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Energy estimates for Love wave in a pre-stressed layered structure

Olivian Simionescu-Panait

Abstract - This paper continues the study of the problem of propagation of TH guided waves in pre-stressed anisotropic layered structures. One derives here the energy estimates for Love type wave, for different classes of anisotropy. We obtain and analyze the energy density and the energy flux distribution, the mean energy density and mean energy flux, resp. the total mean energy density and total mean energy flux.

Key words and phrases : Love wave, pre-stressed anisotropic layered structure, energy estimates.

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1. Introduction

Last period the problems analyzing the behavior of electroelastic materials subject to incremental fields superposed on initial mechanical and electric fields have attracted considerable attention, due their complexity and to multiple applications. Last decade we dealt with various problems in the field, such as progressive waves and attenuated waves propagation in piezoelectric crystals subject to an electromechanical bias, and the propagation of waveguides in monoclinic crystals subject to initial fields (see papers [7]-[21], or the chapter [6] for an overview of our results).

The present paper continues the analysis presented in paper [22], concerning the TH waves propagation in anisotropic layered structures subject to initial mechanical fields. One derives here the energy estimates for Love type wave, for different classes of anisotropy. We obtain and analyze the energy density and the energy flux distribution, the mean energy density and mean energy flux, resp. the total mean energy density and total mean energy flux. Our results generalize, for initial mechanical fields, classical results from seismology concerning Love waves propagation (see works [4] and [5]). Other results concerning the analysis of electromechanical problems may be found in papers [1]-[3], resp. in works [23]-[25].

2. Fundamental equations. Geometric hypotheses

As physical hypotheses, we assume the material to be an elastic dielectric, which is nonmagnetizable and conducts neither heat, nor electricity. Consequently, we shall use the quasi-electrostatic approximation of the equations of balance in electrodynamics of continua. Furthermore, we assume that the elastic dielectric is homogeneous, and that we apply on initial large homogeneous deformations and an initial large homogeneous electric field.

To describe this situation we use three different configurations : the *reference configuration* B_R in which at time $t = 0$ the body is undeformed and free of all fields; the *initial configuration* $\overset{\circ}{B}$ in which the body is deformed statically and carries the large initial fields; the *present (current)configuration* B_t obtained from $\overset{\circ}{B}$ by applying time dependent incremental deformations and fields. In what follows, all the fields related to the initial configuration $\overset{\circ}{B}$ will be denoted by a superposed "$\circ$".

In this case the *homogeneous field equations* take the following form:

$$\begin{gathered} \overset{\circ}{\rho}\, \ddot{\boldsymbol{u}} = \operatorname{div} \boldsymbol{\Sigma},\ \operatorname{div} \boldsymbol{\Delta} = 0 \\ \operatorname{rot} \boldsymbol{e} = 0 \ \Leftrightarrow\ \boldsymbol{e} = -\operatorname{grad} \varphi \end{gathered} \tag{2.1}$$

where $\overset{\circ}{\rho}$ is the mass density, $\boldsymbol{u}$ is the incremental displacement from $\overset{\circ}{B}$ to B_t, $\boldsymbol{\Sigma}$ is the incremental electromechanical nominal stress tensor, $\boldsymbol{\Delta}$ is the incremental electric displacement vector, $\boldsymbol{e}$ is the incremental electric field vector and φ is the incremental electric potential. All incremental fields involved into the above equations depend on the spatial variable $\boldsymbol{x}$ and on time t.

We suppose the following *incremental constitutive equations*:

$$\begin{gathered} \Sigma_{kl} = \overset{\circ}{\Omega}_{klmn}\, u_{m,n} + \overset{\circ}{\Lambda}_{mkl}\, \varphi_{,m} \\ \Delta_k = \overset{\circ}{\Lambda}_{kmn}\, u_{n,m} + \overset{\circ}{\epsilon}_{kl}\, e_l = \overset{\circ}{\Lambda}_{kmn}\, u_{n,m} - \overset{\circ}{\epsilon}_{kl}\, \varphi_{,l}. \end{gathered} \tag{2.2}$$

In these equations $\overset{\circ}{\Omega}_{klmn}$ are the components of the instantaneous elasticity tensor, $\overset{\circ}{\Lambda}_{kmn}$ are the components of the instantaneous coupling tensor and $\overset{\circ}{\epsilon}_{kl}$ are the components of the instantaneous dielectric tensor. The instantaneous coefficients can be expressed in terms of the classical moduli of the material and on the initial applied fields as follows:

$$\overset{\circ}{\Omega}_{klmn}=\overset{\circ}{\Omega}_{nmlk}= c_{klmn}+ \overset{\circ}{S}_{kn}\,\delta_{lm} - e_{kmn}\,\overset{\circ}{E}_l - e_{nkl}\,\overset{\circ}{E}_m - \eta_{kn}\,\overset{\circ}{E}_l\overset{\circ}{E}_m,$$
$$\overset{\circ}{\Lambda}_{mkl}= e_{mkl} + \eta_{mk}\,\overset{\circ}{E}_l, \overset{\circ}{\epsilon}_{kl}=\overset{\circ}{\epsilon}_{lk}= \delta_{kl} + \eta_{kl}, \tag{2.3}$$

where c_{klmn} are the components of the constant elasticity tensor, e_{kmn} are the components of the constant piezoelectric tensor, ϵ_{kl} are the components of the constant dielectric tensor, $\overset{\circ}{E}_i$ are the components of the initial applied electric field and $\overset{\circ}{S}_{kn}$ are the components of the initial applied symmetric (Cauchy) stress tensor.

From the previous field and constitutive equations we obtain the following *fundamental system of equations*:

$$\overset{\circ}{\rho}\,\ddot{u}_l = \overset{\circ}{\Omega}_{klmn}\,u_{m,nk}+ \overset{\circ}{\Lambda}_{mkl}\,\varphi_{,mk},\ \overset{\circ}{\Lambda}_{kmn}\,u_{n,mk}- \overset{\circ}{\epsilon}_{kn}\,\varphi_{,nk} = 0,\ l = \overline{1,3}. \tag{2.4}$$

In what follows we shall describe the *geometric hypotheses* for our problem. The material is assumed to be semi-infinite, occupying the region $x_2 > 0$, and the waves are supposed to propagate along x_1 axis. The plane x_1x_2 containing the surface normal and the propagation direction is called *sagittal plane*. Furthermore, we suppose that the guide of waves has the properties invariant with time t and with x_1 variable. In these conditions, if the material behaves linearly and without attenuation, the *normal modes* are supposed to have the form:

$$u_j(\boldsymbol{x}, t) = u_j^0(x_2, x_3)\exp[\mathrm{i}(\omega t - kx_1)], \quad j = \overline{1,4}. \tag{2.5}$$

Here u_1, u_2, u_3 are the mechanical displacements, while u_4 stands for the electric potential φ. In the previous relations k represents the *wave number*, ω defines the *frequency* of the wave and $\mathrm{i}^2 = -1$ is the complex unit. Using these hypotheses, the equations (2.4) become:

$$\overset{\circ}{\Omega}_{klmn}\,u_{m,nk}+ \overset{\circ}{\Lambda}_{mkl}\,\varphi_{,mk} = -\,\overset{\circ}{\rho}\,\omega^2 u_l,\ \overset{\circ}{\Lambda}_{kmn}\,u_{n,mk} =\overset{\circ}{\epsilon}_{kn}\,\varphi_{,nk},\ l = \overline{1,3}. \tag{2.6}$$

We define the non-dimensional variable $X_2 = kx_2$ and we neglect the effects of diffraction in x_3 direction, so that $\partial/\partial x_3 = 0$. From the other hypotheses it yields the derivation rules $\partial/\partial x_1 = -\mathrm{i}k$ and $\partial/\partial x_2 = k\partial/\partial X_2$. Finally, we introduce the *phase velocity* of the guided wave as $V = \omega/k$.

3. Coupling conditions for waveguide propagation in anisotropic solids

To analyze the coupling of plane waveguide, using the previous hypotheses, we introduce the *differential operators* with complex coefficients, as follows:

$$\begin{gathered}\overset{\circ}{\Gamma}_{il}=\overset{\circ}{\Omega}_{1il1}-\overset{\circ}{\Omega}_{2il2}\,\frac{\partial^2}{\partial X_2^2}+\mathrm{i}(\overset{\circ}{\Omega}_{1il2}+\overset{\circ}{\Omega}_{1li2})\frac{\partial}{\partial X_2},\\ \overset{\circ}{\gamma}_{l}=\overset{\circ}{\Lambda}_{11l}-\overset{\circ}{\Lambda}_{22l}\,\frac{\partial^2}{\partial X_2^2}+\mathrm{i}(\overset{\circ}{\Lambda}_{12l}+\overset{\circ}{\Lambda}_{21l})\frac{\partial}{\partial X_2},\\ \overset{\circ}{\epsilon}=\overset{\circ}{\epsilon}_{11}-\overset{\circ}{\epsilon}_{22}\,\frac{\partial^2}{\partial X_2^2}+2\mathrm{i}\,\overset{\circ}{\epsilon}_{12}\,\frac{\partial}{\partial X_2}.\end{gathered}\tag{3.1}$$

In these conditions, after a lengthy, but elementary calculus, we obtain that the differential system (2.6) has the following form:

$$\begin{pmatrix}\overset{\circ}{\Gamma}_{11}-\overset{\circ}{\rho}\,V^2 & \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{13} & \overset{\circ}{\gamma}_{1}\\ \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{22}-\overset{\circ}{\rho}\,V^2 & \overset{\circ}{\Gamma}_{23} & \overset{\circ}{\gamma}_{2}\\ \overset{\circ}{\Gamma}_{13} & \overset{\circ}{\Gamma}_{23} & \overset{\circ}{\Gamma}_{33}-\overset{\circ}{\rho}\,V^2 & \overset{\circ}{\gamma}_{3}\\ \overset{\circ}{\gamma}_{1} & \overset{\circ}{\gamma}_{2} & \overset{\circ}{\gamma}_{3} & -\overset{\circ}{\epsilon}\end{pmatrix}\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix}=0.\tag{3.2}$$

Here the coefficients are defined by relations (3.1). The system (3.2) is a homogeneous differential system of four equations with four unknowns, i.e. the components of the mechanical displacement and the electric potential, having as coefficients complex differential operators in non-dimensional variable X_2. It generalizes the similar system from the case without initial fields, derived in [5].

In what follows we shall analyze the coupling conditions of the guided plane wave propagation in two particular cases.

3.1. Sagittal plane normal to a direct axis of order two

In this case, we suppose that *the sagittal plane x_1x_2 is normal to a dyad axis* (x_3 in our case). Thus, the material belongs to the class 2 of the monoclinic system ($A_2 \,||\, x_3$). So, we derive the following result concerning the decomposition of the fundamental system (3.2).

If the axis x_3 is a direct dyad axis and if $\overset{\circ}{E}_1=\overset{\circ}{E}_2=0$, the system (3.2) *reduces to two independent subsystems*, as follows:

a) The first subsystem:

$$\begin{pmatrix}\overset{\circ}{\Gamma}_{11}-\overset{\circ}{\rho}\,V^2 & \overset{\circ}{\Gamma}_{12}\\ \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{22}-\overset{\circ}{\rho}\,V^2\end{pmatrix}\begin{pmatrix}u_1\\u_2\end{pmatrix}=0\tag{3.3}$$

defines a non-piezoelectric guided wave, polarized in the sagittal plane x_1x_2, which depends on the initial stress field, only. We shall denote it by $\overset{\circ}{P}_2$.

b) The second subsystem:

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{33} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\gamma}_3 \\ \overset{\circ}{\gamma}_3 & -\overset{\circ}{\epsilon} \end{pmatrix} \begin{pmatrix} u_3 \\ u_4 \end{pmatrix} = 0 \tag{3.4}$$

has as solution a transverse-horizontal wave, with polarization after the axis x_3, which is piezoelectric and electrostrictive active, and depends on the initial mechanical and electrical fields. It is denoted by $\overline{\overset{\circ}{TH}}$ and generalizes the famous Bleustein-Gulyaev wave (see [5], to compare).

3.2. Sagittal plane parallel to a mirror plane

We suppose now that *the sagittal plane* x_1x_2 *is normal to an inverse dyad axis* (x_3 in our case) or, equivalently, that *the sagittal plane is parallel to a mirror plane* M. It follows that the material belongs to the class m of the *monoclinic system* ($M \perp x_3$).

Thus, *if the axis* x_3 *is an inverse dyad axis and if* $\overset{\circ}{E}_3 = 0$*, the fundamental system* (3.2) *splits into two parts*, as follows.

a) The first subsystem has the form:

$$\begin{pmatrix} \overset{\circ}{\Gamma}_{11} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\gamma}_1 \\ \overset{\circ}{\Gamma}_{12} & \overset{\circ}{\Gamma}_{22} - \overset{\circ}{\rho}\, V^2 & \overset{\circ}{\gamma}_2 \\ \overset{\circ}{\gamma}_1 & \overset{\circ}{\gamma}_2 & -\overset{\circ}{\epsilon} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_4 \end{pmatrix} = 0. \tag{3.5}$$

It has as solution a guided wave with sagittal plane polarization, associated with the electric field (*via* the electric potential $u_4 = \varphi$), providing piezoelectric and electrostrictive effects, and depending on the initial stress and electric fields. It is denoted by $\overline{\overset{\circ}{P}}_2$. The electric field, associated with this wave, is contained in the sagittal plane, since $E_3 = \partial\varphi/\partial x_3 = 0$. This fact is consistent with the hypothesis $\overset{\circ}{E}_3 = 0$.

b) The second subsystem reduces to a single equation, as follows:

$$(\overset{\circ}{\Gamma}_{33} - \overset{\circ}{\rho}\, V^2)u_3 = 0. \tag{3.6}$$

Its root corresponds to a transverse-horizontal wave, non-piezoelectric, and influenced by the initial stress field, only. It is called $\overset{\circ}{TH}$ wave.

In this equation:

$$\overset{\circ}{\Gamma}_{33} = c_{55} + \overset{\circ}{S}_{11} + 2\mathrm{i}(c_{45} + \overset{\circ}{S}_{12})\frac{\partial}{\partial X_2} - (c_{44} + \overset{\circ}{S}_{22})\frac{\partial^2}{\partial X_2^2}. \tag{3.7}$$

The corresponding mechanical boundary condition has the form

$$\Sigma_{23} = k[(-\mathrm{i})(c_{45}+ \overset{\circ}{S}_{12}) + (c_{44}+ \overset{\circ}{S}_{22})\frac{\partial}{\partial X_2}]u_3, \tag{3.8}$$

on the plane $x_2 = 0$.

4. Love wave propagation

Now, we focus on the study of $\overset{\circ}{T}H$ wave propagation. Substituting $X_2 = kx_2$ and $V = \omega/k$ the previous differential equation has the form, denoting by $u_3^0 = u(x_2)$:

$$(c_{44}+ \overset{\circ}{S}_{22})\frac{\mathrm{d}^2u}{\mathrm{d}x_2^2} - 2k\mathrm{i}(c_{45}+ \overset{\circ}{S}_{12})\frac{\mathrm{d}u}{\mathrm{d}x_2} + [\overset{\circ}{\rho}\,\omega^2 - k^2(c_{55}+ \overset{\circ}{S}_{11})]u = 0. \tag{4.1}$$

In order to simplify the resolution of equation (4.1), we suppose that the medium is *isotropic in relation with TH wave*. This is the case when x_3 is along a tetrad or hexad symmetry axis, or when the whole medium is isotropic. So that, in this particular case $c_{45} = 0$ and $c_{55} = c_{44}$.

Consequently, the equation (4.1) becomes:

$$(1 + \frac{\overset{\circ}{S}_{22}}{c_{44}})u'' - 2k\mathrm{i}\frac{\overset{\circ}{S}_{12}}{c_{44}}u' + [\frac{\omega^2}{v_T^2} - k^2(1 + \frac{\overset{\circ}{S}_{11}}{c_{44}})]u = 0, \tag{4.2}$$

where $v_T = \sqrt{\dfrac{c_{44}}{\overset{\circ}{\rho}}}$ is the TH wave velocity in the case without initial fields. This equation is related to the boundary condition:

$$(-k\mathrm{i})\,\overset{\circ}{S}_{12}\,u + (c_{44}+ \overset{\circ}{S}_{22})u' = \Sigma_{23},\ \text{on } x_2 = 0. \tag{4.3}$$

The equation (4.2), with the boundary condition (4.3), is to be solved for the following particular problem.

4.1. Layer on a substrate. Love type wave

We suppose an elastic layer ($-h < x_2 < 0$) bonded to an elastic substrate ($x_2 > 0$), which are both isotropic in relation with TH wave. Moreover, we suppose that $\overset{\circ}{S}_{12}= 0$.

In the substrate the displacement must vanish for $x_2 \to \infty$, so the solution of the equation (4.2) has the form:

$$u(x_2) = u_0\exp(-k\chi x_2), \quad \mathrm{Re}[k\chi] > 0, \quad x_2 > 0. \tag{4.4}$$

The corresponding characteristic equation is:

$$k^2\chi^2(1+\frac{\overset{\circ}{S}_{22}}{c_{44}})+\frac{\omega^2}{v_T^2}-k^2(1+\frac{\overset{\circ}{S}_{11}}{c_{44}})=0, \tag{4.5}$$

which implies

$$V=\frac{\omega}{k}<v_T\sqrt{1+\frac{\overset{\circ}{S}_{11}}{c_{44}}} \tag{4.6}$$

under the hypotheses $|\overset{\circ}{S}_{11}|/c_{44}<1$ and $|\overset{\circ}{S}_{22}|/c_{44}<1$.

Moreover, we obtain:

$$\chi=\sqrt{\frac{1+\dfrac{\overset{\circ}{S}_{11}}{c_{44}}-\dfrac{V^2}{v_T^2}}{1+\dfrac{\overset{\circ}{S}_{22}}{c_{44}}}} \tag{4.7}$$

For the layer, we distinguish the variables using a hat. We seek the solution of the equation (4.2) in a sinusoidal form:

$$u(x_2)=\hat{u}_0\cos k\hat{\chi}(x_2+h),\quad -h<x_2<0. \tag{4.8}$$

For this solution the mechanical stress $\Sigma_{32}=(\hat{c}_{44}+\overset{\circ}{S}_{22})u'(x_2)=0$, at the free surface $x_2=-h$, where the displacement is maximal.

From the characteristic equation we obtain:

$$(k\hat{\chi})^2=\frac{\dfrac{\omega^2}{\hat{v}_T^2}-k^2(1+\dfrac{\overset{\circ}{S}_{11}}{\hat{c}_{44}})}{1+\dfrac{\overset{\circ}{S}_{22}}{\hat{c}_{44}}}>0. \tag{4.9}$$

This implies that

$$V=\omega/k>\hat{v}_T\sqrt{1+\frac{\overset{\circ}{S}_{11}}{\hat{c}_{44}}}, \tag{4.10}$$

where $\hat{v}_T=\sqrt{\dfrac{\hat{c}_{44}}{\hat{\overset{\circ}{\rho}}}}$.

The inequalities (4.6), (4.10) show us that the Love wave velocity V satisfies the following fundamental inequalities:

$$\hat{v}_T\sqrt{1+\frac{\overset{\circ}{S}_{11}}{\hat{c}_{44}}}<V<v_T\sqrt{1+\frac{\overset{\circ}{S}_{11}}{c_{44}}}. \tag{4.11}$$

Consequently, the velocity of the TH wave in the substrate is greater than the velocity of the TH wave in the layer $v_T > \hat{v}_T$. Moreover, we obtain:

$$\hat{\chi} = \sqrt{\frac{\frac{V^2}{\hat{v}_T^2} - (1 + \frac{\overset{\circ}{S}_{11}}{\hat{c}_{44}})}{1 + \frac{\overset{\circ}{S}_{22}}{\hat{c}_{44}}}}. \tag{4.12}$$

At the interface $x_2 = 0$ between layer and substrate we suppose the continuity of displacement, so we obtain

$$\hat{u}_0 = \frac{u_0}{\cos(k\hat{\chi}h)},$$

and the continuity of stress components, which yields the relation

$$-(c_{44} + \overset{\circ}{S}_{22})u_0\chi = -(\hat{c}_{44} + \overset{\circ}{S}_{22})\hat{u}_0\hat{\chi}\sin(k\hat{\chi}h).$$

Hence, we obtain the *dispersion relation* in the form:

$$\tan(k\hat{\chi}h) = \frac{(c_{44} + \overset{\circ}{S}_{22})\chi}{(\hat{c}_{44} + \overset{\circ}{S}_{22})\hat{\chi}}, \tag{4.13}$$

which is influenced by the initial mechanical fields, via the previous forms of χ and $\hat{\chi}$. This dispersion equation has an infinite number of solutions given by:

$$(kh)_n = \frac{1}{\hat{\chi}}\tan^{-1}[\frac{(c_{44} + \overset{\circ}{S}_{22})\chi}{(\hat{c}_{44} + \overset{\circ}{S}_{22})\hat{\chi}}] + n\frac{\pi}{\hat{\chi}}, \quad n = 0, 1, 2, ... \tag{4.14}$$

5. Energy estimates for Love wave

In this paragraph we obtain and analyze the energy estimates for a Love wave propagating in a pre-stressed layered structure. We derive here the energy density and energy flux vector, the mean energy density and mean energy flux vector, resp. the total mean energy density and total mean energy flux vector.

5.1. Energy density and energy flux vector

Now, we compute the energy density and the energy flux vector for the Love wave analyzed in the previous paragraph. Here the energy density is defined by

$$e = \frac{1}{2}(\overset{\circ}{\rho}\ \dot{u}_l\dot{u}_l + \overset{\circ}{\Omega}_{klmn}\ u_{l,k}u_{m,n}), \tag{5.1}$$

resp. the energy flux vector has the components

$$\Phi_k = -\Sigma_{kl}\dot{u}_l. \tag{5.2}$$

In the substrate $x_2 > 0$ we have the displacement of the wave in the form (4.4), while in the layer $-h < x_2 < 0$ the wave motion is given by (4.8).

Consequently, we obtain **for the substrate** the energy density in the form

$$\begin{gathered} e = \tfrac{1}{2}k^2 u_0^2 \exp(-2\chi k x_2)[(\overset{\circ}{\rho}\, V^2 + c_{44} + \overset{\circ}{S}_{11})\sin^2 k(x_1 - Vt) + \\ (c_{44} + \overset{\circ}{S}_{22})\chi^2 \cos^2 k(x_1 - Vt)], \end{gathered} \tag{5.3}$$

resp. the energy flux vector components are

$$\begin{gathered} \Phi_1 = (c_{44} + \overset{\circ}{S}_{11}) V k^2 u_0^2 \exp(-2\chi k x_2)\sin^2 k(x_1 - Vt), \\ \Phi_2 = (c_{44} + \overset{\circ}{S}_{22}) V \chi k^2 u_0^2 \exp(-2\chi k x_2) \\ \cdot \sin k(x_1 - Vt)\cos k(x_1 - Vt), \quad \Phi_3 = 0. \end{gathered} \tag{5.4}$$

After an elementary computation, we find that the energy density and the energy flux vector satisfy in the substrate the balance equation

$$\frac{\mathrm{d}e}{\mathrm{d}t} + \frac{\partial \Phi_k}{\partial x_k} = 0. \tag{5.5}$$

As regards the energy density distribution **into the layer**, we obtain

$$\begin{gathered} \hat{e} = \tfrac{1}{2}k^2 \hat{u}_0^2[(\hat{\overset{\circ}{\rho}} V^2 + \hat{c}_{44} + \overset{\circ}{S}_{11})\cos^2 k\hat{\chi}(x_2 + h)\sin^2 k(x_1 - Vt) + \\ (\hat{c}_{44} + \overset{\circ}{S}_{22})\hat{\chi}^2 \sin^2 k\hat{\chi}(x_2 + h)\cos^2 k(x_1 - Vt)]. \end{gathered} \tag{5.6}$$

Moreover, we derive the energy flux vector components, into the layer, having the form

$$\begin{gathered} \hat{\Phi}_1 = (\hat{c}_{44} + \overset{\circ}{S}_{11}) V k^2 \hat{u}_0^2 \cos^2 k\hat{\chi}(x_2 + h)\sin^2 k(x_1 - Vt), \\ \hat{\Phi}_2 = (\hat{c}_{44} + \overset{\circ}{S}_{22}) V \hat{\chi} k^2 \hat{u}_0^2 \sin k\hat{\chi}(x_2 + h)\cos k\hat{\chi}(x_2 + h) \\ \cdot \sin k(x_1 - Vt)\cos k(x_1 - Vt), \quad \hat{\Phi}_3 = 0. \end{gathered} \tag{5.7}$$

It is easy to see that the energy density and the energy flux vector satisfy, in the layer, the balance equation

$$\frac{\mathrm{d}\hat{e}}{\mathrm{d}t} + \frac{\partial \hat{\Phi}_k}{\partial x_k} = 0. \tag{5.8}$$

5.2. Mean energy density and mean energy flux vector

Mean energy density and mean energy flux vector are defined to be the averages over a period of time, for a fixed position

$$\begin{aligned} &< e > (\mathbf{x}) = (\omega/2\pi)\int_0^{2\pi/\omega} e(\mathbf{x},t)\mathrm{d}t, \\ &< \Phi_k > (\mathbf{x}) = (\omega/2\pi)\int_0^{2\pi/\omega} \Phi_k(\mathbf{x},t)\mathrm{d}t. \end{aligned} \tag{5.9}$$

Using (5.3) and (5.4), we obtain the mean energy density and the mean energy flux **into the substrate** as

$$< e >= \frac{1}{2}(c_{44}+ \overset{\circ}{S}_{11})k^2 u_0^2 \exp(-2\chi k x_2), \tag{5.10}$$

resp.

$$\begin{aligned} &< \Phi_1 >= \tfrac{1}{2}(c_{44}+ \overset{\circ}{S}_{11})k^2 u_0^2 V \exp(-2\chi k x_2), \\ &< \Phi_2 >=< \Phi_3 >= 0. \end{aligned} \tag{5.11}$$

From (5.6) and (5.7) we derive the mean energy density and the mean energy flux vector **into the layer** in the form

$$< \hat{e} >= \frac{1}{4}k^2 \hat{u}_0^2[\hat{\overset{\circ}{\rho}}V^2 + (\hat{c}_{44}+ \overset{\circ}{S}_{11})\cos 2k\hat{\chi}(x_2 + h)] \tag{5.12}$$

and

$$\begin{aligned} &< \hat{\Phi}_1 >= \tfrac{1}{2}(\hat{c}_{44}+ \overset{\circ}{S}_{11})k^2 \hat{u}_0^2 V \cos^2 k\hat{\chi}(x_2 + h), \\ &< \hat{\Phi}_2 >=< \hat{\Phi}_3 >= 0. \end{aligned} \tag{5.13}$$

We can observe that the mean energy flux vectors in the layer and in the substrate are parallel to the interface $x_2 = 0$, which coincides with the propagation direction Ox_1.

We find that at the interface $x_2 = 0$ we have a jump condition for the mean energy flux from the substrate vs. the mean energy flux from the layer

$$\frac{< \hat{\Phi}_1 >_0}{< \Phi_1 >_0} = \frac{\hat{c}_{44}+ \overset{\circ}{S}_{11}}{c_{44}+ \overset{\circ}{S}_{11}}, \tag{5.14}$$

respectively, for the mean energy densities

$$\frac{< \hat{e} >_0}{< e >_0} = \frac{\hat{\overset{\circ}{\rho}}V^2 + (\hat{c}_{44}+ \overset{\circ}{S}_{11})\cos 2k\hat{\chi}h}{(c_{44}+ \overset{\circ}{S}_{11})(1 + \cos 2k\hat{\chi}h)}. \tag{5.15}$$

On the other hand, if we define *the energy flux velocity* as the ratio between the mean energy flux vector and the mean energy density, we obtain its components, for the layer, in the form

$$\hat{g}_1 = \frac{V(\hat{c}_{44}+ \overset{\circ}{S}_{11})[1+\cos 2k\hat{\chi}(x_2+h)]}{\overset{\circ}{\hat{\rho}}V^2 + (\hat{c}_{44}+ \overset{\circ}{S}_{11})\cos 2k\hat{\chi}(x_2+h)},\ \hat{g}_2 = \hat{g}_3 = 0, \tag{5.16}$$

resp. in the substrate

$$g_1 = V,\ g_2 = g_3 = 0. \tag{5.17}$$

We easily observe that the energy flux velocity depends on the depth x_2 in the layer, while the energy flux velocity is constant in the substrate, and equals the Love wave velocity.

5.3. Total mean energy density and total mean energy flux vector

We define *the total mean energy density*, resp. *the total mean energy flux vector* as

$$< \hat{e} >_T = \int_{-h}^{0} < \hat{e} > (\mathbf{x})\mathrm{d}x_2, \quad < \hat{\Phi}_k >_T = \int_{-h}^{0} < \hat{\Phi}_k > (\mathbf{x})\mathrm{d}x_2, \tag{5.18}$$

in the layer, resp.

$$< e >_T = \int_{0}^{\infty} < e > (\mathbf{x})\mathrm{d}x_2, \quad < \Phi_k >_T = \int_{0}^{\infty} < \Phi_k > (\mathbf{x})\mathrm{d}x_2, \tag{5.19}$$

in the substrate.

For a Love wave motion, **into the layer** we obtain

$$< \hat{e} >_T = \frac{1}{4}k^2\hat{u}_0^2[\overset{\circ}{\hat{\rho}}V^2h + (\hat{c}_{44}+ \overset{\circ}{S}_{11})\frac{\sin 2k\hat{\chi}h}{2k\hat{\chi}}] \tag{5.20}$$

and

$$\begin{gathered} < \hat{\Phi}_1 >_T = \frac{1}{4}k^2\hat{u}_0^2V(\hat{c}_{44}+ \overset{\circ}{S}_{11})[h + \frac{\sin 2k\hat{\chi}h}{2k\hat{\chi}}], \\ < \hat{\Phi}_2 >_T = < \hat{\Phi}_3 >_T = 0. \end{gathered} \tag{5.21}$$

In the substrate, we find

$$\begin{gathered} < e >_T = \frac{1}{4\chi}ku_0^2(c_{44}+ \overset{\circ}{S}_{11}), \\ < \Phi_1 >_T = \frac{1}{4\chi}ku_0^2V(c_{44}+ \overset{\circ}{S}_{11}), \\ < \Phi_2 >_T = < \Phi_3 >_T = 0. \end{gathered} \tag{5.22}$$

The previous expressions give the dependency of the total mean energy density and total mean energy flux vector on the initial fields and on the dimensionless parameter kh, which is defined by the ratio between the layer depth and the wavelength.

6. Conclusions

In this paper we obtained the energy estimates for a Love type wave propagating in pre-stressed anisotropic layered structures. We derived and analyzed here the energy density and the energy flux distribution, the mean energy density and mean energy flux, resp. the total mean energy density and total mean energy flux for this problem.

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Olivian Simionescu-Panait
University of Bucharest, Department of Mathematics
Str. Academiei 14, 010014 Bucharest, Romania
E-mail: osimion@fmi.unibuc.ro