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Domains of class C: properties and applications

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Abstract - We discuss general shape optimization problems associated to elliptic equations with Dirichlet, Neumann or mixed boundary conditions. We develop an analysis of convergence properties of domains of class C (with the segment property) and the stability of the corresponding Sobolev spaces. The results on shape optimization problems concern existence theory, discretization and approximation properties via finite dimensional optimization problems.

Key words and phrases : segment property, stability, local convergence, existence, approximation.

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1. Introduction

The properties of Sobolev spaces defined in a domain Ω in R^d, $d \in \mathbb{N}$ or of partial differential equations defined in Ω, depend on the regularity properties of Ω.

In many classical texts (Sobolev [21], Agmon [2], Protter and Weinberger [20]), the boundary properties of some domain $\Omega \subset R^n$ are described in a straightforward manner: segment property, cone property, interior ball property, etc. Another point of view is to consider the boundary $\partial\Omega$ to be locally the graph of some function; then the properties of $\partial\Omega$ are specified through the properties of the corresponding local representation: continuity, Lipschitzianity, differentiability of various orders (Necas [15], Adams [1], Miranda [14]). Other approaches are also possible and a nice introduction with very interesting examples may be found in the book of Grisvard [7].

It is the aim of this paper to discuss the weakest property mentioned above, namely the case of domains of class C, i.e. the case of continuous boundaries. We use the following definition

Definition 1.1. *We say that a bounded open subset $\Omega \subset R^d$ is of class C if there exists a family $\mathcal{F}_\Omega$ of continuous functions $g : B(0, k_\Omega) \to R$, where $B(0, k_\Omega) \subset R^{d-1}$ is the open ball of radius $k_\Omega > 0$ centered at the origin,* *such that*

$$\partial\Omega = \bigcup_{g\in\mathcal{F}_\Omega} \{R_g(s,0) + o_g + g(s)y_g,\ s \in B(0,k_\Omega)\},$$

with $y_g = R_g(0,0,\ldots,0,1)$ *for some rotation* R_g *of* R^d *and with some* $o_g \in R^d$.

This definition may be extended to unbounded domains immediately. If $\partial\Omega$ is compact (as above), it is possible to require that the (finitely many) local charts are defined on the same ball in R^{d-1} (depending on Ω) and we shall use this "normalization" in the sequel, in an esential manner.

Domains of class C are locally on one side of the boundary. They are of Charathéodory type, i.e. $\Omega = \text{int}\ \overline{\Omega}$. It follows that they have no cuts or cracks, but their boundary may have cusps (when g is just Hölder continuous) or corners (when g is Lipschitz). It turns out that domains of class C have the (interior and exterior) segment property (Adams [1], Maz'ya [13]): there is some $a_\Omega > 0$ such that, for any local chart $g \in \mathcal{F}_\Omega$:

$$R_g(s,0) + o_g + (g(s) - t)y)y_g \in \Omega,$$

$$R_g(s,0) + o_g + (g(s) + t)y_g \in R^d \setminus \overline{\Omega},$$
$$\forall\ t \in]0, a_\Omega[,\ \forall\ s \in B(0,k_\Omega).$$

Conversely, the segment property allows the introduction of a local system of coordinates (with "vertical" direction given by the segment) and the definition of a mapping describing locally the boundary since every segment "cuts" the boundary in just one point. This mapping is continuous due to a simple contradiction argument.

Again, by the compactness of $\partial\Omega$, we can choose the same constant $a_\Omega > 0$ for all the local charts $g \in \mathcal{F}_\Omega$.

Notice that in Definition 1.1, one can always choose a constant $0 < r_\Omega < k_\Omega$ such that the "restricted" local charts defined on $B(0,r_\Omega) \subset R^{d-1}$ still "cover" the whole $\partial\Omega$. Not all the domains that intuitively have a continuous boundary are of class C as the following example shows:

Example 1.1.

$$g_1(x) = \begin{cases} x - \dfrac{1}{3^k}, & x \in \left[\dfrac{1}{3^k}, \dfrac{2}{3^k}\right] \\ \dfrac{1}{3^{k-1}} - x, & x \in \left[\dfrac{2}{3^k}, \dfrac{1}{3^{k-1}}\right] \end{cases},$$

$k \in N$, and $g_2(x) = g_1(x) - \dfrac{x}{2}$. The segment property is not satisfied around $x = 0$.

Moreover, Definition 1.1 forces that all the turning points from one local chart are of the "same type" (with "axis" given by y_g):

The following extension of the Meyers-Serrin theorem, Adams [1], is one of the main classical applications of domains of class C:

Theorem 1.1. *If Ω has the segment property, then the set of restrictions to Ω of functions in $C_0^\infty(R^d)$ is dense in $W^{m,p}(\Omega)$ for $1 \leq p < \infty$, $m \in \mathbb{N}$.*

In the sequel, we shall use the Hausdorff-Pompeiu [9], [18] complementary metric for open sets $\Omega_i \subset D \subset R^d$, where D is some given "hold all" bounded domain:

$$d(\Omega_1, \Omega_2) = \max\{ \max_{x \in \overline{D} \setminus \Omega_1} \min_{y \in \overline{D} \setminus \Omega_2} |x - y|; \max_{y \in \overline{D} \setminus \Omega_2} \min_{x \in \overline{D} \setminus \Omega_1} |x - y| \}.$$

The family of open subsets of D equipped with the metric $d(\cdot, \cdot)$ forms a complete metric space, which is sequentially compact [12].

The plan of the paper is as follows. the next section investigates stability and compactness properties of domains of class C with respect to various operations. The applications concern general shape optimization problems and will be discussed in Section 3 (existence theory) and in Section 4 (discretization and approximation results). For complete proofs and more details, we quote author's works [8], [16], [17], [23], [24], [25].

2. Stability and compactness

We collect in this section some properties related to the convergence of bounded open sets that play an outstanding role in the applications. The continuity assumption on the boundary is essential in many of them.

Theorem 2.1. (the Γ-property)

Let Ω_n, Ω be open subsets of the bounded domain $D \subset R^d$ such that $d(\Omega_n, \Omega) \to 0$. If $K \subset \Omega$ is compact, there is $n_0 \in N$ such that $K \subset \Omega_n$, $n \geq n_0$.

In the monograph of Delfour and Zolesio [6], the name "compactivorous property" is used. This property appears already in Necas [15] and it is valid for general open subsets.

We indicate now the main compactness result and the following uniform class C assumptions will be of frequent use in the sequel:

Theorem 2.2. *Let Ω_n, Ω be subdomains of $D \subset R^d$ such that $d(\Omega_n, \Omega) \to 0$ for $n \to \infty$. If in addition*

$$k_{\Omega_n} \geq k > 0, \ r_{\Omega_n} \leq r < k, \ a_{\Omega_n} \geq a > 0, \ \forall\, n \in N$$

and if the family $\mathcal{F} = \bigcup_{n \in N} \mathcal{F}_{\Omega_n}$ *of all the corresponding local charts is equicontinous and equibounded on* $\overline{B(0,k)}$*, then* Ω *is of class* C *with* $k_\Omega \geq k$*,* $r_\Omega \leq r$*,* $a_\Omega \geq a$*.*

Moreover, the associated characteristic functions satisfy $\chi_{\Omega_n} \to \chi_\Omega$ *a.e. in* D*, on a subsequence.*

Corollary 2.1. *Under the above assumptions, we also have* $\overline{\Omega_n} \to \overline{\Omega}$ *in the Hausdorff-Pompeiu metric for compact sets. The converse is also true.*

In the absence of the uniform class C assumptions, this property is not valid as the following simple example shows:

Example 2.1. In R^2, let $C_n = \overline{B(0,1)} \cup K_n$, where K_n is the union of n closed rays of length 2 starting from the origin and dividing the plane into sectors of equal angles. Let $\Omega_n = B(0,2) \setminus C_n$. Then $d(\Omega_n, \emptyset) \to 0$ while $\overline{\Omega_n} \to \overline{B(0,2)} - B(0,1)$ in the Hausdorff-Pompeiu metric for compact set.

Consider now another sequence of open sets uniformly of class C, $A_n \subset D$, $d(A_n, A) \to 0$. If some compatibility conditions are imposed (see the next section), then $\Omega_n \cup A_n$ are open subsets of class C, uniformly with respect to $n \in N$, in the sense of Theorem 2.2. We have

Corollary 2.2. $\Omega_n \cup A_n \to \Omega \cup A$, $\Omega_n \cap A_n \to \Omega \cap A$ *in the complementary Hausdorff-Pompeiu metric and the limit sets are of class* C*.*

Again a simple example shows that this property may fail for general open sets:

Example 2.2. Let $X \in R^2$ be the closed rectangle with vertices $(1,1), (-1,1), (1,\frac{1}{2}), (-1,\frac{1}{2})$ and C_n be the closed disc of radius 1, centered at $(1+\frac{1}{n}, 0)$. Denote $U_n = X \cup C_n$ and V_n be obtained from U_n by symmetry with respect to the vertical axis. Clearly $U_n \cap V_n = X = \lim(U_n \cap V_n)$ in the Hausdorff-Pompeiu topology for compact subsets. However $\lim U_n \cap \lim V_n = X \cup (0,0)$, in the same topology. By taking the complementary subsets with respect to some fixed sufficiently big disc, we obtain the desired counter example. This is due to the fact that $(\mathrm{int} U_n) \cup (\mathrm{int} V_n)$ is not uniformly of class C (the outside segment is not bounded from below by a uniform constant with respect to n). We continue with a stability result for domains class C, in the sense of Hedberg-Keldys.

Theorem 2.3. *Let* $\Omega \subset R^d$ *be an open set of class* C*. If* $z \in H^1(R^d)$ *and* $z = 0$ *a.e. in* $R^d - \Omega$*, then* $z\,|_\Omega \in H^1_0(\Omega)$*.*

This result may be immediately extended to higher order Sobolev spaces. It is to be noted that for domains of class C, trace theorems cannot be applied and the above theorem may be viewed as an alternative argument to trace theorems. Related results may be found in Heinonen, Kilpeläinen and Martio [10], via capacity theory methods and under different assumptions.

We close this section with a lower semicotinuity result for integral functionals defined in variable domains:

Theorem 2.4. *Let $l : R^d \times R \times R^d \to R$ be nonnegative and measurable, let $l(x,\cdot,\cdot)$ be continuous on $R \times R^d$ and $l(x,s,\cdot)$ be convex. If $d(\Omega_n,\Omega) \to 0$ and if $y_n \in H^1(\Omega_n)$, $y \in H^1(\Omega)$ satisfy $\{|y_n|_{H^1(\Omega_n)}\}$ bounded and $y_n|_K \to y|_K$ weakly in $H^1(K)$, for any domain $K \subset \overline{K} \subset \Omega$, then*

$$\int_\Omega l(x, y(x), \nabla y(x) dx) \leq \liminf_{n\to\infty} \int_{\Omega_n} l(x, y_n(x), \nabla y_n(x) dx).$$

3. Existence in shape optimization

A general shape optimization problem is defined by a family of subdomains $\mathcal{O}$ in $D \subset R^d$, called admissible domains. In each $\Omega \in \mathcal{O}$, a partial differential equation (usually of elliptic type) is given. For simplicity of writing we shall work with the simple equation ($f \in L^2(D)$ given):

$$-\Delta y_\Omega + y_\Omega = f \quad \text{in } \Omega. \tag{3.1}$$

Various boundary conditions may be added to (3.1). A cost functional to be minimized, usually of integral type is associated

$$\operatorname*{Min}_{\Omega} \int_\Lambda l(x, y_\Omega(x), \nabla y_{\Omega(x)dx}) \tag{3.2}$$

where y_Ω is the solution (in the weak sense) of (3.1), l satisfies the assumptions of Theorem 2.4 and Λ is either Ω or some given domain $\omega \subset R^d$, such that $\omega \subset \Omega$ for any $\Omega \in \mathcal{O}$.

The family $\mathcal{O}$ of admissible subdomains is obtained by fixing some positive constants, k, r, a in Theorem 2.2 and the boundedness and the continuity properties of the corresponding local charts. Constraints of the type

$$\omega \subset \Omega \subset D,\ \forall\, \Omega \in \mathcal{O} \tag{3.3}$$

may be also imposed. State constraints (i.e. on y_Ω) are not considered here. If they are given, they may be penalized in the cost functional as it is standard in optimal control theory.

It is clear that for the study of the problem (3.1) - (3.3), the properties of the mapping $\Omega \to y_\Omega$ are crucial. In particular, the continuity (in a sense to be defined) is essential for the existence theory. Even the convergence of a sequence of solutions $\{y_{\Omega_n}\}$ is not straightforward to define since the y_{Ω_n} are given in different domains $\Omega_n \in \mathcal{O}$.

There are several ways to treat the problem and we shall discuss basically the extension and the local techniques, with brief remarks on other approaches.

3.1. Dirichlet boundary conditions

If null boundary conditions are added to (3.1), the extension method is particularly simple: $y_\Omega \in H_0^1(\Omega)$ may be extended by 0 to $\tilde{y}_\Omega \in H_0^1(D)$.

Taking $y_\Omega \in H_0^1(\Omega)$ as test function in the definition of the weak solution of (3.1), we get $\{|y_\Omega|_{H_0^1(\Omega)}\}$ bounded. Consequently $\{\tilde{y}_\Omega\}$ is bounded for any $\Omega \in \mathcal{O}$, by a constant depending just on $f \in L^2(D)$.

Let $\{\Omega_n\} \subset \mathcal{O}$ be a minimizing sequence of domains for the optimal design problem (3.1) - (3.3). On a subsequence, we may assume that

$$\tilde{y}_{\Omega_n} \to \tilde{y} \quad \text{weakly in } H_0^1(D), \tag{3.4}$$

$$d(\Omega_n, \Omega) \to 0$$

and $\Omega \in \mathcal{O}$ by Theorem 2.2.

The difficulty is to show that $y = \tilde{y}|_\Omega$ is in $H_0^1(\Omega)$ and satisfies (3.1) in Ω. Here, Theorem 2.1 and Theorem 2.3 play the important role: any test function $\varphi \in C_0^\infty(\Omega)$ has the support supp $\varphi \subset \Omega_n$, for $n \geq n_\varphi$ and may be used as test function in Ω_n as well. This gives the definition of the weak solution in Ω by a simple passage to the limit. Again a distributions argument shows that $\tilde{y}|_{D-\Omega} = 0$ a.e. and Theorem 2.3 concludes that $\tilde{y}|_\Omega \in H_0^1(\Omega)$.

Theorem 3.1. *The shape optimization problem* (3.1) - (3.3) *has at least one optimal pair* $[\Omega^*, y^*] \in \mathcal{O} \times H_0^1(\Omega^*)$.

The argument is as above, completed with the application of (3.4) via Theorem 2.4. The uniqueness of the optimal pair is not valid in general since the dependence of y_Ω on Ω has a strongly nonlinear character and the shape optimization problems are nonconvex, in general. Some rare exceptions when uniqueness is valid are, however, known [11], [16].

Remark 3.1. One should note here as well the existence results of Sverak [22], in dimension two, for general open sets $\Omega \subset D$, with $D \setminus \Omega$ having a bounded number of connected components.

3.2. Neumann boundary conditions

The extension technique works in this case under supplementary regularity conditions (uniform Lipschitz property) on the boundary of any admissible $\Omega \in \mathcal{O}$. This is due to Chenais [3] and it is based on the observation that the norm of the extension operators $T_\Omega : H^1(\Omega) \to H^1(D)$ is bounded by some constant depending just on the Lipschitz constant of $\partial\Omega$. Then, as in the previous subsection, the extensions $\{\tilde{y}_\Omega\}$ are bounded in $H^1(D)$ and the argument may proceed somehow similarly.

However, by using the local technique, it is possible to prove existence results for shape optimization problems, for general admissible domains $\Omega \in \mathcal{O}$ of class C.

The assumptions on $\mathcal{O}$ are as in the previous subsection and as in Theorem 2.2. We have immediately that $\{|y_\Omega|_{H^1(\Omega)}\}$ is bounded, y_Ω being the weak solution of (3.1) with Neumann boundary conditions:

$$\int_\Omega \nabla y_\Omega \cdot \nabla v + \int_\Omega y_\Omega v = \int_\Omega fv,\ \forall\, v \in H^1(\Omega). \tag{3.5}$$

Let $\{y_{\Omega_n}\}$ be a minimizing sequence for the optimal design problem (3.2), (3.3), (3.5) and let $d(\Omega_n, \Omega) \to 0$, $\Omega \in \mathcal{O}$ due to Theorem 2.2.

For any $K \subset \overline{K} \subset \Omega$, we have $K \subset \Omega_n$, $n \geq n_K$ and $\{y_{\Omega_n}|_K\}$ bounded in $H^1(K)$. On a subsequence we may assume $y_{\Omega_n} \to \tilde{y}$ weakly in $H^1(K)$. By taking further subsequences and letting $K \to \Omega$, we can construct $\tilde{y}$ in the whole domain Ω. Some distributions argument shows that $\tilde{y} \in H^1(\Omega)$.

We rewrite (3.5), in Ω_n, in the form

$$\int_K \nabla y_{\Omega_n} \cdot \nabla v + \int_K y_{\Omega_n} v - \int_{\Omega_n} fv = \int_{\Omega_n \setminus K} [\nabla y_{\Omega_n} \cdot \nabla v + y_{\Omega_n} v],\ \forall v \in C^1(\overline{D}) \tag{3.6}$$

The right-hand side in (3.6) may be estimated by

$$\left| \int_{\Omega_n \setminus K} [\nabla y_{\Omega_n} \cdot \nabla v + y_{\Omega_n} v] \right| \leq M |v|_{C^1(D)} \mu(\Omega_n \setminus K)^{\frac{1}{2}} \tag{3.7}$$

where M bounds $|y_{\Omega_n}|_{H^1(\Omega_n)}$ and μ is the Lebesgue measure in R^d.

Due to the pointwise convergence of the characteristic functions $\chi_{\Omega_n} \to \chi_\Omega$ a.e. in D, we can pass to the limit $n \to \infty$ in (3.6), (3.7) to obtain

$$\left| \int_K [\nabla \tilde{y} \cdot \nabla v + \tilde{y} v] - \int_\Omega fv \right| \leq M |v|_{C^1(D)} \mu(\Omega \setminus K)^{\frac{1}{2}}.$$

By taking $K \to \Omega$ and by using Theorem 1.2, we see that $\tilde{y} \in H^1(\Omega)$ is indeed the solution of the Neumann problem in Ω, $\tilde{y} = y_\Omega$.

Again by Theorem 2.4, we get

Theorem 3.2. *The shape optimization problem* (3.2), (3.3), (3.5) *has at least one optimal pair* $[\Omega^*, y^*] \in \mathcal{O} \times H^1(\Omega^*)$.

Remark 3.2. The local technique may be applied in the case of Dirichlet boundary conditions as well. In [16], the case of general nonlinear elliptic operators of Leray-Lions type (the generalized divergence operator) is discussed by this method.

Remark 3.3. Due to the very weak assumptions on $\mathcal{O}$, the extension to general nonhomogeneous Neumann conditions is unclear since (3.5) would involve an integral on $\partial\Omega$.

3.3. Mixed boundary conditions

In this case, it is necessary to add to the general hypotheses on $\mathcal{O}$ some compatibility conditions in the boundary points where the Dirichlet condition changes into the Neumann condition and conversely.

For each $\Omega \in \mathcal{O}$, we associate an open set $D_\Omega \subset D$, $D_\Omega \not\subset \Omega$, not necessarily connected, $D_\Omega = \bigcup_{i=1}^{i_\Omega} D_\Omega^i$ (D_Ω^i are the connected components and $i_\Omega \in N$ is their number depending on Ω). We assume that $\Omega \cup D_\Omega$ is connected and $D_\Omega \cap \partial\Omega$ represents the parts of $\partial\Omega$ where the null Dirichlet condition is valid. This setting allows a considerable freedom in the formulation of the mixed boundary value problem. On the open sets D_Ω^i similar class C hypotheses, with positive constants $k_\Omega^i, r_\Omega^i, a_\Omega^i$ and family of continuous local charts $\mathcal{F}_\Omega^i$ are imposed, as for $\Omega \in \mathcal{O}$.

Since trace theorems are not valid in this setting, it is necessary to introduce a special subspace $V(\Omega) = \overline{\mathcal{M}}$ (the closure in $H^1(\Omega)$),

$$\mathcal{M} = \{w|_\Omega;\ w \in C_0^\infty(R^d) \text{ with } w|_{Q_w \setminus K_w} = 0\}$$

for some open subset $Q_w \supset \overline{D}_\Omega$ and some compact subset $K_w \subset \Omega$.

It is easy to see that $\mathcal{M}$ has indeed a linear structure since one may take $Q_{w_1+w_2} = Q_{w_1} \cap Q_{w_2}$, $K_{w_1+w_2} = K_{w_1} \cup K_{w_2}$.

On $V(\Omega)$ the same inner product and norm as in $H^1(D)$ will be considered. We can now write the weak formulation of the mixed boundary value problem asoociated to (3.1) in Ω:

$$\int_\Omega [\nabla y_\Omega \cdot \nabla v + y_\Omega v] = \int_\Omega fv,\ \forall\, v \in V(\Omega). \tag{3.8}$$

If $\partial\Omega$ is Lipschitzian, then (3.8) is the usual mixed boundary value problem with homogeneous Dirichlet and Neumann conditions on $\partial\Omega \cap D_\Omega$, respectively $\partial\Omega \setminus D_\Omega$.

The corresponding shape optimization problem is given by (3.8), (3.2), (3.3). We indicate now the compatibility hypotheses, necessary due to the complex geometric structure of this problem:

$$\forall\, x \in \partial\Omega \cap \partial D_\Omega^i,\ i = \overline{1, i_\Omega},\ \exists\, V_x \tag{3.9}$$

neighbourhood of x such that both $\partial(\Omega \cup (D \setminus \overline{D}_\Omega^i)) \cap V_x$ and $\partial(\Omega \cup D_\Omega^i) \cap V_x$ can be represented in the same local system of axes by continuous functions that extend the representation of ∂D_Ω^i around x from $\partial D_\Omega^i \setminus \Omega$. That is

$$\partial(\Omega \cup D_\Omega^i) \cap V_x = \{(s, \tilde{g}_x(s));\ s \in B(0, k_\Omega^i)\}$$

$$\partial(\Omega \cup (D \setminus \overline{D}_\Omega^i)) \cap V_x = \{(s, \hat{g}_s(s));\ s \in B(0, k_\Omega^i)\}$$

and $\hat{g}_x(s) \le \tilde{g}_x(s)$, $\forall\, s \in B(0, k_\Omega^i)$.

On the subset of $B(0, k_\Omega^i)$ corresponding to

$$\partial D_\Omega^i \cap [\partial(\Omega \cup D_\Omega^i) \cap V_x] = \partial D_\Omega^i \cap [\partial(\Omega \cup (D \setminus \overline{D}_\Omega^i)) \cap V_x]$$

these two mappings coincide.

$$\begin{aligned} z \in \partial\Omega \setminus \bigcup_{x \in \partial\Omega \cap \partial D_\Omega} [\{(s, \tilde{g}_x(s));\ s \in \overline{B(0, r_\Omega^i)}\} \cup \\ \cup \{(s, \hat{g}_x(s));\ s \in \overline{B(0, r_\Omega^i)}\}] \Rightarrow d(z, \partial D_\Omega^i) \ge c > 0,\ 1 \le i \le i_\Omega. \end{aligned} \tag{3.10}$$

Roughly speaking, condition (3.10) says that any $\Omega \in \mathcal{O}$ is not "developping fingers" far from $\partial\Omega \cap \partial D_\Omega$ that enter in small neighbourhoods of ∂D_Ω. The constant $c > 0$ is uniform for all $\Omega \in \mathcal{O}$. While conditions (3.9), (3.10) may look complicated, we have to recall that D_Ω is at our choice, just in order to represent the parts $\partial\Omega \cap D_\Omega$ where the Dirichlet condition is valid. This allows a big flexibility in the choice of D_Ω such that (3.9), (3.10) are fulfilled as well.

If for any $\Omega \in \mathcal{O}$, $a_\Omega \ge a > 0$, $a_\Omega^i \ge a > 0$, $k_\Omega \ge k > 0$, $k_\Omega^i \ge k > 0$, $r_\Omega \le r < k$, $r_\Omega^i \le r$, $1 \le i \le i_\Omega$ and the family of all the local charts $\bigcup_{\Omega \in \mathcal{O}} [\mathcal{F}_\Omega \cup (\bigcup_{i=1}^{i_\Omega} \mathcal{F}_\Omega^i)]$ is equibounded and equicontinuous on $\overline{B(0, k)}$ and assuming (3.9), (3.10) a compactness theorem similar to Thm 2.2 may be proved in this case as well, [16]. Namely, if $d(\Omega_n, \Omega) \to 0$ and $d(D_{\Omega_n}, D_\Omega) \to 0$ then Ω, D_Ω satisfy all the above assumptions, including (3.9), (3.10).

Moreover, $\chi_{\Omega_n} \to \chi_\Omega$ a.e. in D, on a subsequence.

We also need an extension of the Hedberg-Keldys stability result, Theorem 2.3, to this situation and providing a characterization of the space $V(\Omega)$, [16].

Theorem 3.3. *Let $\Omega \in \mathcal{O}$ be given and let $\tilde{v}$ denote the extension of $v \in V(\Omega)$ to $\Omega \cup D_\Omega$, by zero. Then $\tilde{v} \in H^1(\Omega \cup D_\Omega)$. Conversely, if $w \in H^1(\Omega \cup D_\Omega)$ and $w = 0$ a.e. in $D_\Omega \setminus \Omega$, then $w|_\Omega \in V(\Omega)$.*

In this setting, it is possible to prove local continuity results with respect to Ω, of the solution of (3.8), similar to the previous subsection. However, the argument is much more technical and will be omitted.

By using Theorem 2.4, one can establish existence results for the shape optimization problem (3.8), (3.2), (3.3) as well.

Remark 3.4. It is possible to prove results of the same type when variational inequalities appear as the state equation.

4. Approximation

In this section, we perform an analysis of the discretization via finite elements of general shape optimization problems as discussed in §3. The case of Dirichlet boundary conditions was considered by Zuazua and Chenais [4].

Applications to problems governed by stationary Navier-Stokes equations, under Lipschitz conditions on $\partial\Omega$, are discussed in [8], [25].

Let $\{\mathcal{T}_h\}_{h\geq 0}$ be a family of uniformly regular finite element meshes in D, the bounded open set containing the family of all admissible domains of class C, $\Omega \in \mathcal{O}$. We define two types of discrete approximations of Ω (exterior and interior):

$$\widehat{\Omega}_h = \text{int}\{\cup\overline{T_h},\ T_h \in \mathcal{T}_h,\ T_h \cap \Omega \neq \emptyset\}, \tag{4.1}$$

$$\widetilde{\Omega}_h = \text{int}\{\cup\overline{T_h},\ T_h \in \mathcal{T}_h,\ T_h \subset \Omega\}. \tag{4.2}$$

We denote by $\widehat{\mathcal{O}}_h$, respectively $\widetilde{\mathcal{O}}_h$, the family of admissible discretized domains obtained by applying to $\mathcal{O}$ rule (4.1), respectively (4.2).

If no confusion may arise, we write shortly $\Omega_h, \mathcal{O}_h$. Notice that $\mathcal{O}_h$ is always finite since any $\Omega_h \in \mathcal{O}_h$ is a combination of elements $T_h \in \mathcal{T}_h$. A discretized admissible domain $\Omega_h \in \mathcal{O}_h$ may correspond to "many" admissible $\Omega \in \mathcal{O}$.

Some properties of (4.1), (4.2) under the uniform class C assumptions (see §3) for $\mathcal{O}$:

Theorem 4.1. *(a)* $d(\Omega_h, \Omega) \leq h$,

(b) meas$(\widehat{\Omega}_h \setminus \Omega) \leq c_\Omega(h + o(h))$,

(c) meas$(\Omega \setminus \widetilde{\Omega}_h) \leq c_\Omega(h + o(h))$,

where $o(h)$ *is the uniform continuity modulus of the family* $\mathcal{F}$ *of local charts,* $\Omega \in \mathcal{O}$ *and* $c_\Omega > 0$ *is a constant independent of* $h > 0$*, but depending on* Ω.

Theorem 4.2. *Let* $U_h \in \mathcal{O}_h$ *satisfy* $d(U_h, \Omega) \to 0$ *as* $h \to 0$*. Then*

(a) $\Omega \in \mathcal{O}$,

(b) meas$[(U_h \setminus \Omega) \cup (\Omega \setminus U_h)] \to 0$.

Remark 4.1. Neither $U_h \subset \Omega$, nor $\Omega \subset U_h$ is assumed in Theorem 4.2. $U_h \in \mathcal{O}_h$ is of type (4.1) or (4.2) but not necessarily constructed from $\Omega \in \mathcal{O}$.

We consider now the case of the Neumann boundary conditions (3.5), (3.2), (3.3). The discretization of (3.5) is

$$\int_{\widehat{\Omega}_h} (\nabla \tilde{y}_h \cdot \nabla v_h + \tilde{y}_h v_h) dx = \int_{\widehat{\Omega}_h} f v_h dx, \ \forall\, v_h \in V_h. \tag{4.3}$$

In (4.3) we use the variant (4.1) of the discretization of Ω and V_h is a finite element space in $\widehat{\Omega}_h$ obtained as the restriction of a finite element space V_h' on the mesh $\mathcal{T}_h$ in D.

Notice that on V_h' we impose that for any v smooth in D, its projection $v_h \in V_h'$ satisfies $v_h \to v$ in $W^{1,\infty}(D)$. Such finite element spaces of higher order are described in Ciarlet and Raviart [5], Oden and Reddy [19].

Theorem 4.3. *(a) Assume that $d(\widehat{U}_h, \Omega) \to 0$ and let y_h denote the solution of (4.3) in $\widehat{U}_h \in \widehat{\mathcal{O}}_h$. Then, on a subsequence, we have*

$$y_h|_\omega \to y_\Omega|_\omega \quad \text{weakly in } H^1(\omega). \tag{4.4}$$

(b) If $f \in L^\infty(D)$ and $\widehat{U}_h \supset \Omega\ \forall\, h > 0$, then

$$y_h|_\Omega \to y_h \quad \text{strongly in } H^1(D).$$

Remark 4.2. Since $\widehat{U}_h$ are of type (4.1), we get automatically that $\omega \subset \widehat{U}_h$ by (3.3) and (4.4) makes sense.

We consider the cost functional

$$J(\Omega) = \int_\omega j(x, y_\Omega(x)) dx \tag{4.5}$$

that defines the shape optimization problem together with (3.3), (3.5) and we assume that $j(x, y_h(x)) \to j(x, y(x))$ weakly in $L^1(\omega)$ if $y_h \to y$ weakly in $H^1(\omega)$. The discrete variant of (4.5) is obtained simply by replacing y_Ω with $\tilde{y}_h$, the solution of (4.3). We denote by J_h the discrete cost functional. Notice that the discrete variant of (3.3) is automatically satisfied by the construction of $\widehat{\Omega}_h$ in (4.1).

Theorem 4.4. *J and J_h reach their minimum point (not necessarily unique) on $\mathcal{O}$, respectively $\widehat{\mathcal{O}}_h$.*

(a) Any accumulation point of any sequence $\{\widehat{\Omega}_h^*\}$ of discrete minimizers on $\widehat{\mathcal{O}}_h$ is a continuous minimizer $\Omega^* \in \mathcal{O}$.

(b) $J_h(\widehat{\Omega}_h^*) \to J(\Omega^*)$ for $h \to 0$, on the initial sequence.

The existence property for J is discussed in the previous section, while for J_h it is obvious since $\widehat{\mathcal{O}}_h$ is always a finite family. Property (a) is a consequence of Theorem 4.3 and property (b) is given by the uniqueness of the optimal value.

We continue with the case of mixed boundary conditions. The hypotheses are the same as in the previous section, §3.3. The discretization of domains $\Omega \in \mathcal{O}$, respectively $D_\Omega = \bigcup_{i=1}^{i_\Omega} D_\Omega^i$ is of type $\widetilde{\Omega}_h$, $(\widehat{D}_\Omega^i)_h$ (interior - exterior). Clearly, $\widetilde{\Omega}_h \subset \Omega$, $(\widehat{D}_\Omega^i)_h \supset D_\Omega^i$ for any $h > 0$. The finite element space defined in $\widetilde{\Omega}_h$ is denoted by $\overset{\circ}{V}_h$ and consist of all the elements $v_h \in V_h''$ (a finite element space defined in $D \supset \Omega$) such that $v_h|_{(\widehat{D}_\Omega)_h \setminus \widetilde{\Omega}_h} = 0$ and by taking their restriction to $\widetilde{\Omega}_h$. This is the discrete form of the condition in the definition of $\mathcal{M}$ in §3.3. For the finite element space V_h'' in D, it is assumed that the projection w_h of any element $w \in \mathcal{M} \subset V(\Omega)$ satisfies $w_h \to w$ in $W^{1,\infty}(D)$.

The discretization of (3.8) is

$$\int_{\widetilde{\Omega}_h} [\nabla y_h \cdot \nabla v_h + y_h v_h] dx = \int_{\widetilde{\Omega}_h} f v_h dx, \ \forall\, v_h \in \overset{\circ}{V}_h\,. \tag{4.6}$$

Theorem 4.5. *Let $\widehat{U}_h, \widetilde{A}_h$ be such that $d(\widehat{U}_h, D_\Omega) \to 0$, $d(\widetilde{A}_h, \Omega) \to 0$ and $\widehat{U_h}$, $\widetilde{A}_h$ are constructed via* (4.1), (4.2) *starting from the open sets D_Ω, $\Omega \in \mathcal{O}$, respectively.*

Let $y_h \in \overset{\circ}{V}_h$ be the solution of (4.6) *corresponding to $\widetilde{A}_h$. Then, for any subdomain $K \to \Omega$, we have*

$$y_h|_K \to y_\Omega|_K \quad \text{weakly in } H^1(K), \tag{4.7}$$

where y_Ω is the solution of (3.8).

The cost functional J has the form (4.5) (with the same hypothesis) and the discrete cost functional J_h is again obtained by replacing y_Ω with y_h in (4.5).

We strengthen slightly the constraint (3.3):

$$\omega \to \omega_1 \subset \Omega \subset D, \ \forall\, \Omega \in \mathcal{O}. \tag{4.8}$$

By (4.8) y_h (defined in $\widetilde{\Omega}_h \subset \Omega$) is defined in ω for h sufficiently small and J_h makes sense.

Theorem 4.6. *J and J_h reach their minimum (not necessarily unique) on $\mathcal{O}$, respectively $\mathcal{O}_h$, $h > 0$.*

(a) Any accumulation point of any sequence $\{\widetilde{\Omega}_h^*\}$ of discrete minimizers on $\mathcal{O}_h$ (together with the corresponding $(\widehat{D}_\Omega)_h^*$ and their accumulation point) is a continuous minimizer Ω^* (together with D_Ω^*) in $\mathcal{O}$.

(b) $J_h(\widetilde{\Omega}_h^*) \to J(\Omega^*)$ on the initial sequence.

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