^0, \mathbb{D}^0, Q^0).$$

for some $Z^0 = (U^0, \mathbb{D}^0, Q^0) \in \mathbf{Z}$. Then there is a subsequence $(u_{\varepsilon'}, D_{\varepsilon'}, q_{\varepsilon'})_{\varepsilon'}$ such that

$$(u_{\varepsilon'}(t), D_{\varepsilon'}(t), q_{\varepsilon'}(t)) \overset{\text{w2c}}{\rightharpoonup} Z(t) = (U(t), \mathbb{D}(t), Q(t)) \text{ in } \mathbf{Z}, \text{ for every } t \in [0, T],$$

where $Z : [0, T] \to \mathbf{Z}$ is a solution of $(\mathbf{S})\,\&\,(\mathbf{E})$ with the initial condition $Z(0) = Z^0$.

Proof. Let us first show that $(u_\varepsilon, D_\varepsilon, q_\varepsilon)_\varepsilon$ has a subsequence weakly two-scale cross-convergent to some $(U, \mathbb{D}, Q) \in \mathbf{Z}$. Using successively the coercivity of $W + \alpha$ and Korn's inequality leads to the following estimate for $\mathcal{E}_\varepsilon$ (see also [20, Lemma 4.4])

$$\mathcal{E}_\varepsilon(t, u_\varepsilon, D_\varepsilon, q_\varepsilon) \ge c_0(\|u_\varepsilon\|^2_{\mathrm{H}^1} + \|D_\varepsilon\|^2_{\mathrm{L}^2} + \|q_\varepsilon\|^2_{\mathrm{H}^1}) - \|\ell(t)\|_* \|(u_\varepsilon, D_\varepsilon)\|_{\mathrm{H}^1\times\mathrm{L}^2} - C_0, \tag{5.2}$$

for some $c_0, C_0 > 0$. Since the constants from Korn's inequality and from the coercivity for $W + \alpha$ do not depend on ε, the constants c_0 and C_0 are ε-independent. But $\mathcal{E}_\varepsilon(t, u_\varepsilon, D_\varepsilon, q_\varepsilon) \le M_T$ for some ε-independent constant $M_T > 0$ (for details on the provenience of this constant see Theorem 3.2 and Lemma 4.5 from [20]). Consequently, for some $M > 0$ we have

$$\|(u_\varepsilon(t), D_\varepsilon(t), q_\varepsilon(t))\|_{\mathrm{H}^1\times\mathrm{L}^2\times\mathrm{H}^1} < M \quad \text{for all } t \in [0, T], \varepsilon > 0.$$

As $(D_\varepsilon(t))_\varepsilon$ is bounded in $\mathrm{L}^2(\mathbb{R}^d)$, Theorem 4.2 allows us to extract a subsequence (for which we use the same notation), such that $D_\varepsilon(t) \overset{\text{w2}}{\rightharpoonup} \mathbb{D}(t) \in \mathrm{L}^2(\mathbb{R}^d \times Y)$ for every $t \in [0, T]$. Since $(D_\varepsilon(t))_\varepsilon \subset \mathrm{L}^2_{\mathrm{div}}(\mathbb{R}^d)$, the two-scale limit $\mathbb{D}(t)$ belongs to $\mathrm{L}^2_{\mathrm{Div}}(\mathbb{R}^d)$ (see [2, Prop. 1.14]). As $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d)$ is reflexive, there exists a subsequence of $(u_\varepsilon(t))_\varepsilon$, which converges weakly to some $u_0(t)$ in $\mathrm{H}^1_{\Gamma_{\mathrm{Dir}}}(\Omega; \mathbb{R}^d)$. By Theorem 4.1 we have $u_\varepsilon(t) \overset{\text{s2}}{\rightharpoonup} u_0(t)$, and there exists a subsequence and $U_1(t) \in \mathrm{L}^2(\Omega;\mathrm{H}^1_{\mathrm{av}}(Y))$, such that $\nabla u_\varepsilon(t) \stackrel{\mathrm{w2}}{\rightharpoonup} E\nabla u_0(t) + \nabla_y U_1(t)$. The same arguments apply to $(q_\varepsilon(t))_\varepsilon$. We have thus proved (for a subsequence) that

$$(u_\varepsilon(t), D_\varepsilon(t), q_\varepsilon(t)) \stackrel{\mathrm{w2c}}{\rightharpoonup} Z(t) = (U(t), \mathbb{D}(t), Q(t)) \text{ in } \mathbf{Z}.$$

We claim that $(U, \mathbb{D}, Q)$ is the solution for $(\mathbf{S})\,\&\,(\mathbf{E})$.
Step 1. We first show that the stability condition $(\mathbf{S})$ is satisfied. This will follow by Proposition 5.4 if we show that for any test state $(\widetilde{U}, \widetilde{\mathbb{D}}, \widetilde{Q}) \in \mathbf{Z}$, there is a sequence of test functions $(\widetilde{u}_\varepsilon, \widetilde{D}_\varepsilon, \widetilde{q}_\varepsilon)_\varepsilon \subset \mathcal{Y}$ such that

$$\begin{aligned} \limsup\nolimits_{\varepsilon\to0} \big[\mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{D}_\varepsilon, \widetilde{q}_\varepsilon) + \mathcal{R}_\varepsilon(\widetilde{q}_\varepsilon - q_\varepsilon) - \mathcal{E}_\varepsilon(t, u_\varepsilon, D_\varepsilon, q_\varepsilon)\big] \\ \le \mathbf{E}(t, \widetilde{U}, \widetilde{\mathbb{D}}, \widetilde{Q}) + \mathbf{R}(\widetilde{Q} - Q) - \mathbf{E}(t, U, \mathbb{D}, Q). \end{aligned} \tag{5.3}$$

As $(u_\varepsilon, D_\varepsilon, q_\varepsilon) \stackrel{\mathrm{w2c}}{\rightharpoonup} (U, \mathbb{D}, Q)$, by Proposition 5.3(i) we get

$$\limsup_{\varepsilon\to0}(-\mathcal{E}_\varepsilon(t, u_\varepsilon, D_\varepsilon, q_\varepsilon)) \le -\mathbf{E}(t, U, \mathbb{D}, Q). \tag{5.4}$$

Let us construct a *joint recovery sequence* $(\widetilde{u}_\varepsilon, \widetilde{D}_\varepsilon, \widetilde{q}_\varepsilon)_\varepsilon \subset \mathcal{Y}$. By Corollary 5.1 we get a sequence $(\widetilde{u}_\varepsilon, \widetilde{D}_\varepsilon, \widetilde{q}_\varepsilon)_\varepsilon \subset \mathcal{Y}$, such that $(\widetilde{u}_\varepsilon, \widetilde{D}_\varepsilon, \widetilde{q}_\varepsilon) \stackrel{\mathrm{s2c}}{\rightharpoonup} (\widetilde{U}, \widetilde{\mathbb{D}}, \widetilde{Q})$. By Theorem 5.3, we see that

$$\lim_{\varepsilon\to0} \mathcal{E}_\varepsilon(t, \widetilde{u}_\varepsilon, \widetilde{D}_\varepsilon, \widetilde{q}_\varepsilon) = \mathbf{E}(t, \widetilde{U}, \widetilde{\mathbb{D}}, \widetilde{Q}). \tag{5.5}$$

We have $Q = (q_0, Q_1)$ and $\widetilde{Q} = (\widetilde{q}_0, \widetilde{Q}_1)$. As $q_\varepsilon \rightharpoonup q_0$ in $\mathrm{H}^1(\Omega;\mathbb{R}^{d_q})$, by Theorem 4.1 we deduce that $q_\varepsilon \stackrel{\mathrm{s2}}{\rightharpoonup} Eq_0$. Hence $\widetilde{q}_\varepsilon - q_\varepsilon \stackrel{\mathrm{s2}}{\rightharpoonup} E\widetilde{q}_0 - Eq_0$. By Proposition 5.2(ii),

$$\lim_{\varepsilon\to0} \mathcal{R}_\varepsilon(\widetilde{q}_\varepsilon - q_\varepsilon) = \mathbf{R}(\widetilde{Q} - Q). \tag{5.6}$$

Combining (5.4)–(5.6) yields (5.3). We conclude that the stability condition $(\mathbf{S})$ holds.
Step 2. We next establish the energy balance $(\mathbf{E})$. For this we consider (E^ε) as $\varepsilon \to 0$. The first term on the right-hand side converges to the corresponding term in $(\mathbf{E})$, due to the hypothesis. The second term converges by Lebesgue's dominated convergence theorem as the integrands are uniformly bounded and converge pointwise. In order to handle the left-hand side of (E^ε), set

$$e_\varepsilon(t) = \mathcal{E}_\varepsilon(t, u_\varepsilon(t), D_\varepsilon(t), q_\varepsilon(t)), \qquad d_\varepsilon(t) = \int_0^t \mathcal{R}_\varepsilon(\dot{q}_\varepsilon(s))\,\mathrm{d}s.$$

By the above, we see that $r_\varepsilon(t) = e_\varepsilon(t) + d_\varepsilon(t)$ converges to $r_0(t)$, which is the limit of the right-hand side of (E^ε). For $e^*(t) = \limsup_{\varepsilon\to0} e_\varepsilon(t)$ and $d_*(t) = \liminf_{\varepsilon\to0} d_\varepsilon(t)$, we have $e^*(t) + d_*(t) = r_0(t)$. We next use the lower estimates for the functionals. For the stored energy, by Proposition 5.3 we get

$$\mathbf{E}(t, U(t), \mathbb{D}(t), Q(t)) \le \liminf_{\varepsilon\to 0} e_\varepsilon(t) \le \limsup_{\varepsilon\to 0} e_\varepsilon(t) = e^*(t).$$

For the dissipation integral we have
$\int_0^t \mathbf{R}(\dot{Q}(s))\, \mathrm{d}s = \sup \sum_{j=1}^N \mathbf{R}(Q(t_j)-Q(t_{j-1}))$, where the supremum is taken over all finite partitions of $[0,t]$ (see [16]). By Proposition 5.3 it follows that

$$\begin{aligned}\textstyle\sum_{j=1}^N \mathbf{R}(Q(t_j)-Q(t_{j-1})) &\le \liminf_{\varepsilon\to 0} \textstyle\sum_{j=1}^N \mathcal{R}_\varepsilon(q_\varepsilon(t_j)-q_\varepsilon(t_{j-1})) \\ &\le \liminf_{\varepsilon\to 0} \textstyle\int_0^t \mathcal{R}_\varepsilon(\dot{q}_\varepsilon(s))\,\mathrm{d}s = d_*(t).\end{aligned} \tag{5.7}$$

Since $e^* + d_* = r_0$, we get the lower energy estimate

$$\begin{aligned}\mathbf{E}(t, U(t), \mathbb{D}(t), Q(t)) \ &+ \textstyle\int_0^t \mathbf{R}(\dot{Q}(s))\, \mathrm{d}s \le e^*(t) + d_*(t) \\ &= \mathbf{E}(0, U(0), \mathbb{D}(0), Q(0)) - \int_0^t \langle \dot{\ell}(s), (u_0(s), \overline{\mathbb{D}}(s))\rangle\, \mathrm{d}s.\end{aligned}$$

The upper energy estimate (just replace "$\le$" by "$\ge$") follows from the already proved stability of $(U, \mathbb{D}, Q)$ (see [19, Th. 2.5]). This establishes the energy balance (**E**) and completes the proof. ☐

Remark 5.1. The condition $\mathcal{E}_\varepsilon(0, u_\varepsilon(0), D_\varepsilon(0), q_\varepsilon(0)) \to \mathbf{E}(0, U^0, \mathbb{D}^0, Q^0)$ from Theorem 5.1 is satisfied whenever $(u_\varepsilon(0), D_\varepsilon(0), q_\varepsilon(0)) \overset{\mathrm{s2c}}{\rightharpoonup} (U^0, \mathbb{D}^0, Q^0)$.

Indeed, this follows easily by Proposition 5.3(ii).

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