Annals of the University of Bucharest (mathematical series)
4 (LXII) (2013), 243–252

A contact problem with normal compliance and adhesion

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Abstract - The aim of this work is to study a quastistatic contact problem between an elastic body and a deformable foundation. The behavior of the material is modeled by a nonlinear elastic law and the contact is modeled with normal compliance and adhesion. The evolution of the bonding field is described by a nonlinear differential equation. We state the classical formulation of the problem, then we derive its variational formulation. Next, we prove the existence of a uniquene the weak solution to the problem. The proof is based on arguments on variational inequalities, the Cauchy-Lipschitz theorem and the Banach fixed point argument.

Key words and phrases : elastic material, contact, adhesion, normal compliance, differential equation, variational inequality, weak solution, fixed point.

Mathematics Subject Classification (2010) **:** 74M15, 74M10, 49J40, 35J85.

1. Introduction

Contact phenomena with adhesion between two deformable bodies or between a deformable body and a rigid foundation abound in industry and in everyday life. The contact with adhesion between the different layers of a composite material and between the pistons and sleeves are current examples. Because of the importance of the adhesive contact process in mechanical systems and structures, considerable efforts have been made in its mathematical modeling, mathematical analysis and numerical simulations.

To model the process of contact with adhesion, Frémond [1], [2] introduced a new internal variable of surface, the bonding field, denoted in this paper by β. It describes the fractional density of active bonds on the contact surface Γ_3. When $\beta = 1$ at a point of contact surface the adhesion is complete and all the bonds are active; when $\beta = 0$ all the bonds are inactive, severed, and there is no adhesion; when $0 < \beta < 1$ the adhesion is partial and only a fraction of the bonds is active. The adhesive on the contact surface introduces tension that opposes the separation of the surfaces in the normal direction and opposes the relative motion in the tangential direction. The adhesive tensile traction is assumed to be proportional to β^2 and to the normal displacement u_ν. Problems of contact with adhesion were studied by several authors. Significant results on these problems can be found in [4], [6], [7] and references therein.

The aim of this work is to continue the study of adhesive problems presented in [5], [8]. The novelty consists in the fact that here we consider a quasistatic process of contact and we model the material's behavior with a nonlinear elastic law. The main contribution of this study lies in the proof of the existence and uniqueness of the weak solution of the mechanical problem.

The rest of the manuscript is organized as follows. In Section 2 we present some notations and preliminaries. In Section 3 we describe the mechanical problem, state the assumptions on the data and deduce its variational formulation. Finally, in Section 4 we prove the existence and uniqueness of the weak solution. The proof is carried out in several steps; it is based on arguments of variational inequalities, ordinary differential equations and fixed point.

2. Preliminaries

We denote by r_+ the positive part of $r \in \mathbb{R}$, $\mathbb{S}^N$ is the space of second order symmetric tensors on $\mathbb{R}^N$ ($N = 2, 3$), while "$\cdot$" and $|\cdot|$ represent the inner product and the Euclidean norm on $\mathbb{R}^N$ and $\mathbb{S}^N$, respectively. Thus, for every $u, v \in \mathbb{R}^N$ and $\sigma, \tau \in \mathbb{S}^N$ we have

$$u \cdot v = u_i v_i, \quad |u| = (u \cdot u)^{\frac{1}{2}}, \quad \sigma \cdot \tau = \sigma_{ij}\tau_{ij}, \quad |\sigma| = (\sigma \cdot \sigma)^{\frac{1}{2}}$$

Here and everywhere in what follows the indices i, j vary between 1, N and the summation convention over repeated indices is adopted.

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with a Lipschitz boundary Γ and let ν denote the unit outer normal on Γ. We need the spaces

$$H = \{ u = (u_i) \ : \ u_i \in L^2(\Omega)\}, \quad \mathcal{H} = \{ \sigma = (\sigma_{ij}) \ : \ \sigma_{ij} = \sigma_{ji} \in L^2(\Omega) \},$$

$$H_1 = \{u \in H \ : \ \varepsilon(u) \in \mathcal{H}\}, \quad \mathcal{H}_1 = \{\sigma \in \mathcal{H} \ : \ Div\,\sigma \in H\}$$

where $\varepsilon : H_1 \longrightarrow \mathcal{H}$, $Div : \mathcal{H} \longrightarrow H$ are the deformation and the divergence operators, respectively, defined by

$$\varepsilon(u) = (\varepsilon_{ij}(u)), \quad \varepsilon_{ij}(u) = \frac{1}{2}(\partial_j u_i + \partial_i u_j), \quad Div\,\sigma = (\partial_j \sigma_{ij}).$$

The spaces H, $\mathcal{H}$, H_1 and $\mathcal{H}_1$ are real Hilbert spaces endowed with the canonical inner products given by

$$\langle u, v\rangle_H = \int_\Omega u_i v_i dx \quad \forall u, v \in H, \quad \langle \sigma, \tau\rangle_{\mathcal{H}} = \int_\Omega \sigma_{ij}\tau_{ij} dx \quad \forall \sigma, \tau \in \mathcal{H},$$

$$\langle u,v\rangle_{H_1} = \langle u,v\rangle_H + \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}} \quad \forall u,v \in H_1,$$

$$\langle \sigma,\tau\rangle_{\mathcal{H}_1} = \langle \sigma,\tau\rangle_{\mathcal{H}} + \langle Div\sigma, Div\tau\rangle_H \quad \forall \sigma,\tau \in \mathcal{H}_1.$$

The associated norms are denoted by $|.|_H$, $|.|_{\mathcal{H}}$, $|.|_{H_1}$and $|.|_{\mathcal{H}_1}$ respectively. Since the boundary Γ is Lipschitz continuous, the unit outward normal vector ν on the boundary is defined almost everywhere. For every vector field $u \in H_1$ we still use the notation u for the trace of u on Γ. The normal and tangential components of u on the boundary Γ are given by $u_\nu = u \cdot \nu$, $u_\tau = u - u_\nu \nu$. We define, similarly, the normal and tangential components of the Cauchy-stress vector $\sigma\nu$ on the boundary Γ by equalities $\sigma_\nu = (\sigma\nu)\cdot\nu$, $\sigma_\tau = \sigma\nu - \sigma_\nu \nu$ and we recall that the following Green's formula holds:

$$\langle \sigma, \varepsilon(u)\rangle_{\mathcal{H}} + \langle Div\,\sigma, u\rangle_H = \int_\Gamma \sigma\nu\, u ds \quad \forall u \in H_1.$$

Let Γ_1 be a measurable part of Γ such that $meas\,\Gamma_1 > 0$ and let V be the closed subset of H_1 defined by

$$V = \{v \in H_1 \ : \ v = 0 \quad \text{on } \Gamma_1\}$$

Since $meas\,\Gamma_1 > 0$, then Korn's inequality holds, and thus there exists a constant $c > 0$, depending on Ω and Γ_1, such that :

$$|\varepsilon(u)|_{\mathcal{H}} \geq c\,|u|_{H_1} \quad \forall u \in V$$

A proof of Korn's inequality may be found in [3], p. 79. Over the space V we consider the inner product

$$\langle u,v\rangle_V = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}} \quad \forall u,v \in V$$

It follows from Korn's inequality that $|.|_V$ and $|.|_{H_1}$are equivalent norms on V. Therefore, $(V, |.|_V)$ is a real Hilbert space. Moreover, from the Sobolev trace theorem, there exists a positive constant $c > 0$, depending on Ω, Γ_1 and Γ_3 such that

$$|v|_{L^2(\Gamma_3)^N} \leq c\,|v|_V \quad \forall v \in V$$

Finally, we use the standard notation for the spaces of fonctions defined on the time interval $[0,T]$ $(T > 0)$ with values in a real normed space X. In particular, we shall use the spaces $C(0,T;X)$, $L^p(0,T;X)$ and $W^{k,p}(0,T;X)$. Moreover, we define the set $\mathcal{Q}$ by

$$\mathcal{Q} = \left\{\beta \in C\left(0,T;L^2(\Gamma_3)\right) : 0 \leq \beta(t) \leq 1 \quad \forall t \in [0,T] \text{ a.e. on } \Gamma_3\right\}$$

We end this preliminary with the following version of the classical theorem of Cauchy-Lipschitz which can be found in [9], p. 60.

Theorem 2.1. *Let $(X, |.|_X)$, be a real Banach space and $F(t,.) : X \longrightarrow X$ an operator defined almost everywhere on $(0,T)$ which satisfies the following conditions:*

1) There exists $L_F > 0$ such that $|F(t,u) - F(t,v)|_X \leq L_F |u-v|_X$ $\forall u, v \in X$ a.e. $t \in (0,T)$.

2) There exists $p \geq 1$ such that $t \mapsto F(t,u) \in L^p(0,T;X) \quad \forall u \in X$.

Then, for every $u_0 \in X$, there exists a unique function $u \in W^{1,p}(0,T;X)$ such that

$$\dot{u}(t) = F(t, u(t)) \quad a.e.\ t \in (0,T), \qquad u(0) = u_0.$$

This theorem will be used to prove the existence and uniqueness of the weak solution of the contact problem we introduce in the next section.

3. The model

We consider an elastic body which occupies a bounded domain $\Omega \subset \mathbb{R}^N$ ($N = 2, 3$ for applications) and we assume that its boundary Γ is regular and partitioned into three disjoint measurable parts Γ_1, Γ_2 and Γ_3 such that $meas\,\Gamma_1 > 0$. We are interested in the deformation of the body in the time interval $(0,T)$, where $T > 0$. The body is clamped on $\Gamma_1 \times (0,T)$ and, therefore, the displacement field vanishes there. We also assume that a volume force of density f_0 acts in $\Omega \times (0,T)$ and a surface traction of density f_2 acts on $\Gamma_2 \times (0,T)$. On $\Gamma_3 \times (0,T)$ the body is in adhesive frictionless contact with a deformable foundation. Moreover, the process is quasistatic and the evolution of the bonding field is described by a nonlinear differential equation. The classical formulation of the problem is as follows.

Problem P. *Find a displacement field $u : \Omega \times [0,T] \longrightarrow \mathbb{R}^N$, a stress field $\sigma : \Omega \times [0,T] \longrightarrow \mathbb{S}^N$ and a bonding field $\beta : \Gamma_3 \times [0,T] \longrightarrow [0,1]$ such that*

$$\sigma(t) = F(\varepsilon(u(t))) \quad in\ \Omega \times (0,T), \tag{3.1}$$

$$Div\sigma(t) + f_0(t) = 0 \quad in\ \Omega \times (0,T), \tag{3.2}$$

$$u = 0 \quad on\ \Gamma_1 \times (0,T), \tag{3.3}$$

$$\sigma\nu = f_2 \quad on\ \Gamma_2 \times (0,T), \tag{3.4}$$

$$-\sigma_\nu = p_\nu(u_\nu) - \gamma_\nu \beta^2 (-R(u_\nu))_+ \quad on\ \Gamma_3 \times (0,T), \tag{3.5}$$

$$\sigma_\tau = 0 \quad on\ \Gamma_3 \times (0,T), \tag{3.6}$$

$$\dot{\beta} = -\left(\gamma_\nu \beta \left[(-R(u_\nu))_+\right]^2 - \epsilon_a\right)_+ \quad on\ \Gamma_3 \times (0,T), \tag{3.7}$$

$$\beta(0) = \beta_0 \quad on\ \Gamma_3. \tag{3.8}$$

For a detailed description of the equations and boundary conditions in Problem **P** we refer to [7]. Here we restrict ourselves to recall that (3.1) is the nonlinear elastic constitutive law, (3.2) represents the equilibrium equation, (3.3) and (3.4) are the displacement and the traction boundary conditions, respectively; condition (3.5) represents the normal compliance contact with adhesion where p_ν is a given positive function, γ_ν and ϵ_a are given adhesion coefficients and R is the truncation operator defined by

$$R(s) = \begin{cases} L \quad \text{if } s \geq L \\ s \text{ if } |s| < L \\ -L \quad \text{if } s \leq -L, \end{cases} \tag{3.9}$$

$L > 0$ being the characteristic length of the bonds; condition (3.6) represents the frictionless contact and shows that the tangential stress vanish on the contact surface Γ_3, during the process; finally, equation (3.7) describes the evolution of the bonding field and (3.8) represents the initial condition for the bonding field.

In the study of Problem **P** we assume that the elasticity operator F, and the normal compliance function p_ν, satisfy the following conditions.

$$\begin{cases} \text{(a) } F : \Omega \times \mathbb{S}^N \longrightarrow \mathbb{S}^N. \\ \text{(b) } \exists m > 0 \text{ such that } (F(x, \varepsilon_1) - F(x, \varepsilon_2)) \cdot (\varepsilon_1 - \varepsilon_2) \geq m\,|\varepsilon_1 - \varepsilon_2|^2 \\ \qquad \text{a.e. } x \in \Omega \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^N. \\ \text{(c) } \exists L > 0 \text{ such that } |F(x, \varepsilon_1) - F(x, \varepsilon_2)| \leq L\,|\varepsilon_1 - \varepsilon_2| \\ \qquad \text{a.e. } x \in \Omega \quad \forall \varepsilon_1, \varepsilon_2 \in \mathbb{S}^N. \\ \text{(d) The mapping } x \mapsto F(x, \varepsilon) \text{ is Lebesgue measurable} \\ \qquad \text{a.e. } x \in \Omega, \quad \forall \varepsilon \in \mathbb{S}^N. \\ \text{(e) The mapping } x \mapsto F(x, 0) \text{ belongs to } \mathcal{H}. \end{cases} \tag{3.10}$$

$$\begin{cases} \text{(a) } p_\nu : \Gamma_3 \times \mathbb{R} \longrightarrow \mathbb{R}_+. \\ \text{(b) } \exists L_\nu > 0 \text{ such that } |p_\nu(x, r_1) - p_\nu(x, r_2)| \leq L_\nu\,|r_1 - r_2| \\ \qquad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3. \\ \text{(c) } (p_\nu(x, r_1) - p_\nu(x, r_2))(r_1 - r_2) \geq 0 \\ \qquad \forall r_1, r_2 \in \mathbb{R}, \text{ a.e. } x \in \Gamma_3. \\ \text{(d) The mapping } x \mapsto p_\nu(x, r) \text{ is Lebesgue measurable} \\ \qquad \text{on } \Gamma_3, \forall r \in \mathbb{R}. \\ \text{(e) } p_\nu(x, r) = 0 \quad \forall r \leq 0, \quad \text{a.e. } x \in \Gamma_3. \end{cases} \tag{3.11}$$

We also suppose that the adhesion coefficients satisfy

$$\gamma_\nu \in L^\infty(\Gamma_3), \quad \epsilon_a \in L^2(\Gamma_3), \quad \gamma_\nu,\, \epsilon_a \geq 0 \quad \text{a.e. on } \Gamma_3, \tag{3.12}$$

the densities of the body forces and surface traction have the regularity

$$f_0 \in W^{1,\infty}(0, T; H), \quad f_2 \in W^{1,\infty}(0, T; L^2(\Gamma_2^N)) \tag{3.13}$$

and, finally, the initial data satisfies

$$\beta_0 \in L^2(\Gamma_3), \quad 0 \leq \beta_0 \leq 1 \quad a.e.\ on\ \Gamma_3. \tag{3.14}$$

It follows from (3.13) and Riesz-Frechet's representation theorem that there exists a unique function $f : [0,T] \longrightarrow V$ such that :

$$\langle f(t), v\rangle_V = \langle f_0(t), v\rangle_H + \langle f_2(t), v\rangle_{L^2(\Gamma_2)^N} \quad \forall v \in V, t \in (0,T) \tag{3.15}$$

and, moreover,

$$f \in W^{1,\infty}(0,T;V). \tag{3.16}$$

Finally, we define the adhesion functional $j : L^\infty(\Gamma_3) \times V \times V \longrightarrow \mathbb{R}$ and the normal compliance functional $k : V \times V \longrightarrow \mathbb{R}$ by equalities

$$j(\beta, u, v) = -\int_{\Gamma_3} \gamma_\nu \beta^2 (-R(u_\nu))_+ v_\nu ds, \tag{3.17}$$

$$k(u,v) = \int_{\Gamma_3} p_\nu(u_\nu) v_\nu ds \tag{3.18}$$

for all $u,\ v \in V$, $\beta \in L^\infty(\Gamma_3)$.

Applying Green's formula and using the equilibrium equation and the boundary conditions, we can easily deduce the following variational formulation of the mechanical problem.

Problem PV. *Find a displacement* $u : [0,T] \longrightarrow V$*, and a bonding field* $\beta : [0,T] \longrightarrow L^\infty(\Gamma_3)$ *such that :*

$$\begin{aligned} &\langle F(\varepsilon(u(t))), \varepsilon(v)\rangle_{\mathcal{H}} + j(\beta(t), u(t), v) + k(u(t), v) \\ &\quad = \langle f(t), v - u(t)\rangle_V \quad \forall v \in V, \quad t \in (0,T) \end{aligned} \tag{3.19}$$

$$\dot{\beta}(t) = -\left(\gamma_\nu \beta(t)\left[(-R(u_\nu(t)))_+\right]^2 - \epsilon_a\right)_+ \quad a.e. \quad t \in (0,T) \tag{3.20}$$

$$\beta(0) = \beta_0. \tag{3.21}$$

The unique solvability of Problem **PV** will be proved in the next section.

4. Existence and uniqueness

Our main result in the study of Problem **PV** is the following.

Theorem 4.1. *Assume that* (3.10)–(3.14) *hold. Then, there exists a unique solution* (u, β) *to Problem* **PV** *which satisfies*

$$u \in W^{1,\infty}(0,T;V), \qquad \beta \in W^{1,\infty}(0,T;L^\infty(\Gamma_3)) \cap \mathcal{Q}. \tag{4.1}$$

Proof. The proof of Theorem 4.1 will be carried out in four steps that we sketch in what follows.

(i) Assume that $\beta \in \mathcal{Q}$ is given. In the first step we consider auxiliary problem of finding a displacement $u_\beta : [0,T] \longrightarrow V$ such that

$$\begin{aligned} &\langle F\left(\varepsilon\left(u_\beta\left(t\right)\right)\right), \varepsilon\left(v\right)\rangle_{\mathcal{H}} + j\left(\beta\left(t\right), u_\beta\left(t\right), v\right) \\ &\quad + k\left(u_\beta\left(t\right), v\right) = \langle f\left(t\right), v\rangle_V \quad \forall v \in V, \quad t \in (0,T). \end{aligned} \tag{4.2}$$

We prove that this problem has a unique solution which satisfies $u_\beta \in C\left(0,T;V\right)$. To this end we fix $t \in [0,T]$ and we consider the operator $A_t : V \longrightarrow V$ defined by

$$\langle A_t u, v\rangle_V = \langle F\left(\varepsilon\left(u\right)\right), \varepsilon\left(v\right)\rangle_{\mathcal{H}} + j\left(\beta\left(t\right), u\left(t\right), v\right) + k\left(u, v\right)$$

for all $u, v \in V$. It is easy to see that A_t is a Lipschitz continuous operator and, therefore, it follows from standard results that there exists a unique element $u_\beta(t)$ such that $A_t(u_\beta(t)) = f(t)$. We conclude from here that $u_\beta(t)$ satisfies (4.2). A standard computation shows that $u_\beta \in C\left(0,T;V\right)$.

(ii) In the second step we use the displacement field u_β obtained in the previous step and we consider the auxiliary problem of finding a bonding field $\theta_\beta : [0,T] \longrightarrow L^\infty\left(\Gamma_3\right)$ such that

$$\dot{\theta_\beta}\left(t\right) = -\left(\gamma_\nu \theta_\beta\left(t\right)\left[\left(-R\left(u_{\beta\nu}\left(t\right)\right)\right)_+\right]^2 - \epsilon_a\right)_+ \quad \text{a.e.} \quad t \in (0,T), \tag{4.3}$$

$$\theta_\beta\left(0\right) = \beta_0. \tag{4.4}$$

We show that there exists a unique solution to the Cauchy problem (4.3)–(4.4) which satisfies $\theta_\beta \in W^{1,\infty}\left(0,T;L^\infty\left(\Gamma_3\right)\right) \cap \mathcal{Q}$. Indeed, consider the mapping $F_\beta : [0,T] \times L^\infty\left(\Gamma_3\right) \longrightarrow L^\infty\left(\Gamma_3\right)$ defined as

$$F_\beta\left(t, \theta_\beta\right) = -\left(\gamma_\nu \theta_\beta\left(t\right)\left[\left(-R\left(u_{\beta\nu}\left(t\right)\right)\right)_+\right]^2 - \epsilon_a\right)_+.$$

It follows from the properties of the truncation operator R, that F_β is Lipschitz continuous with respect to the second argument, uniformly in time. Moreover, for any $\theta_\beta \in L^\infty\left(\Gamma_3\right)$, the mapping : $t \mapsto F_\beta\left(t, \theta_\beta\right)$ belongs to $L^\infty\left(0,T;L^\infty\left(\Gamma_3\right)\right)$. Then, from Theorem 2.1 we deduce the existence of a unique function $\theta_\beta \in W^{1,\infty}\left(0,T;L^\infty\left(\Gamma_3\right)\right)$ which satisfies (4.3)–(4.4). The regularity $\theta_\beta \in \mathcal{Q}$, follows from (4.3)–(4.4) and assumption $0 \le \beta_0 \le 1$ a.e. on Γ_3. Indeed; equation (4.3) implies that for a.e. $x \in \Gamma_3$, the function $t \mapsto \theta_\beta\left(x,t\right)$ is decreasing and its derivative vanishes when $\gamma_\nu \theta_\beta\left(t\right)\left[\left(-R\left(u_{\beta\nu}\left(t\right)\right)\right)_+\right]^2 \le \epsilon_a$. Combining these properties with the inequality $0 \le \beta_0 \le 1$ we deduce that $0 \le \theta_\beta\left(t\right) \le 1$, for all $t \in [0,t]$, a.e. on Γ_3, which shows that $\theta_\beta \in \mathcal{Q}$.

(iii) In the third step we denote by u_β and θ_β the solution of the auxiliary problems defined in the previous steps, for every $\beta \in \mathcal{Q}$. Moreover, we define the operator $\Lambda : \mathcal{Q} \longrightarrow \mathcal{Q}$ by equality

$$\Lambda\beta = \theta_\beta. \tag{4.5}$$

We prove that the operator Λ has a unique fixed point β^*. To this end we suppose in what follows that β_i are two functions of $\mathcal{Q}$ and we denote by u_i, θ_i the functions obtained in steps **(i)** and **(ii)**, respectively, for $\beta = \beta_i$, $(i = 1, 2)$. Let $t \in [0, T]$. We use (4.2) and the properties of F, j and k to deduce that

$$|u_1(t) - u_2(t)|_V \leq c\,|\beta_1(t) - \beta_2(t)|_{L^2(\Gamma_3)} \tag{4.6}$$

which implies that

$$\int_0^t |u_1(s) - u_2(s)|_V\, ds \leq c \int_0^t |\beta_1(s) - \beta_2(s)|_{L^2(\Gamma_3)}\, ds. \tag{4.7}$$

Here and below c denotes a positive constant which does not depend on t and whose value may change from place to place.

On the other hand, it follows from (4.3) and (4.4) that

$$\theta_i(t) = \beta_0 - \int_0^t \left(\gamma_\nu \theta_i(s)\left[(-R(u_{i\nu}(s)))_+\right]^2 - \epsilon_a\right)_+ ds \quad i = 1, 2$$

and then

$$\begin{aligned} &|\theta_1(t) - \theta_2(t)|_{L^2(\Gamma_3)} \leq \\ &\quad c\int_0^t \left|\theta_1(s)\left[(-R(u_{1\nu}(s)))_+\right]^2 - \theta_2(s)\left[(-R(u_{2\nu}(s)))_+\right]^2\right|_{L^2(\Gamma_3)} ds. \end{aligned}$$

Using the definition (3.9) and writing $\theta_1 = \theta_1 - \theta_2 + \theta_2$, we get

$$\begin{aligned} |\theta_1(t) - \theta_2(t)|_{L^2(\Gamma_3)} \quad \leq \quad & c\int_0^t |\theta_1(s) - \theta_2(s)|_{L^2(\Gamma_3)}\, ds + \\ & + c\int_0^t |u_{1\nu}(s) - u_{2\nu}(s)|_{L^2(\Gamma_3)}\, ds \end{aligned}$$

By Gronwall's inequality and the Sobolev trace theorem, it follows that

$$|\theta_1(t) - \theta_2(t)|_{L^2(\Gamma_3)} \leq c\int_0^t |u_1(s) - u_2(s)|_V\, ds \tag{4.8}$$

and, substituting the definition (4.5) into (4.8), we get

$$|\Lambda\beta_1(t) - \Lambda\beta_2(t)|_{L^2(\Gamma_3)} \leq c\int_0^t |u_1(s) - u_2(s)|_V\, ds. \tag{4.9}$$

We now combine (4.7) and (4.9) to obtain

$$|\Lambda\beta_1(t) - \Lambda\beta_2(t)|_{L^2(\Gamma_3)} \leq c \int_0^t |\beta_1(s) - \beta_2(s)|_{L^2(\Gamma_3)}\, ds.$$

Reiterating this inequality p times yields

$$|\Lambda^p\beta_1 - \Lambda^p\beta_2|_{C(0,T;L^2(\Gamma_3))} \leq \frac{c^pT^p}{p!} |\beta_1 - \beta_2|_{C(0,T;L^2(\Gamma_3))} \quad \forall p \in \mathbb{N}. \quad (4.10)$$

The previous inequality shows that for p sufficiently large the operator Λ^p is contractive. Since $\mathcal{Q}$ is closed subset in the Banach subspace in $C\left(0, T; L^2(\Gamma_3)\right)$, it follows that Λ^p has a unique fixed point $\beta^* \in \mathcal{Q}$. Therefore, the operator Λ has a unique fixed point $\beta^* \in \mathcal{Q}$, which concludes the proof.

(iv) In the fourth step we prove the existence part in Theorem 4.1. Let $\beta^* \in \mathcal{Q}$ be the fixed point of Λ and u^* be the solution of equation (4.2) for $\beta = \beta^*$, i.e. $u^* = u_{\beta^*}$. Then, (4.6) implies that

$$|u^*(t_1) - u^*(t_2)|_V \leq c\,|\beta^*(t_1) - \beta^*(t_2)|_{L^2(\Gamma_3)} \quad \forall t_1, t_2 \in [0, T]. \quad (4.11)$$

Since $\beta^* = \theta_{\beta^*}$ it follows from step **(ii)** that $\beta^* \in W^{1,\infty}(0, T; L^\infty(\Gamma_3))$ and therefore (4.11) implies that $u^* \in W^{1,\infty}(0, T; V)$. From (4.2), (4.3) and (4.4) we conclude that (u^*, β^*) is a solution of the Problem **PV** whit regularity (4.1).

(v) In the fifth step we prove the uniqueness of the solution. It is a consequence of the uniqueness of the fixed point of the operator Λ and the uniqueness of the auxiliary problems studied in the steps **(i)** and **(ii)**. Indeed, let (u, β) be a solution of Problem **PV** which satisfies (4.1). Since $\beta \in \mathcal{Q}$, it follows from (3.19) that u is a solution to (4.2); on the other hand, step **(i)** implies that this problem has a unique solution, denoted u_β. Thus,

$$u = u_\beta \quad (4.12)$$

Letting $u = u_\beta$ in (3.20) and using the initial condition (3.21), we can see that β is a solution to problem (4.3)–(4.4). Therefore, since step **(ii)** implies that this last problem has a unique solution, denoted θ_β, we deduce that

$$\beta = \theta_\beta. \quad (4.13)$$

We now use (4.5) and (4.13) to see that $\Lambda\beta = \beta$, i.e. β is a fixed point of the operator Λ. It follows from step **(iii)** that

$$\beta = \beta^* \quad (4.14)$$

The uniqueness of the solution is now a consequence of (4.12) and (4.14). □

A triple (u, σ, β) which satisfies (3.1) and (3.19)–(3.21) is called a *weak solution* of the mechanical problem **P**. It follows from Theorem 4.1 we that the mechanical problem **P** has a unique weak solution. Note that the regularity of the weak solution is $\sigma \in W^{1,\infty}(0,T;\mathcal{H}_1)$. Indeed, taking $v = \varphi \in D^{\infty}(\Omega)$ in (3.19) and using (3.1), (3.15) we find that $Div\, \sigma(t) + f_0(t) = 0$, for all $t \in [0,T]$. This equality and (3.13) imply that $Div\sigma \in W^{1,\infty}(0,T;H)$, which in its turn implies $\sigma \in W^{1,\infty}(0,T;\mathcal{H}_1)$.

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