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Finite-dimensional attractors for thin film models

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Abstract - Our aim in this paper is to prove the existence of finite-dimensional attractors for a class of equations which contains some thin film models.

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1. Introduction

In [12] and [13], the authors considered the following equations:

$$\frac{\partial u}{\partial t} + \epsilon \Delta^2 u + \mathrm{div}(\frac{\nabla u}{1 + |\nabla u|^2}) = 0 \tag{1.1}$$

and

$$\frac{\partial u}{\partial t} + \epsilon \Delta^2 u - \mathrm{div}(|\nabla u|^2 \nabla u) + \Delta u = 0 \tag{1.2}$$

in order to model epitaxial growth of thin films. Here, $\epsilon > 0$ is a small parameter and, in two space dimensions, u is a scaled height of the thin film. Furthermore, the fourth-order term accounts for diffusion, while the second-order ones account for the so-called Ehrlich-Schowoebel effect: adatoms (i.e., atoms which are absorbed by the surface, but have not yet become part of the crystal) diffuse on a terrace and likely hit a terrace boundary; then, in order to stick to the boundary from an upper terrace, they must overcome a higher energy barrier, the Ehrlich-Schowoebel barrier (see [12] and [13] for more details and further references).

We can also note that, typically, in an epitaxial growth which starts with a flat substrate, one observes the occurrence of surface morphological instabilities as the film thickness reaches a critical value. This can be seen as some kind of spinodal decomposition. This is then followed by some nucleation process, in which nuclei (which appear on the film surface) evolve into mounds whose structure coarsens (see [13] and the references therein for more details). This bears some resemblance with the spinodal decomposition and coarsening process in binary alloys described by the Cahn-Hilliard equation (see, e.g., [2] and [17]).

These two equations are associated with the energy functionals

$$E_1(u) = \int_\Omega (-\frac{1}{2}\ln(1+|\nabla u|^2) + \frac{\epsilon}{2}|\Delta u|^2)\,dx \tag{1.3}$$

and

$$E_2(u) = \int_\Omega (\frac{1}{4}(|\nabla u|^2 - 1)^2 + \frac{\epsilon}{2}|\Delta u|^2)\,dx, \tag{1.4}$$

respectively, where Ω is the spatial domain. In particular, the first term in $E_2(u)$ selects the slope of the film surface, hence the denomination growth equation with slope selection for (1.2) and, accordingly, growth equation without slope selection for (1.1).

We can also note that, assuming that $|\nabla u|$ is small with respect to 1 and writing, at first approximation,

$$\frac{1}{1+|\nabla u|^2} \approx 1 - |\nabla u|^2$$

in (1.1), we recover (1.2) (see also Remark 2.2 below for further approximations of (1.1)).

Furthermore, we can rewrite (1.1) and (1.2) in the form

$$\frac{\partial u}{\partial t} + \epsilon\Delta^2 u - \operatorname{div}(\varphi(|\nabla u|^2)\nabla u) = 0, \tag{1.5}$$

where $\varphi(s) = -\frac{1}{1+s}$ and $\varphi(s) = s-1$, $s \geq 0$, respectively.

In [12], the authors proved the existence and uniqueness of weak solutions to (1.1) and (1.2), for regular initial data and periodic boundary conditions. In what follows, we will consider Neumann boundary conditions, but all results can easily be adapted to Dirichlet and periodic boundary conditions. Equation (1.1) was further studied in [7], [8], [9] and [10]; in particular, in [10], the authors proved the existence of finite-dimensional attractors and the convergence of single trajectories to steady states. We also refer the interested reader to [3], [4] and [20] for the numerical analysis of the two models.

An equation of the form (1.5) (containing (1.2), but not (1.1)) was considered in [11]. There, the authors studied the well-posedness and the regularity of solutions, as well as the structure of ω-limit sets and stationary solutions.

In this paper, we are interested in the study of the asymptotic behavior of the more general equation (1.5) which, as already mentioned, contains the two thin film models. More precisely, we prove the existence of the global attractor which is the smallest compact set which is invariant by the flow and attracts all bounded sets of initial data as time goes to infinity. Then, under some restrictions on the growth of the nonlinear term (which are satisfied by the thin film models), we prove the existence of an exponential attractor which is a compact and positively invariant set which contains the global attractor, has, by definition, finite fractal dimension and attracts exponentially fast the bounded sets of initial data.

2. Setting of the problem

We consider the following initial and boundary value problem (for simplicity, we take $\epsilon = 1$ in (1.5)):

$$\frac{\partial u}{\partial t} + \Delta^2 u - \operatorname{div}(\varphi(|\nabla u|^2)\nabla u) = 0, \tag{2.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{2.2}$$

$$u|_{t=0} = u_0, \tag{2.3}$$

in a bounded and regular domain of $\mathbb{R}^n$, $n = 1$, 2 or 3, with boundary Γ.

As far as the nonlinear term φ is concerned, we make the following assumptions:

$$\varphi \text{ is of class } \mathcal{C}^1, \tag{2.4}$$

$$\phi'(x)h.h \geq -c_0|h|^2,\ c_0 \geq 0,\ x,\ h \in \mathbb{R}^n, \tag{2.5}$$

where

$$\phi(x) = \varphi(|x|^2)x,\ x \in \mathbb{R}^n,$$

and $\phi'(x)h = \varphi(|x|^2)h + 2\varphi'(|x|^2)(x \cdot h)x,\ x,\ h \in \mathbb{R}^n$,

$$c_1 s^p - c_2 \leq \varphi(s)s \leq c_3(s^p + 1),\ c_1,\ c_3 > 0,\ c_2 \geq 0,\ s \geq 0, \tag{2.6}$$

$$c_4 s^p - c_5 s - c_6 \leq \psi(s) \leq c_7(s^p + 1),\ c_4,\ c_7 > 0,\ c_5,\ c_6 \geq 0,\ s \geq 0, \tag{2.7}$$

where

$$\psi(s) = \int_0^s \varphi(\tau)\, d\tau,\ s \geq 0.$$

Here, $p \geq 0$ is given. Possible restrictions on p will be given when needed.

Remark 2.1. a) In particular, the functions $\varphi_1(s) = -\frac{1}{1+s}$ (which corresponds to the thin film model without slope selection) and $\varphi_2(s) = s - 1$ (which corresponds to the thin film model with slope selection) satisfy the above assumptions, for $p = 0$ and $p = 2$, respectively. In concrete situations, the only difficulty is to prove that (2.5) holds. This is however straightforward for the above examples. Indeed, we have

$$\phi_1'(x)h.h = -\frac{|h|^2}{1+|x|^2} + \frac{2(x \cdot h)^2}{(1+|x|^2)^2} \geq -|h|^2$$

and

$$\phi_2'(x)h.h = (|x|^2 - 1)|h|^2 + 2(x \cdot h)^2 \geq -|h|^2.$$

b) Assumption (2.5), which allows to prove the uniqueness of solutions, can be replaced by the weaker assumption

$$(\phi(x_1) - \phi(x_2)) \cdot (x_1 - x_2) \geq -c_0|x_1 - x_2|^2,\ c_0 \geq 0,\ x_1,\ x_2 \in \mathbb{R}^n. \quad (2.8)$$

This assumption is again satisfied for both thin film models. We will however need the stronger assumption (2.5) to obtain further regularity on $\frac{\partial u}{\partial t}$ in Remark 4.1 below.
c) Assumption (2.6) can also be weakened as follows:

$$c_1 s^p - c_2 s - c_3 \leq \varphi(s)s \leq c_4(s^p + 1),\ c_1,\ c_4 > 0,\ c_2,\ c_3 \geq 0,\ s \geq 0. \quad (2.9)$$

However, this does not allow to prove the dissipativity of the associated dynamical system when $p \leq 1$ and $c_2 > 0$.

Remark 2.2. Assuming again that $|\nabla u| \ll 1$ in (1.1) and writing, at first approximation,

$$\frac{1}{1+|\nabla u|^2} \approx \theta_k(|\nabla u|^2),\ \theta_k(s) = \sum_{i=0}^{2k-1} (-1)^i s^i,\ k \in \mathbb{N},$$

we can define a whole family of equations approximating (1.1) and generalizing (1.2). For instance, when $k = 2$, we obatin the following thin film model (for $\epsilon = 1$):

$$\frac{\partial u}{\partial t} + \Delta^2 u - \operatorname{div}(|\nabla u|^6 \nabla u) + \operatorname{div}(|\nabla u|^4 \nabla u) - \operatorname{div}(|\nabla u|^2 \nabla u) + \Delta u = 0.$$

Here, the function $\varphi_k = -\theta_k$ satisfies (2.4)-(2.7), for $p = 2k$. Again, the only difficulty is to prove that (2.5) holds and we have

$$\phi_k'(x)h.h = -|h|^2 + \sum_{i=1}^{2k-1}(-1)^{i+1}[|x|^{2i}|h|^2 + 2|x|^{2i-2}(x\cdot h)^2]$$

$$\geq -|h|^2 + c|x|^{4k-2}|h|^2,\ c>0,$$

hence (2.5).

We denote by $((\cdot,\cdot))$ the usual L^2-scalar product, with associated norm $\|\cdot\|$, and we denote by $\|\cdot\|_X$ the norm in the Banach space X.

Setting

$$\langle\cdot\rangle = \frac{1}{\mathrm{Vol}(\Omega)}\int_\Omega \cdot\, dx,$$

we note that

$$v \mapsto (\|\nabla v\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

$$v \mapsto (\|\Delta v\|^2 + \langle v\rangle^2)^{\frac{1}{2}},$$

$$v \mapsto (\|\nabla\Delta v\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

and

$$v \mapsto (\|\Delta^2 v\|^2 + \langle v\rangle^2)^{\frac{1}{2}}$$

are norms on $H^i(\Omega)$, $i=1$, 2, 3 and 4, respectively, which are equivalent to the usual ones. Furthermore, $v \mapsto (\|\nabla v\|^{2p}_{L^{2p}(\Omega)} + \langle v\rangle^{2p})^{\frac{1}{2p}}$ is a norm on $W^{1,2p}(\Omega)$ which is equivalent to the usual one.

Remark 2.3. Of course, here and in what follows, the $W^{1,2p}$-regularity (and also the L^{2p}-one) only makes sense when $p \geq \frac{1}{2}$. When $p < \frac{1}{2}$, it is understood in what follows that we do not take into account such a regularity (we can also note that it is not difficult to adapt the estimates below in that case).

Throughout this paper, the same letter c (and, sometimes, c') denotes constants which may vary from line to line. Similarly, the same letter Q denotes monotone increasing (with respect to each argument) functions which may vary from line to line.

3. A priori estimates

We first note that, integrating (formally) (2.1) over Ω, we have, owing to (2.2),

$$\frac{d}{dt}\int_\Omega u\,dx = 0,$$

hence

$$\langle u(t)\rangle = \langle u_0\rangle,\ t \geq 0. \tag{3.1}$$

We then multiply (2.1) by u and obtain, integrating over Ω and by parts,

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 + \int_\Omega \varphi(|\nabla u|^2)|\nabla u|^2\,dx = 0,$$

which yields, owing to (2.6),

$$\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 + c\int_\Omega |\nabla u|^{2p}\,dx \leq c'$$

and, finally,

$$\frac{d}{dt}\|u\|^2 + c(\|u\|^2_{H^2(\Omega)} + \|u\|^{2p}_{W^{1,2p}(\Omega)}) \leq Q(|\langle u_0\rangle|),\ c > 0. \tag{3.2}$$

We then multiply (2.1) by $\frac{\partial u}{\partial t}$ and find

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \|\frac{\partial u}{\partial t}\|^2 + \int_\Omega \varphi(|\nabla u|^2)\nabla u\cdot\nabla\frac{\partial u}{\partial t}\,dx = 0,$$

hence

$$\frac{d}{dt}(\|\Delta u\|^2 + \int_\Omega \psi(|\nabla u|^2)\,dx) + 2\|\frac{\partial u}{\partial t}\|^2 = 0. \tag{3.3}$$

In particular, this yields that the energy decreases along the trajectories, as expected.

We now assume that $p \leq 2$ when $n = 3$. We multiply (2.1) by $-\Delta u$ and have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u\|^2 + \|\nabla\Delta u\|^2 + \int_\Omega \varphi(|\nabla u|^2)\nabla u\cdot\nabla\Delta u\,dx = 0.$$

Noting that, owing to (2.6),

$$|\varphi(s)| \leq c(|s|^{p-1} + 1),\ s \geq 0, \tag{3.4}$$

being understood that, when $p < 1$, φ is bounded (this case being easier to treat), we obtain

$$|\int_\Omega \varphi(|\nabla u|^2)\nabla u\cdot\nabla\Delta u\,dx| \le c\int_\Omega (|\nabla u|^{2p-1}+1)|\nabla\Delta u|\,dx$$

$$\le \frac{1}{2}\|\nabla\Delta u\|^2 + c(\|\nabla u\|^{4p-2}_{L^{4p-2}(\Omega)}+1)$$

$$\le \frac{1}{2}\|\nabla\Delta u\|^2 + c(\|u\|^{4p-2}_{H^2(\Omega)}+1),$$

owing to standard Sobolev embeddings. Therefore,

$$\frac{d}{dt}\|\nabla u\|^2 + c\|u\|^2_{H^3(\Omega)} \le Q(|\langle u_0\rangle|) + c'\|u\|^{4p-2}_{H^2(\Omega)}. \tag{3.5}$$

We finally assume that $p \le 2$ when $n = 2$ or 3 and that

$$|\varphi'(s)| \le c(|s|^{p-2}+1),\ s \ge 0, \tag{3.6}$$

being again understood that, when $p < 2$, φ' is bounded (this case is also easier to treat). We multiply (2.1) by $\Delta^2 u$ and find

$$\frac{1}{2}\frac{d}{dt}\|\Delta u\|^2 + \|\Delta^2 u\|^2 - \int_\Omega \operatorname{div}(\varphi(|\nabla u|^2)\nabla u)\Delta^2 u\,dx = 0.$$

Noting that

$$\operatorname{div}(\varphi(|\nabla u|^2)\nabla u) = \varphi(|\nabla u|^2)\Delta u + 2\varphi'(|\nabla u|^2)\nabla\nabla u\cdot\nabla u\cdot\nabla u,$$

we have, owing to (3.4) and (3.6),

$$|\int_\Omega \operatorname{div}(\varphi(|\nabla u|^2)\nabla u)\Delta^2 u\,dx| \le c\int_\Omega (|\nabla u|^{2p-2}+1)(|\Delta u|+|\nabla\nabla u|)|\Delta^2 u|\,dx.$$

We consider the most difficult case $p = 2$ and $n = 2$ or 3 (in one space dimension, we can use the continuous embedding $H^1(\Omega) \subset L^\infty(\Omega)$). We obtain, owing to Agmon's inequality,

$$|\int_\Omega \operatorname{div}(\varphi(|\nabla u|^2)\nabla u)\Delta^2 u\,dx| \le \frac{1}{2}\|\Delta^2 u\|^2 + c(\|\nabla u\|^4_{L^\infty(\Omega)}+1)\|u\|^2_{H^2(\Omega)}$$

$$\le \frac{1}{2}\|\Delta^2 u\|^2 + c(\|u\|^2_{H^2(\Omega)}\|u\|^2_{H^3(\Omega)}+1)\|u\|^2_{H^2(\Omega)},$$

hence

$$\frac{d}{dt}\|\Delta u\|^2 + c\|u\|^2_{H^4(\Omega)} \le Q(|\langle u_0\rangle|) + c'\|u\|^4_{H^2(\Omega)}(\|u\|^2_{H^3(\Omega)}+1),\ c > 0. \tag{3.7}$$

4. Existence and uniqueness of solutions

We have the

Theorem 4.1. *(i) We assume that* (2.4)-(2.7) *hold and that* $u_0 \in L^2(\Omega)$. *Then,* (2.1)-(2.3) *possesses a unique solution u such that there holds* $u \in L^\infty(\mathbb{R}^+; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega)) \cap L^{2p}(0,T; W^{1,2p}(\Omega)) \cap L^\infty(\tau, +\infty; H^2(\Omega) \cap W^{1,2p}(\Omega))$ *and* $\frac{\partial u}{\partial t} \in L^2(\tau, T; L^2(\Omega))$, $\forall 0 < \tau < T$.
(ii) If we further assume that $p \le 2$ *when* $n = 3$, *then we have the additional regularity* $u \in L^2(\tau, T; H^3(\Omega))$, $\forall 0 < \tau < T$.
(iii) If we further assume that $p \le 2$ *when* $n = 2$ *or* 3, *then we have the additional regularity* $u \in L^2(\tau, T; H^4(\Omega))$, $\forall 0 < \tau < T$.

Proof.
(i) **a) Uniqueness:**

Let u_1 and u_2 be two solutions to (2.1)-(2.2) with initial data $u_{1,0}$ and $u_{2,0}$, respectively. We set $u = u_1 - u_2$ and $u_0 = u_{1,0} - u_{2,0}$ and have

$$\frac{\partial u}{\partial t} + \Delta^2 u - \operatorname{div}(\varphi(|\nabla u_1|^2)\nabla u_1 - \varphi(|\nabla u_2|^2)\nabla u_2) = 0, \tag{4.1}$$

$$\frac{\partial u}{\partial \nu} = \frac{\partial \Delta u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{4.2}$$

$$u|_{t=0} = u_0. \tag{4.3}$$

We multiply (4.1) by u and obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 + \int_\Omega (\phi(\nabla u_1) - \phi(\nabla u_2)) \cdot \nabla u \, dx = 0. \tag{4.4}$$

Noting that

$$\int_\Omega (\phi(\nabla u_1) - \phi(\nabla u_2)) \cdot \nabla u \, dx = \int_\Omega dx \int_0^1 \tau \phi'(\tau \nabla u_1 + (1-\tau)\nabla u_2)\nabla u \cdot \nabla u \, d\tau,$$

it follows from (2.5) and (4.4) that

$$\frac{1}{2}\frac{d}{dt}\|u\|^2 + \|\Delta u\|^2 \le c_0 \|\nabla u\|^2.$$

Employing finally the interpolation inequality

$$\|u\|_{H^1(\Omega)} \le c\|u\|^{\frac{1}{2}} \|u\|^{\frac{1}{2}}_{H^2(\Omega)}, \tag{4.5}$$

we find, noting that $|\langle u \rangle|^2 \le c\|u\|^2$,

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2_{H^2(\Omega)} \le c'\|u\|^2,\ c > 0. \tag{4.6}$$

We thus deduce from (4.6) and Gronwall's lemma that

$$\|u_1(t) - u_2(t)\| \le e^{ct}\|u_{1,0} - u_{2,0}\|, \tag{4.7}$$

hence the uniqueness, as well as the continuous dependence with respect to the initial data in the L^2-norm.

b) Existence:

The proof of existence is based on (3.2) and a standard Galerkin scheme (see also [12]).

The only difficulty here is to pass to the limit in the nonlinear term. To do so, we note that, for an approximated solution u_m constructed by a Galerkin scheme,

$$|\varphi(|\nabla u_m|^2)\nabla u_m| \le c(|\nabla u_m|^{2p-1} + 1)$$

(here, we treat the case $p \ge 1$; the case $p < 1$, which yields that φ is bounded, is easier to treat), so that $\varphi(|\nabla u_m|^2)\nabla u_m$ is bounded in the space $L^{\frac{2p}{2p-1}}(0,T; L^{\frac{2p}{2p-1}}(\Omega)^n)$, $T > 0$, independently of m. Thus, up to a subsequence which we do not relabel,

$$\varphi(|\nabla u_m|^2)\nabla u_m \to \overline{\varphi} \text{ in } L^{\frac{2p}{2p-1}}(0,T; L^{\frac{2p}{2p-1}}(\Omega)^n) \text{ weak.}$$

We then note that u_m is bounded in $L^2(0,T; H^2(\Omega) \cap W^{1,2p}(\Omega))$ and $\frac{\partial u_m}{\partial t}$ is bounded in $L^{\frac{2p}{2p-1}}(0,T; H^{-2}(\Omega) + W^{-1,\frac{2p}{2p-1}}(\Omega))$ and it follows from classical Aubin-Lions compactness results that (again up to a subsequence which we do not relabel)

$$u_m \to u \text{ in } L^2(0,T; H^1(\Omega)),\ \nabla u_m \to \nabla u \text{ a.e.}$$

and, thus,

$$\varphi(|\nabla u_m|^2)\nabla u_m \to \varphi(|\nabla u|^2)\nabla u \text{ a.e.,}$$

hence $\overline{\varphi} = \varphi(|\nabla u|^2)\nabla u$.

In order to obtain the desired regularity, we note that it follows from (3.2) that

$$\int_t^{t+r} (\|u\|^2_{H^2(\Omega)} + \|u\|^{2p}_{W^{1,2p}(\Omega)})\, d\tau \le Q(r, \|u_0\|),\ t \ge 0, \tag{4.8}$$

$r > 0$ fixed arbitrarily. It thus follow from (2.7), (3.3) and the uniform Gronwall lemma (see, e.g., [19]) that

$$\|u(t)\|^2_{H^2(\Omega)} + \|u(t)\|^{2p}_{W^{1,2p}(\Omega)} \le Q(r, \|u_0\|),\ t \ge r. \tag{4.9}$$

Indeed, we note that it follows from (2.7) and (4.5) that

$$\|\Delta u\|^2 + \int_\Omega \psi(|\nabla u|^2)\, dx \ge \|\Delta u\|^2 + c_4\|\nabla u\|^{2p}_{L^{2p}(\Omega)} - c\|u\|\|u\|_{H^2(\Omega)} - c',$$

hence

$$\|\Delta u\|^2 + < u >^2 + \int_\Omega \psi(|\nabla u|^2)\, dx \ge c\|u\|^2_{H^2(\Omega)} \tag{4.10}$$

$$+ c_4\|\nabla u\|^{2p}_{L^{2p}(\Omega)} - c'(\|u\|^2 + 1),\ c > 0.$$

The regularity on $\frac{\partial u}{\partial t}$ then again follows from (3.3).

(ii) This follows from (3.5) and (4.9).

(iii) This follows from (3.7), (4.9) and (ii).

□

Remark 4.1. a) Under the assumptions of (i), if $u_0 \in H^2(\Omega) \cap W^{1,2p}(\Omega)$, with $\frac{\partial u_0}{\partial \nu} = 0$ on Γ, then we have $u \in L^\infty(\mathbb{R}^+; H^2(\Omega) \cap W^{1,2p}(\Omega))$ and $\frac{\partial u}{\partial t} \in L^2(0,T; L^2(\Omega))$, $\forall T > 0$. Indeed, in that case, we deduce from (3.3) that $u \in L^\infty(0,T; H^2(\Omega) \cap W^{1,2p}(\Omega))$, which we combine with the above regularity. Furthermore, if $p \le 2$ when $n = 2$ or 3, then $u \in L^2(0,T; H^4(\Omega))$, $\forall T > 0$.
b) We can also prove that, if $\frac{\partial u}{\partial t}(0) \in L^2(\Omega)$ (note that $\frac{\partial u}{\partial t}(0)$ can be read from (2.1)), then $\frac{\partial u}{\partial t} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; H^2(\Omega))$, $\forall T > 0$. Indeed, differentiating (2.1) with respect to time, we have

$$\frac{\partial}{\partial t}\frac{\partial u}{\partial t} + \Delta^2 \frac{\partial u}{\partial t} - \operatorname{div}(\phi'(\nabla u)\nabla \frac{\partial u}{\partial t}) = 0.$$

Multiplying the above equation by $\frac{\partial u}{\partial t}$, we find, in view of (2.5),

$$\frac{1}{2}\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + \|\Delta \frac{\partial u}{\partial t}\|^2 \le c_0\|\nabla \frac{\partial u}{\partial t}\|^2,$$

hence, employing (4.5) and noting that $\langle \frac{\partial u}{\partial t} \rangle = 0$,

$$\frac{d}{dt}\|\frac{\partial u}{\partial t}\|^2 + \|\Delta \frac{\partial u}{\partial t}\|^2 \le c\|\frac{\partial u}{\partial t}\|^2.$$

5. Existence of finite-dimensional attractors

It follows from Theorem 4.1 that we can define the family of solving operators

$$S(t) : L^2(\Omega) \to L^2(\Omega),\ u_0 \mapsto u(t),\ t \geq 0,$$

where u is the unique solution to (2.1)-(2.3). Furthermore, these solving operators form a continuous semigroup, i.e., $S(0) = \mathrm{Id}$, Id denoting the identity operator, and $S(t) \circ S(s) = S(t+s)$, $t,\ s \geq 0$.

Actually, in view of the conservation property (3.1), we study the existence of compact attractors on the subset

$$\Phi_M = \{v \in L^2(\Omega),\ |\langle v \rangle| \leq M\}$$

of $L^2(\Omega)$.

We have the

Theorem 5.1. *The semigroup $S(t)$ acting on Φ_M possesses the global attractor $\mathcal{A}_M$ in $L^2(\Omega)$, i.e.,*
(i) $\mathcal{A}_M$ is compact in $L^2(\Omega)$ and bounded in $H^2(\Omega) \cap W^{1,2p}(\Omega)$,
(ii) $\mathcal{A}_M$ is invariant, $S(t)\mathcal{A}_M = \mathcal{A}_M$, $\forall t \geq 0$,
(iii) $\mathcal{A}_M$ attracts the bounded sets of initial data in the following sense: $\forall B \subset \Phi_M$ bounded,

$$\lim_{t \to +\infty} \mathrm{dist}(S(t)B, \mathcal{A}_M) = 0,$$

where dist *denotes the Hausdorff semi-distance between sets defined by*

$$\mathrm{dist}(A, B) = \sup_{a \in A} \inf_{b \in B} \|a - b\|.$$

This is equivalent to the following: $\forall B \subset \Phi_M$ bounded, $\forall \epsilon > 0$, $\exists t_0 = t_0(B, \epsilon) \geq 0$ such that $t \geq t_0 \implies S(t)B \subset \mathcal{U}_\epsilon$, where $\mathcal{U}_\epsilon$ is the ϵ-neighborhood of $\mathcal{A}_M$.

Remark 5.1. It follows from the definition that the global attractor, if it exists, is indeed unique. Furthermore, it is the smallest (for the inclusion) closed set which enjoys the attraction property and thus appears as a suitable object in view of the study of the asymptotic behavior of the system.

Proof.

It follows from (3.2) that

$$\frac{d}{dt}\|u\|^2 + c(\|u\|^2_{H^2(\Omega)} + \|u\|^{2p}_{W^{1,2p}(\Omega)}) \leq c_M,\ c > 0, \tag{5.1}$$

which yields

$$\frac{d}{dt}\|u\|^2 + c\|u\|^2 \le c_M,\ c > 0. \tag{5.2}$$

We thus deduce from (5.2) and Gronwall's lemma the existence of a bounded absorbing set $\mathcal{B}_0$ for $S(t)$ on Φ_M, i.e., $\forall B \subset \Phi_M$ bounded, $\exists t_0 = t_0(B) \ge 0$ such that $t \ge t_0 \implies S(t)B \subset \mathcal{B}_0$ (the existence of such a bounded absorbing set is often used as a mathematical definition of dissipation).

Let then B be a bounded subset of Φ_M and t_0 be such that $t \ge t_0 \implies S(t)B \subset \mathcal{B}_0$. Then, it follows from (5.1) that, if $t \ge t_0$,

$$\int_t^{t+r} (\|u\|^2_{H^2(\Omega)} + \|u\|^{2p}_{W^{1,2p}(\Omega)})\, d\tau \le c_{M,\mathcal{B}_0,r},\ t \ge t_0, \tag{5.3}$$

$r > 0$ fixed arbitrarily. It thus follows from (3.3), (4.10), (5.3) and the uniform Gronwall lemma that

$$\|u(t)\|^2_{H^2(\Omega)} + \|u(t)\|^{2p}_{W^{1,2p}(\Omega)} \le c_{M,\mathcal{B}_0,r},\ t \ge t_0 + r. \tag{5.4}$$

In particular, (5.4) yields the existence of a bounded absorbing set $\mathcal{B}_2$ for $S(t)$ on Φ_M which is bounded in $H^2(\Omega) \cap W^{1,2p}(\Omega)$ and thus compact in $L^2(\Omega)$. The existence of the global attractor then follows from standard results (see, e.g., [1], [16] and [19]).

□

Remark 5.2. Replacing, if necessary, the bounded absorbing set $\mathcal{B}_2$ by $\cup_{t\ge t_0} S(t)\mathcal{B}_2$, where t_0 is such that $t \ge t_0$ implies $S(t)\mathcal{B}_2 \subset \mathcal{B}_2$, we can assume, without loss of generality, that $\mathcal{B}_2$ is bounded in $H^2(\Omega) \cap W^{1,2p}(\Omega)$ and positively invariant by $S(t)$, i.e., $S(t)\mathcal{B}_2 \subset \mathcal{B}_2$, $\forall t \ge 0$.

We now assume that (3.6) holds, i.e.,

$$|\varphi'(s)| \le c(|s|^{p-2} + 1),\ c \ge 0,\ s \ge 0,$$

when $p \ge 2$. When $p < 2$, it is once more understood that φ' is bounded. We also assume that $p \le 2$ when $n = 2$ or 3.

We have the

Theorem 5.2. *Under the above assumptions, there holds*

$$t\|S(t)u_{1,0} - S(t)u_{2,0}\|_{H^1(\Omega)} \le ce^{c't}\|u_{1,0} - u_{2,0}\|,\ t > 0, \tag{5.5}$$

$\forall u_{1,0},\ u_{2,0} \in \mathcal{B}_2$ and where the positive constants c and c' only depend on M and $\mathcal{B}_2$.

Proof.

We multiply (4.1) by $-t\Delta u$ and obtain

$$\frac{1}{2}\frac{d}{dt}(t\|\nabla u\|^2)+t\|\nabla\Delta u\|^2 \tag{5.6}$$

$$+t\int_\Omega(\varphi(|\nabla u_1|^2)\nabla u_1-\varphi(|\nabla u_2|^2)\nabla u_2)\cdot\nabla\Delta u\,dx$$

$$=\frac{1}{2}\|\nabla u\|^2.$$

Noting that

$$(\varphi(|\nabla u_1|^2)\nabla u_1-\varphi(|\nabla u_2|^2)\nabla u_2)\cdot\nabla\Delta u=(\phi(\nabla u_1)-\phi(\nabla u_2))\cdot\nabla\Delta u$$

$$=\int_0^1\tau\phi'(\tau\nabla u_1+(1-\tau)\nabla u_2)\nabla u\cdot\nabla\Delta u\,d\tau$$

$$=\int_0^1\tau[\varphi(|\tau\nabla u_1+(1-\tau)\nabla u_2|^2)\nabla u\cdot\nabla\Delta u$$

$$+2\varphi'(|\tau\nabla u_1+(1-\tau)\nabla u_2|^2)$$

$$\times((\tau\nabla u_1+(1-\tau)\nabla u_2)\cdot\nabla u)((\tau\nabla u_1+(1-\tau)\nabla u_2)\cdot\nabla\Delta u)]\,d\tau,$$

we deduce from (3.4) and (3.6) that

$$|\int_\Omega(\varphi(|\nabla u_1|^2)\nabla u_1-\varphi(|\nabla u_2|^2)\nabla u_2)\cdot\nabla\Delta u\,dx|$$

$$\le c\int_\Omega(|\nabla u_1|^{2p-2}+|\nabla u_2|^{2p-2}+1)|\nabla u||\nabla\Delta u|\,dx$$

$$\le\frac{1}{2}\|\nabla\Delta u\|^2+c(\|\nabla u_1\|^{4p-4}_{L^\infty(\Omega)}+\|\nabla u_2\|^{4p-4}_{L^\infty(\Omega)}+1)\|\nabla u\|.$$

Taking the most difficult case $p=2$ ($n=2$ or 3) and employing Agmon's inequality, we find

$$|\int_\Omega(\varphi(|\nabla u_1|^2)\nabla u_1-\varphi(|\nabla u_2|^2)\nabla u_2)\cdot\nabla\Delta u\,dx| \tag{5.7}$$

$$\le\frac{1}{2}\|\nabla\Delta u\|^2+c(\|u_1\|^2_{H^2(\Omega)}\|u_1\|^2_{H^3(\Omega)}+\|u_2\|^2_{H^2(\Omega)}\|u_2\|^2_{H^3(\Omega)}+1)\|\nabla u\|^2$$

$$\leq \frac{1}{2}\|\nabla\Delta u\|^2 + c(\|u_1\|^2_{H^3(\Omega)} + \|u_2\|^2_{H^3(\Omega)} + 1)\|\nabla u\|^2.$$

It thus follows from (5.6)-(5.7) that

$$\frac{d}{dt}(t\|\nabla u\|^2) + t\|\nabla\Delta u\|^2 \tag{5.8}$$

$$\leq \|\nabla u\|^2 + ct(\|u_1\|^2_{H^3(\Omega)} + \|u_2\|^2_{H^3(\Omega)} + 1)\|\nabla u\|^2.$$

Noting finally that it follows from (3.5) that

$$\int_0^t \|u_i\|^2_{H^3(\Omega)}\,d\tau \leq c_{M,\mathcal{B}_2}(t+1),\ i=1,\ 2,$$

and from (4.6)-(4.7) that

$$\int_0^t \|u(t)\|^2_{H^1(\Omega)}\,d\tau \leq ce^{c't},$$

where c and c' only depend on M and $\mathcal{B}_2$, (5.5) follows from (5.8) and Gronwall's lemma.

□

We also have the

Proposition 5.1. *There holds*

$$\|u(t_1) - u(t_2)\| \leq c_{M,\mathcal{B}_2,T}|t_1 - t_2|^{\frac{1}{2}}, \tag{5.9}$$

for every solution u to (2.1)-(2.3) with initial datum $u_0 \in \mathcal{B}_2$, for every t_1, $t_2 \in [0,T]$, for every $T > 0$.

Proof.

Indeed,

$$\|u(t_1)-u(t_2)\| = \|\int_{t_1}^{t_2}\frac{\partial u}{\partial t}\,d\tau\| \leq |\int_{t_1}^{t_2}\|\frac{\partial u}{\partial t}\|\,d\tau| \leq |t_1-t_2|^{\frac{1}{2}}|\int_{t_1}^{t_2}\|\frac{\partial u}{\partial t}\|^2\,d\tau|^{\frac{1}{2}}$$

and the result follows from (3.3).

□

We deduce from (5.5), (5.9) and standard results (see, e.g., [5], [6] and [16]) the

Theorem 5.3. *The semigroup $S(t)$ acting on Φ_M possesses an exponential attractor $\mathcal{M}_M$ in $L^2(\Omega)$, i.e.,*
(i) $\mathcal{M}_M$ is compact in $L^2(\Omega)$ and bounded in $H^2(\Omega)\cap W^{1,2p}(\Omega)$,
(ii) $\mathcal{M}_M$ is positively invariant, $S(t)\mathcal{M}_M \subset \mathcal{M}_M$, $\forall t \geq 0$,

(iii) $\mathcal{M}_M$ *has finite fractal dimension (for the topology of* $L^2(\Omega)$*),*
(iv) $\mathcal{M}_M$ *attracts the bounded subsets of* Φ_M *exponentially fast in the following sense:* $\forall B \subset \Phi_M$ *bounded,*

$$\mathrm{dist}(S(t)B, \mathcal{M}_M) \leq Q(\|B\|)e^{-ct},\ c > 0,\ t \geq 0,$$

where the constant c *is independent of* B.

Remark 5.3. a) Due to the relaxation from invariance to positive invariance, an exponential attractor, if it exists, is not unique.
b) We can note that the rate of exponential attraction is uniform and can be computed explicitly (in terms of the physical parameters of the problem in concrete situations). Therefore, exponential attractors are expected to be more robust under perturbations. Indeed, the rate of attraction of trajectories to the global attractor may be slow and it is very difficult, if not impossible, to estimate this rate of attraction in general. We refer the reader to [5] and [16] for discussions on this subject.
c) Having finite fractal dimension means, very roughly speaking that, even though the initial phase space is infinite-dimensional, the reduced dynamics can be described by a finite number of parameters. We again refer the interested reader to [5] and [16] for more details.

Since an exponential attractor always contains the global attractor, we deduce from Theorem 5.3 the

Corollary 5.1. *The global attractor* $\mathcal{A}_M$ *has finite fractal dimension for the topology of* $L^2(\Omega)$.

Remark 5.4. Actually, in two space dimensions, we can prove the existence of an exponential attractor without any restriction on p (allowing, in particular, to prove the existence of the finite-dimensional global attractor for the models described in Remark 2.2), by using the so-called l-trajectories method (see, e.g., [14], [15] and [18]). Indeed, it first follows from (4.6) that, if u_1 and u_2 are, as above, two solutions to (2.1)-(2.2) with initial data in $\mathcal{B}_2$ and $u = u_1 - u_2$,

$$\|u\|_{L^2(l,2l;H^2(\Omega))} \leq c\|u\|_{L^2(0,l;L^2(\Omega))}, \tag{5.10}$$

for a proper constant $l > 0$ depending only on the constant c in (4.6) (see [14] and [15] for details). We then note that

$$\frac{\partial u}{\partial t} = -\Delta^2 u + \mathrm{div}(\phi(\nabla u_1) - \phi(\nabla u_2)),$$

which yields

$$\|\frac{\partial u}{\partial t}\|_{L^2(l,2l;H^{-2}(\Omega))} \le c\|u\|_{L^2(l,2l;H^2(\Omega))}$$

$$+ \sup_{\xi\in H^2(\Omega),\ \|\xi\|_{H^2(\Omega)}=1} \int_l^{2l} dt \int_\Omega |\phi(\nabla u_1) - \phi(\nabla u_2)||\nabla \xi|\, dx.$$

Writing

$$\int_\Omega |\phi(\nabla u_1) - \phi(\nabla u_2)||\nabla\xi|\, dx \le c\int_\Omega (|\nabla u_1|^{2p-2} + |\nabla u_2|^{2p-2} + 1)|\nabla u||\nabla \xi|\, dx$$

$$\le c(\|\nabla u_1\|^{2p-2}_{L^{3p-3}(\Omega)} + \|\nabla u_2\|^{2p-2}_{L^{3p-3}(\Omega)} + 1)\|u\|_{H^2(\Omega)}\|\xi\|_{H^2(\Omega)}$$

$$\le c(\|u_1\|^{2p-2}_{H^2(\Omega)} + \|u_2\|^{2p-2}_{H^2(\Omega)} + 1)\|u\|_{H^2(\Omega)}$$

$$\le c\|u\|_{H^2(\Omega)},$$

we deduce that

$$\|\frac{\partial u}{\partial t}\|_{L^2(l,2l;H^{-2}(\Omega))} \le c\|u\|_{L^2(l,2l;H^2(\Omega))},$$

hence, owing to (5.10),

$$\|\frac{\partial u}{\partial t}\|_{L^2(l,2l;H^{-2}(\Omega))} \le c\|u\|_{L^2(0,l;L^2(\Omega))}. \tag{5.11}$$

The two estimates (5.10) and (5.11) finally allow to prove the existence of an exponential attractor (see [15] and [18] for more details). We can note that, in three space dimensions, this only yields a slight improvement, namely, $p \le 3$ (in order to use the continuous embedding $H^1(\Omega) \subset L^6(\Omega)$).

Remark 5.5. It is also important to study the limit problem (corresponding to $\epsilon = 0$ in (1.5)), i.e., the initial and boundary value problem

$$\frac{\partial u}{\partial t} - \mathrm{div}(\varphi(|\nabla u|^2)\nabla u) = 0, \tag{5.12}$$

$$\frac{\partial u}{\partial \nu} = 0 \text{ on } \Gamma, \tag{5.13}$$

$$u|_{t=0} = u_0. \tag{5.14}$$

Multiplying (5.12) by u, we have, owing to (2.6),

$$\frac{d}{dt}\|u\|^2 + c\|\nabla u\|^{2p} \le c',\ c > 0, \tag{5.15}$$

and, multiplying (2.1) by $\frac{\partial u}{\partial t}$, we obtain

$$\frac{d}{dt}\int_\Omega \psi(|\nabla u|^2)\,dx + 2\|\frac{\partial u}{\partial t}\|^2 = 0. \tag{5.16}$$

Unfortunately, this is not sufficient to pass to the limit in the nonlinear term. Indeed, we do not have enough regularity to employ Aubin-Lions compactness results and the operator ϕ is not monotone. Another problem is the uniqueness. However, if $p > 1$ (this contains the thin film model with slope selection), we have (formally) the dissipativity in $L^2(\Omega)$ and $W^{1,2p}(\Omega)$.

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