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History-dependent hemivariational inequalities with applications to Contact Mechanics

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Abstract - In this paper we survey some of our recent results on the existence and uniqueness of solutions to nonconvex and nonsmooth problems which arise in Contact Mechanics. The approach is based on operator subdifferential inclusions and hemivariational inequalities, and focuses on three aspects. First, we report on results on the second order history-dependent subdifferential inclusions and hemivariational inequalities; next, we discuss a class of stationary history-dependent operator inclusions and hemivariational inequalities; finally, we use these abstract results in the study of two viscoelastic contact problems with subdifferential boundary conditions.

Key words and phrases : history-dependent operator, evolutionary inclusion, hemivariational inequality, nonconvex potential, subdifferential mapping, frictional contact, viscoelastic material, weak solution.

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1. Introduction

The present paper represents a shortened version of the talks presented by the authors on the special sessions *Modèles Mathématiques et Numériques en Mécanique des Solides* and *Équations aux Dérivées Partielles et Applications*, organized inside the *11e Colloque Franco-Roumain de Mathématiques Appliquées* which took place at the Faculty of Mathematics and Computer Science of the University of Bucarest in the interval August 24–30, 2012. The goal of this work is to review some recent results concerning nonlinear operator subdifferential inclusions and hemivariational inequalities, as well as to present some applications of these abstract results in the study of two viscoelastic frictional contact problems.

Phenomena of contact between deformable bodies abound in industry and everyday life. Contact of braking pads with wheels, tires with roads, pistons with skirts are just a few simple examples. Common industrial processes such as metal forming and metal extrusion involve contact evolutions. Owing to their inherent complexity, contact phenomena lead to mathematical models expressed in terms of strongly nonlinear elliptic or evolutionary boundary value problems as illustrated in [5, 6, 22, 23], for instance.

Considerable progress has been achieved recently in modelling, mathematical analysis and numerical simulations of various contact processes and, as a result, a general Mathematical Theory of Contact Mechanics is currently emerging. It is concerned with the mathematical structures which underly general contact problems with different constitutive laws, i.e., materials, varied geometries and different contact conditions. Its aim is to provide a sound, clear and rigorous background to the constructions of models for contact, proving existence, uniqueness and regularity results, assigning precise meaning to solutions, among others. To this end, it operates with various mathematical tools which include variational and hemivariational inequalities. The use of variational inequalities arise in the study of contact problems which involve convex energy functions (potentials); in contrast, the use of hemivariational inequalities arise in the study of contact problems which involve nonconvex energy functions (superpotentials).

The notion of hemivariational inequality was introduced by P.D.Panagiotopoulos in the early 1980s as a generalization of the variational inequality. It is based the notion of the generalized gradient of Clarke, introduced in [1] for a class of locally Lipschitz functions. By means of hemivariational inequality, contact problems involving nonmonotone and multivalued constitutive laws and boundary conditions can be treated mathematically. Nowadays the theory of hemivariational inequalities provides powerful methods and mathematical tools which allow to give positive answers to unsolved or partially unsolved problems which arise in the theory of Partial Differential Equations, Contact Mechanics and Engineering Sciences, see [9]–[21] and the references therein.

The present paper is structured as follows. In Section 2 we recall some basic notation and definitions we need in the rest of the manuscript. In Section 3 we state a result on the existence and uniqueness of the solution to a class of second order evolutionary inclusions involving a history-dependent operator. Then, in Section 4, we apply this result in the study of a class of second order hemivariational inequalities. In Section 5 we present similar results for a class of stationary history-dependent inclusions and hemivariational inequalities. Finally, in Section 6 we show how these abstract results can be used in the study of dynamic and quasistatic contact problems with subdifferential conditions. Everywhere in what follows we skip the details in proofs; we restrict ourselves to mention only their main steps and the main arguments used; nevertheless, we indicate the references where the complete proofs can be found.

2. Preliminaries

In this section we recall the basic notation and definition we need in the rest of the paper.

Let V and Z be separable and reflexive Banach spaces with their topological duals V^* and Z^*, respectively. Let H denote a separable Hilbert space that we identify with its dual. We assume that $V \subset H \subset V^*$ and $Z \subset H \subset Z^*$ are evolution triples of spaces where all embedings are continuous, dense and compact (see, e.g., Chapter 23.4 of [24] and Chapter 3.4 of [3]). We also suppose that V is compactly embedded in Z. Let $\|\cdot\|$, $|\cdot|$ and $\|\cdot\|_{V^*}$ denote the norms in V, H and V^*, respectively, and let $\langle\cdot,\cdot\rangle$ be the duality pairing between V^* and V. Given a finite interval of time $(0,T)$, we also introduce the spaces

$$\mathcal{V} = L^2(0,T;V), \quad \mathcal{Z} = L^2(0,T;Z), \quad \widehat{\mathcal{H}} = L^2(0,T;H),$$

$$\mathcal{Z}^* = L^2(0,T;Z^*), \quad \mathcal{V}^* = L^2(0,T;V^*), \quad \mathcal{W} = \{\, v \in \mathcal{V} \mid v' \in \mathcal{V}^* \,\}.$$

Here and below we denote by v' and v'' the first and the second time derivative of v in the sense of vector-valued distributions. The duality pairing between $\mathcal{V}^*$ and $\mathcal{V}$ is denoted by

$$\langle\langle z, w\rangle\rangle = \int_0^T \langle z(t), w(t)\rangle\, dt \quad \text{for} \quad z \in \mathcal{V}^*,\ w \in \mathcal{V}.$$

It is well known that the space $\mathcal{W}$ is embedded continuously in the space of continuous functions on $[0,T]$ with values in H denoted by $C(0,T;H)$. Moreover, since V is embedded compactly in H, then so does $\mathcal{W}$ into $L^2(0,T;H)$, see [3], for instance.

Let X and Y be Banach spaces. A multifunction $F\colon X \to 2^Y \setminus \{\emptyset\}$ is lower semicontinuous (upper semicontinuous, respectively) if for $C \subset Y$ closed, the set $F^+(C) = \{x \in X \mid F(x) \subset C\}$ ($F^-(C) = \{x \in X \mid F(x) \cap C \neq \emptyset\}$, respectively) is closed in X. F is bounded on bounded sets if $F(B) = \cup_{x\in B} F(x)$ is a bounded subset of Y for all bounded sets B in X.

Let Y be a reflexive Banach space and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between Y and its dual. An operator $F\colon Y \to Y^*$ is said to be monotone if

$$\langle Fy - Fz, y - z\rangle \geq 0 \qquad \text{for all } y,\, z \in Y.$$

It is pseudomonotone if $y_n \to y_0$ weakly in Y and $\limsup \langle Fy_n, y_n - y_0\rangle \leq 0$ imply that

$$\langle Fy_0, y_0 - y\rangle \leq \liminf \langle Fy_n, y_n - y\rangle \quad \text{for all } y \in Y.$$

It is said to be demicontinuous if $y_n \to y_0$ in Y implies $Fy_n \to Fy_0$ weakly in Y^*. It is hemicontinuous if the real-valued function $t \to \langle F(y+tv), w\rangle$ is continuous on $[0,1]$ for all y, v, $w \in Y$.

A multivalued mapping $F\colon Y \to 2^{Y^*}$ is said to be pseudomonotone, if it satisfies the following conditions:

(a) for every $y \in Y$, Fy is a nonempty, convex, and weakly compact set in Y^*.
(b) F is upper semicontinuous from every finite dimensional subspace of Y into Y^* endowed with the weak topology.
(c) if $y_n \to y$ weakly in Y, $y_n^* \in Fy_n$, and $\limsup \langle y_n^*, y_n - y\rangle \le 0$, then for each $z \in Y$ there exists $y^*(z) \in Fy$ such that $\langle y^*(z), y - z\rangle \le \liminf \langle y_n^*, y_n - z\rangle$.

Let $L\colon D(L) \subset Y \to Y^*$ be a linear densely defined maximal monotone operator. A mapping $F\colon Y \to 2^{Y^*}$ is said to be L-pseudomonotone (pseudomonotone with respect to $D(L)$) if and only if (a), (b) and the following hold:

(d) if $\{y_n\} \subset D(L)$ is such that $y_n \to y$ weakly in Y, $y \in D(L)$, $Ly_n \to Ly$ weakly in Y^*, $y_n^* \in Fy_n$, $y_n^* \to y^*$ weakly in Y^* and $\limsup \langle y_n^*, y_n - y\rangle \le 0$, then $y^* \in Ty$ and $\langle y_n^*, y_n\rangle \to \langle y^*, y\rangle$.

Given a Banach space $(X, \|\cdot\|_X)$, the symbol w–X is always used to denote the space X endowed with the weak topology. By $\mathcal{L}(X, X^*)$ we denote the class of linear and bounded operators from X to X^*. If $U \subset X$, then we write $\|U\|_X = \sup\{\, \|x\|_X \mid x \in U\,\}$.

Let $\varphi\colon X \to \mathbb{R}$ be a locally Lipschitz function. Then, following [1], the generalized directional derivative of φ at $x \in X$ in the direction $v \in X$, denoted by $\varphi^0(x; v)$, is defined by

$$\varphi^0(x; v) = \limsup_{y \to x,\ t \downarrow 0} \frac{\varphi(y + tv) - \varphi(y)}{t}\,.$$

Moreover, the generalized gradient of φ at x, denoted by $\partial\varphi(x)$, is a subset of a dual space X^* given by

$$\partial\varphi(x) = \{\, \zeta \in X^* \mid \varphi^0(x; v) \ge \langle \zeta, v\rangle_{X^* \times X} \text{ for all } \ v \in X\,\}.$$

Finally, a locally Lipschitz function φ is called regular (in the sense of Clarke) at $x \in X$ if for all $v \in X$ the one–sided directional derivative $\varphi'(x; v)$ exists and satisfies $\varphi^0(x; v) = \varphi'(x; v)$ for all $v \in X$.

3. Second order history-dependent inclusions

In this section we consider a class of second order evolutionary inclusions which can be formulated as follows.

Problem 3.1. Find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} u''(t) + A(t, u'(t)) + B(t, u(t)) + (\mathcal{S}u)(t) + F(t, u'(t)) \ni f(t) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{a.e. } t \in (0,T), \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

We complete Problem 3.1 with the following definition.

Definition 3.1. *A function $u \in \mathcal{V}$ is a solution to Problem* 3.1 *if $u' \in \mathcal{W}$ and there exists $z \in \mathcal{Z}^*$ such that*

$$\begin{cases} u''(t) + A(t, u'(t)) + B(t, u(t)) + (\mathcal{S}u)(t) + z(t) = f(t) \quad \text{a.e. } t \in (0,T), \\ z(t) \in F(t, u'(t)) \quad \text{a.e. } t \in (0,T), \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

We assume the following hypotheses on the data of Problem 3.1.

$\underline{H(A)}$: $A\colon (0,T) \times V \to V^*$ is such that

(a) $A(\cdot, v)$ is measurable on $(0,T)$ for every $v \in V$.
(b) $A(t, \cdot)$ is hemicontinuous for a.e. $t \in (0,T)$.
(c) $A(t, \cdot)$ is strongly monotone for a.e. $t \in (0,T)$, i.e. there exists $m_1 > 0$ such that $\langle A(t,u) - A(t,v), u - v\rangle \geq m_1 \|u - v\|^2$ for a.e. $t \in (0,T)$, all u, $v \in V$.
(d) $\|A(t,v)\|_{V^*} \leq a_0(t) + a_1\|v\|$ for a.e. $t \in (0,T)$, all $v \in V$ with $a_0 \in L^2(0,T)$, $a_0 \geq 0$ and $a_1 > 0$.
(e) $\langle A(t,v), v\rangle \geq \alpha \|v\|^2$ for a.e. $t \in (0,T)$, all $v \in V$ with $\alpha > 0$.

$\underline{H(B)}$: $B\colon (0,T) \times V \to V^*$ is such that

(a) $B(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$.
(b) $B(t, \cdot)$ is Lipschitz continuous for a.e. $t \in (0,T)$, i.e. there exists $L_B > 0$ such that $\|B(t,u) - B(t,v)\|_{V^*} \leq L_B \|u - v\|$ for all u, $v \in V$, a.e. $t \in (0,T)$.
(c) $\|B(t,v)\|_{V^*} \leq b_0(t) + b_1\|v\|$ for all $v \in V$, a.e. $t \in (0,T)$ with $b_0 \in L^2(0,T)$ and b_0, $b_1 \geq 0$.

$\underline{H(\mathcal{S})}$: $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ is such that

$$\|(\mathcal{S}u)(t) - (\mathcal{S}v)(t)\|_{V^*} \leq L_{\mathcal{S}} \int_0^t \|u(s) - v(s)\|\, ds$$

for all u, $v \in \mathcal{V}$, a.e. $t \in (0,T)$ with $L_{\mathcal{S}} > 0$.

$\underline{H(F)}$: $F\colon (0,T) \times V \to 2^{Z^*}$ has nonempty, closed, convex values and

(a) $F(\cdot, v)$ is measurable on $(0,T)$ for all $v \in V$.

(b) $F(t,\cdot)$ is upper semicontinuous from V into w–Z^* for a.e. $t \in (0,T)$, where V is endowed with Z–topology.

(c) $\|F(t,v)\|_{Z^*} \le d_0(t) + d_1\|v\|$ for all $v \in V$, a.e. $t \in (0,T)$ with $d_0 \in L^2(0,T)$ and $d_0,\, d_1 \ge 0$.

(d) $\langle F(t,u) - F(t,v), u - v\rangle_{Z^*\times Z} \ge -m_2\|u-u\|^2$ for all $u, v \in V$, a.e. $t \in (0,T)$ with $m_2 \ge 0$.

$\underline{(H_0)}$: $f \in \mathcal{V}^*$, $u_0 \in V$, $v_0 \in H$.

$\underline{(H_1)}$: $m_1 > m_2$ and $\alpha > 2\sqrt{3}\, c_e\, d_1$, where $c_e > 0$ is the embedding constant of V into Z, i.e. $\|z\|_Z \le c_e\|z\|$ for all $z \in V$.

We note that the hypothesis $H(\mathcal{S})$ is satisfied for the operator $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ given by

$$(\mathcal{S}v)(t) = R\Big(t, \int_0^t v(s)\,ds + u_0\Big) \quad \text{for all } v \in \mathcal{V},\ \text{a.e. } t \in (0,T), \tag{3.1}$$

where $R\colon (0,T)\times V \to V^*$ is such that $R(\cdot,v)$ is measurable on $(0,T)$ for all $v \in V$, $R(t,\cdot)$ is a Lipschitz continuous operator for a.e. $t \in (0,T)$ and $u_0 \in V$. It is also satisfied for the *Volterra operator* $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ given by

$$(\mathcal{S}v)(t) = \int_0^t C(t-s)\,v(s)\,ds \quad \text{for all } v \in \mathcal{V},\ \text{a.e. } t \in (0,T), \tag{3.2}$$

where $C \in L^\infty(0,T;\mathcal{L}(V,V^*))$. In the case of the operators (3.1) and (3.2) the current value $(\mathcal{S}v)(t)$ at the moment t depends on the history of the values of v at the moments $0 \le s \le t$ and, therefore, we refer the operators of the form (3.1) or (3.2) as *history-dependent operators*. We extend this definition to all operators $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$ which satisfy $H(\mathcal{S})$ and, for this reason, we say that Problem 3.1 represents a *second-order history-dependent subdifferential inclusion.*

It follows from Lemma 5.3 of [19] that under the hypothesis $H(F)$, the multifunction $G\colon W^{1,2}(0,T;V) \to 2^{\mathcal{Z}^*}$ defined by

$$G(u) = \{\, z \in \mathcal{Z}^* \mid z(t) \in F(t,u'(t)) \text{ a.e. on } (0,T)\,\},$$

for $u \in W^{1,2}(0,T;V)$, has nonempty, weakly compact and convex values. Hence, the multifunction $t \mapsto F(t,u'(t))$ has a measurable $\mathcal{Z}^*$ selection and, therefore, Definition 3.1 makes sense.

The following is our main result in the study of Problem 3.1.

Theorem 3.1. *Under hypotheses $H(A)$, $H(B)$, $H(\mathcal{S})$, $H(F)$, (H_0) and (H_1), Problem* 3.1 *has a unique solution.*

Proof. The proof is carried out in several steps. In the first one we consider the intermediate evolutionary inclusion

$$\begin{cases} u''(t) + A(t, u'(t)) + F(t, u'(t)) \ni f(t) \quad \text{a.e. } \ t \in (0,T), \\ u(0) = u_0, \quad u'(0) = v_0 \end{cases} \tag{3.3}$$

and, denoting $w = u'$, we obtain

$$\begin{cases} w'(t) + A(t, w(t)) + F(t, w(t)) \ni f(t) \quad \text{a.e. } \ t \in (0,T), \\ w(0) = v_0. \end{cases} \tag{3.4}$$

We define the operator $K\colon \mathcal{V} \to \mathcal{V}$ by equality

$$(Kv)(t) = \int_0^t v(s)\,ds + u_0.$$

Then, it is easy to see that w solves (3.4) if and only if $u = Kw$ is a solution to (3.3). Next, we rewrite (3.4) as an operator inclusion $(L + \mathcal{F})w \ni f$, where $Lw = w'$ denotes the generalized time derivative, $\mathcal{F} = \mathcal{A}_1 + F_1$ with $(\mathcal{A}_1 w)(t) = A(t, w(t) + v_0)$ and

$$F_1 w = \{\, z^* \in \mathcal{Z}^* \mid z^*(t) \in F(t, w(t) + v_0) \text{ a.e. } t \in (0,T) \,\}.$$

We then prove that $\mathcal{F}$ is bounded, coercive and pseudomonotone with respect to the graph norm topology of the domain of L. By exploting the fact that L is closed, densely defined and maximal monotone operator, from Theorem 1.3.73 of [3], we obtain that $L + \mathcal{F}$ is surjective which implies that (3.3) is solvable. Subsequently we show that the solution to (3.3) is unique.

In the second step, we consider the operator $\Lambda\colon \mathcal{V}^* \to \mathcal{V}^*$ defined by

$$(\Lambda\eta)(t) = B(t, u_\eta(t)) + (\mathcal{S}u_\eta)(t) \quad \text{for all } \eta \in \mathcal{V}^*, \ \text{a.e. } t \in (0,T),$$

where u_η is the unique solution to the following inclusion

$$\begin{cases} u''(t) + A(t, u'(t)) + F(t, u'(t)) \ni f(t) - \eta(t) \quad \text{a.e. } \ t \in (0,T), \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases} \tag{3.5}$$

Applying the Banach Contraction Principle, we show that Λ has a unique fixed point η^*. The solution of (3.5) corresponding to η^* is the unique solution to Problem 3.1, which concludes the proof. □

For a detailed proof of Theorem 3.1 we refer to [8] and [11]. Also, for the study of a general case when the multifunction depends on the unknown function u, i.e. $F = F(t, u, u')$, we refer the reader to Chapter 5 of [19].

4. Second order history-dependent hemivariational inequalities

In this section we apply Theorem 3.1 in the study of a class of second order hemivariational inequalities which involve history-dependent operators.

Let $\Omega \subset \mathbb{R}^d$ be a bounded domain with a Lipschitz boundary Γ and let Γ_C be a measurable part of Γ, $\Gamma_C \subseteq \Gamma$. Let V be a closed subspace of $H^1(\Omega;\mathbb{R}^d)$, $H = L^2(\Omega;\mathbb{R}^d)$ and $Z = H^\delta(\Omega;\mathbb{R}^d)$ with a fixed $\delta \in (1/2,1)$. Denoting by $i\colon V \to Z$ the embedding, by $\gamma\colon Z \to L^2(\Gamma;\mathbb{R}^d)$ and $\gamma_0\colon H^1(\Omega;\mathbb{R}^d) \to H^{1/2}(\Gamma;\mathbb{R}^d) \subset L^2(\Gamma;\mathbb{R}^d)$ the trace operators, we get $\gamma_0 v = \gamma(iv)$ for all $v \in V$. For simplicity, in what follows, we omit the notation of the embedding i and we write $\gamma_0 v = \gamma v$ for all $v \in V$. It is well known from the theory of Sobolev spaces (see, for instance, [2, 3, 24]) that (V,H,V^*) and (Z,H,Z^*) form evolution triples of spaces and the embedding $V \subset Z$ is compact. We denote by c_e the embedding constant of V into Z, by $\|\gamma\|$ the norm of the trace in $\mathcal{L}(Z, L^2(\Gamma;\mathbb{R}^d))$ and by $\gamma^*\colon L^2(\Gamma;\mathbb{R}^d) \to Z^*$ the adjoint operator to γ. We are interested in the following problem.

Problem 4.1. Find $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{cases} \langle u''(t) + A(t,u'(t)) + B(t,u(t)) + (\mathcal{S}u)(t), v\rangle + \\ \quad + \displaystyle\int_{\Gamma_C} j^0(x,t,\gamma u'(t);\gamma v)\, d\Gamma \ge \langle f(t), v\rangle \quad \text{for all } v \in V, \text{ a.e. } t \in (0,T), \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

Here $\mathcal{S}$ is an operator which satisfies condition $H(\mathcal{S})$. Therefore, we refer to Problem 4.1 as a *second order history-dependent hemivariational inequality*. In the study of this problem we consider the following hypothesis.

$\underline{H(j)}$: $j\colon \Gamma_C \times (0,T) \times \mathbb{R}^d \to \mathbb{R}$ is such that

(a) $j(\cdot,\cdot,\xi)$ is measurable for all $\xi \in \mathbb{R}$ and $j(\cdot,\cdot,0) \in L^1(\Gamma_C \times (0,T))$.

(b) $j(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t) \in \Gamma_C \times (0,T)$.

(c) $\|\partial j(x,t,\xi)\|_{\mathbb{R}^d} \le \widetilde{c}_0(t) + \widetilde{c}_1 \|\xi\|_{\mathbb{R}^d}$ for all $\xi \in \mathbb{R}^d$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $\widetilde{c}_0$, $\widetilde{c}_1 \ge 0$, $\widetilde{c}_0 \in L^2(0,T)$.

(d) $(\eta_1 - \eta_2, \xi_1 - \xi_2)_{\mathbb{R}^d} \ge -\widetilde{m}_2 \|\xi_1 - \xi_2\|^2_{\mathbb{R}^d}$ for all $\eta_i \in \partial j(x,t,\xi_i)$, $\xi_i \in \mathbb{R}^d$, $i = 1, 2$, a.e. $(x,t) \in \Gamma_C \times (0,T)$ with $\widetilde{m}_2 \ge 0$.

(e) $j^0(x,t,\xi;-\xi) \le \widetilde{d}_0\,(1 + \|\xi\|_{\mathbb{R}^d})$ for a.e. $(x,t) \in \Gamma_C \times (0,T)$, all $\xi \in \mathbb{R}^d$ with $\widetilde{d}_0 \ge 0$.

Here and below, j^0 and ∂j denote the directional derivative and the Clarke generalized gradient of $j(x,t,\cdot)$, respectively.

We consider the functional $J\colon (0,T) \times L^2(\Gamma_C;\mathbb{R}^d) \to \mathbb{R}$ defined by

$$J(t,v) = \int_{\Gamma_C} j(x,t,v(x))\, d\Gamma \quad \text{for all } v \in L^2(\Gamma_C;\mathbb{R}^d), \text{ a.e. } t \in (0,T). \tag{4.1}$$

We recall the following result which corresponds to Lemma 3.1 of [16].

Lemma 4.1. *Assume that $H(j)$ holds. Then the functional J given by* (4.1) *satisfies the following properties:*

(a) $J(\cdot, v)$ *is measurable for all* $v \in L^2(\Gamma_C;\mathbb{R}^d)$ *and* $J(\cdot,0) \in L^1(0,T)$.

(b) $J(t,\cdot)$ *is locally Lipschitz for a.e.* $t \in (0,T)$.

(c) $\|\partial J(t,v)\|_{L^2(\Gamma_C;\mathbb{R}^d)} \le c_0(t) + c_1 \|v\|_{L^2(\Gamma_C;\mathbb{R}^d)}$ *for all* $v \in L^2(\Gamma_C;\mathbb{R}^d)$, *a.e.* $t \in (0,T)$ *with* $c_0 \in L^2(0,T)$, $c_0, c_1 \ge 0$.

(d) $(z_1 - z_2, w_1 - w_2)_{L^2(\Gamma_C;\mathbb{R}^d)} \ge -\widetilde{m}_2 \|w_1 - w_2\|^2_{L^2(\Gamma_C;\mathbb{R}^d)}$ *for all* $z_i \in \partial J(t,w_i)$, $w_i \in L^2(\Gamma_C;\mathbb{R}^d)$, $i=1,2$, *a.e.* $t \in (0,T)$ *with* $\widetilde{m}_2 \ge 0$.

(e) for all u, $v \in L^2(\Gamma_C;\mathbb{R}^d)$, *we have*

$$J^0(t,u;v) \le \int_{\Gamma_C} j^0(x,t,u(x);v(x))\, d\Gamma, \tag{4.2}$$

where $J^0(t,u;v)$ *denotes the directional derivative of* $J(t,\cdot)$ *at a point* $u \in L^2(\Gamma_C;\mathbb{R}^d)$ *in the direction* $v \in L^2(\Gamma_C;\mathbb{R}^d)$.

We now use Theorem 3.1 and Lemma 4.1 to obtain the following existence and uniqueness result.

Theorem 4.1. *Assume that* $H(A)$, $H(B)$, $H(\mathcal{S})$, $H(j)$, (H_0) *hold and*

$$\alpha > 2\sqrt{3}\, c_0\, c_e^2 \|\gamma\|^2, \qquad m_1 > \widetilde{m}_2\, c_e^2\, \|\gamma\|^2.$$

Then Problem 4.1 *has at least one solution. If, in addition to the above hypotheses, either* $j(x,t,\cdot)$ *or* $-j(x,t,\cdot)$ *is regular on* $\mathbb{R}^d$ *for a.e.* $(x,t) \in \Gamma_C \times (0,T)$, *then Problem* 4.1 *has a unique solution.*

Proof. We define the multifunction $F\colon (0,T) \times V \to 2^{Z^*}$ by

$$F(t,v) = \gamma^* \partial J(t,\gamma v) \ \text{ for all } v \in V, \text{ a.e. } t \in (0,T)$$

where J is given by (4.1). Using the linearity and continuity of the trace operator, the properties of the Clarke subdifferential (cf. Propositions 5.6.9 and 5.6.10 of [2]) and Lemma 4.1, we obtain that F satisfies $H(F)$, see [7, 11] for details. Hence, by Theorem 3.1, we know that there exists a unique solution $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ of the evolution inclusion

$$\begin{cases} u''(t) + A(t,u'(t)) + B(t,u(t)) + (\mathcal{S}u)(t) + F(t,u'(t)) \ni f(t) \text{ a.e. } t \in (0,T), \\ u(0) = u_0, \quad u'(0) = v_0. \end{cases}$$

From Definition 3.1, we have

$$u''(t) + A(t,u'(t)) + B(t,u(t)) + (\mathcal{S}u)(t) + \zeta(t) = f(t), \tag{4.3}$$

for a.e. $t \in (0,T)$ with $\zeta(t) = \gamma^* z(t)$ and $z(t) \in \partial J(t, \gamma u'(t))$ for a.e. $t \in (0,T)$. The latter is equivalent to

$$(z(t), w)_{L^2(\Gamma_C;\mathbb{R}^d)} \le J^0(t, \gamma u'(t); w)$$

for all $w \in L^2(\Gamma_C;\mathbb{R}^d)$ and a.e. $t \in (0,T)$. Hence, using (4.3) and (4.2), we deduce

$$\langle f(t) - u''(t) - A(t,u'(t)) - B(t,u(t)) - (\mathcal{S}u)(t), v\rangle = \langle \zeta(t), v\rangle_{Z^*\times Z} =$$

$$= (z(t), \gamma v)_{L^2(\Gamma_C;\mathbb{R}^d)} \le J^0(t, \gamma u'(t); \gamma v) \le \int_{\Gamma_C} j^0(x,t,\gamma u'(t); \gamma v)\, d\Gamma$$

for all $v \in V$ and a.e. $t \in (0,T)$. This means that u is a solution to Problem 4.1.

Next, let u be a solution to Problem 4.1 obtained above. It follows from Theorem 5.6.38 of [2] that if either $j(x,t,\cdot)$ or $-j(x,t,\cdot)$ is regular for a.e. $(x,t) \in \Gamma_C \times (0,T)$, then either $J(t,\cdot)$ or $-J(t,\cdot)$ is regular for a.e. $t \in (0,T)$, respectively, and (4.2) holds with equality. Using the equality in (4.2), it follows that

$$\langle f(t) - u''(t) - A(t,u'(t)) - B(t,u(t)) - (\mathcal{S}u)(t), v\rangle \le J^0(t,\gamma u'(t); \gamma v)$$

for all $v \in V$ and a.e. $t \in (0,T)$. Then, by Proposition 2(i) of [12], we have

$$\langle f(t) - u''(t) - A(t,u'(t)) - B(t,u(t)) - (\mathcal{S}u)(t), v\rangle \le (J\circ\gamma)^0(t,u'(t); v)$$

for all $v \in V$ and a.e. $t \in (0,T)$. Therefore, by Proposition 2(ii) of [12] combined with the definition of the subdifferential we obtain

$$f(t) - u''(t) - A(t,u'(t)) - B(t,u(t)) - (\mathcal{S}u)(t) \in$$
$$\in \partial(J\circ\gamma)(t,u'(t)) = \gamma^*\partial J(t,\gamma u'(t)) = F(t,u'(t))$$

for a.e. $t \in (0,T)$. Thus u is a solution to the evolutionary inclusion in Problem 3.1. The uniqueness of solution to Problem 4.1 follows now from the uniqueness result of Theorem 3.1, which completes the proof. □

5. Stationary history-dependent inclusions and hemivariational inequalities

In this section we extend the results presented in Sections 3 and 4 to problems which do not include the derivative of the unknown, the so-called stationary problems. To this end, let $A\colon (0,T)\times V \to V^*$, $\mathcal{S}\colon \mathcal{V} \to \mathcal{V}^*$, $F\colon (0,T)\times V \to 2^{Z^*}$, $f\colon (0,T) \to V^*$ be given and consider the following inclusion.

Problem 5.1. *Find $u \in \mathcal{V}$ such that*

$$A(t, u(t)) + (\mathcal{S}u)(t) + F(t, u(t)) \ni f(t) \quad \text{a.e. } t \in (0, T). \tag{5.1}$$

Here $\mathcal{S}$ is assumed to be a history-dependent operator and, therefore, we refer to Problem 5.1 as a *stationary history-dependent inclusion.* We complete this problem with the following definition.

Definition 5.1. *A function $u \in \mathcal{V}$ is called a solution to Problem* 5.1 *if there exists $z \in \mathcal{Z}^*$ such that*

$$\begin{cases} A(t, u(t)) + (\mathcal{S}u)(t) + z(t) = f(t) & a.e.\ t \in (0, T), \\ z(t) \in F(t, u(t)) & a.e.\ t \in (0, T). \end{cases}$$

Before we provide the result on the existence and uniqueness for Problem 5.1, we first state a result on the unique solvability of an inclusion in which the time variable plays the role of a parameter.

Lemma 5.1. *Assume that $H(A)$, $H(F)$ and (H_1) hold, and $f \in \mathcal{V}^*$. Then the problem*

$$A(t, u(t)) + F(t, u(t)) \ni f(t) \quad a.e.\ t \in (0, T) \tag{5.2}$$

has a unique solution $u \in \mathcal{V}$.

Proof. We provide main steps of the proof without details. First, since the operator $A(t, \cdot)$ satisfies $H(A)$(b)–(c), it is pseudomonotone for a.e. $t \in (0, T)$. This follows from the facts that every strongly monotone operator is monotone and every bounded, hemicontinuous and monotone operator is pseudomonotone (cf. Proposition 27.6 of [24]). Subsequently, we define the multivalued map $\mathcal{F}\colon (0, T) \times V \to 2^{V^*}$ by $\mathcal{F}(t, v) = A(t, v) + F(t, v)$ for all $v \in V$ and a.e. $t \in (0, T)$. From $H(A)$(a) and $H(F)$(a), it is clear that $\mathcal{F}(\cdot, v)$ is a measurable multifunction for all $v \in V$. Exploiting Proposition 6.3.66 of [3], we show that $\mathcal{F}(t, \cdot)$ is pseudomonotone and coercive for a.e. $t \in (0, T)$. Therefore, applying the fundamental surjectivity result (cf. e.g. Theorem 6.3.70 of [3]), it follows that $\mathcal{F}(t, \cdot)$ is surjective. This implies that for a.e. $t \in (0, T)$ there exists a solution $u(t) \in V$ of the problem (5.2). Furthermore, owing to the coercivity of $\mathcal{F}(t, \cdot)$, we deduce the following estimate

$$\|u(t)\| \le c\ (1 + \|f(t)\|_{V^*}) \ \text{ for a.e. } t \in (0, T) \text{ with } c > 0. \tag{5.3}$$

Using the strong monotonicity of $A(t, \cdot)$, $H(F)$(d) and the hypothesis $m_1 > m_2$, we prove now that the solution to the problem (5.2) is unique. Also, we prove that the solution of the problem (5.2) is a measurable function on $(0, T)$. Since $f \in \mathcal{V}^*$, from the estimate (5.3), we conclude that $u \in \mathcal{V}$ and (5.2) holds, which completes the proof of the lemma. □

The existence and uniqueness result for Problem 5.1 reads as follows.

Theorem 5.1. *Assume $H(A)$, $H(F)$, $H(\mathcal{S})$, (H_1) and $f \in \mathcal{V}^*$. Then Problem* 5.1 *has a unique solution.*

Proof. We use a fixed point argument. Let $\eta \in \mathcal{V}^*$. We denote by $u_\eta \in \mathcal{V}$ the solution of the following problem

$$A(t, u_\eta(t)) + F(t, u_\eta(t)) \ni f(t) - \eta(t) \quad \text{a.e. } t \in (0, T). \tag{5.4}$$

By Lemma 5.1 we know that $u_\eta \in \mathcal{V}$ exists and it is unique. Next, we consider the operator $\Lambda \colon \mathcal{V}^* \to \mathcal{V}^*$ defined by

$$\Lambda\eta(t) = (\mathcal{S}u_\eta)(t) \quad \text{for all } \ \eta \in \mathcal{V}^*, \text{ a.e. } t \in (0, T).$$

We show by using the Banach Contraction Principle that the operator Λ has a unique fixed point $\eta^* \in \mathcal{V}^*$. Then u_{η^*} is a solution to Problem 5.1, which concludes the existence part of the theorem. The uniqueness part follows from the uniqueness of the fixed point of Λ. Namely, let $u \in \mathcal{V}$ be a solution to Problem 5.1 and define the element $\eta \in \mathcal{V}^*$ by $\eta(t) = (\mathcal{S}u)(t)$ for a.e. $t \in (0, T)$. It follows that u is the solution to the problem (5.4) and, by the uniqueness of solutions to (5.4), we obtain $u = u_\eta$. This implies $\Lambda\eta = \mathcal{S}u_\eta = \mathcal{S}u = \eta$ and by the uniqueness of the fixed point of Λ we have $\eta = \eta^*$, so $u = u_{\eta^*}$, which completes the proof. □

Next, we provide a result on existence and uniqueness of a solution to a class of hemivariational inequalities associated with Problem 5.1, the so-called *stationary history-dependent variational inequalities.* With the notation in Section 4, the problem under consideration reads as follows.

Problem 5.2. *Find $u \in \mathcal{V}$ such that*

$$\langle A(t, u(t)) + (\mathcal{S}u)(t), v\rangle + \int_{\Gamma_C} j^0(x, t, \gamma u(t); \gamma v)\, d\Gamma \geq \langle f(t), v\rangle \tag{5.5}$$

for all $v \in V$ and a.e. $t \in (0, T)$.

From Theorem 5.1, we deduce the following existence and uniqueness result for Problem 5.2.

Theorem 5.2. *Assume that $H(A)$ and $H(\mathcal{S})$ hold, and $f \in \mathcal{V}^*$. If one of the following hypotheses*

i) $\quad H(j)$(a)–(d) *and* $m_1 > \max\{\sqrt{3}\,\bar{c}_1, m_2\}\, c_e^2\, \|\gamma\|^2$

ii) $\quad H(j)$ *and* $m_1 > m_2\, c_e^2\, \|\gamma\|^2$

is satisfied, then Problem 5.2 *has a solution $u \in \mathcal{V}$. If, in addition, either $j(x, t, \cdot)$ or $-j(x, t, \cdot)$ is regular on $\mathbb{R}^d$ for a.e. $(x, t) \in \Gamma_C \times (0, T)$, then the solution of Problem* 5.2 *is unique.*

For the proof of Theorem 5.2 we refer the reader to [18].

6. Applications to contact problems

In this section we study two frictional contact problems which are described by nonmonotone and possibly multivalued boundary conditions of subdifferential type. We show that the contact problems under consideration lead to history-dependent hemivariational inequalities for the displacement and for the velocity, respectively.

The physical setting is as follows. A viscoelastic body occupies a subset Ω of $\mathbb{R}^d$, $d = 2$, 3 in applications. The body is acted upon by volume forces and surface tractions and, as a result, its state is evolving. We are interested in evolution process of the mechanical state of the body on the time interval $[0, T]$ with $0 < T < \infty$. The boundary Γ of Ω is supposed to be Lipschitz continuous and therefore the unit outward normal vector ν exists a.e. on Γ. It is assumed that Γ is divided into three mutually disjoint measurable parts Γ_D, Γ_N and Γ_C such that the measure of Γ_D is positive. We suppose that the body is clamped on Γ_D, so the displacement field vanishes there. Volume forces of density f_0 act in Ω and surface tractions of density f_N are applied on Γ_N. The body may come in contact with an obstacle over the potential contact surface Γ_C. Let $\mathbb{S}^d$ be the linear space of second order symmetric tensors on $\mathbb{R}^d$ (equivalently, the space $\mathbb{R}^{d\times d}_s$ of symmetric matrices of order d) and let $Q = \Omega \times (0, T)$, $\Sigma_D = \Gamma_D \times (0, T)$, $\Sigma_N = \Gamma_N \times (0, T)$ and $\Sigma_C = \Gamma_C \times (0, T)$. For simplicity we skip the dependence of various functions on the spatial variable $x \in \Omega \cup \Gamma$.

The first problem of contact we consider is dynamic. Following [16], its classical formulation is as follows: find the displacement field $u\colon Q \to \mathbb{R}^d$ and the stress tensor $\sigma\colon Q \to \mathbb{S}^d$ such that

$$u''(t) - \operatorname{div} \sigma(t) = f_0(t) \quad \text{in} \quad Q \tag{6.1}$$

$$\sigma(t) = \mathcal{A}(t, \varepsilon(u'(t))) + \mathcal{B}(t, \varepsilon(u(t))) + \int_0^t \mathcal{C}(t-s)\varepsilon(u(s))\, ds \quad \text{in} \quad Q \tag{6.2}$$

$$u(t) = 0 \quad \text{on} \quad \Sigma_D \tag{6.3}$$

$$\sigma(t)\nu = f_N(t) \quad \text{on} \quad \Sigma_N \tag{6.4}$$

$$-\sigma_\nu(t) \in \partial j_\nu(t, u'_\nu(t)), \quad -\sigma_\tau(t) \in \partial j_\tau(t, u'_\tau(t)) \quad \text{on} \quad \Sigma_C \tag{6.5}$$

$$u(0) = u_0, \quad u'(0) = v_0 \quad \text{in} \quad \Omega. \tag{6.6}$$

Note that equation (6.2) represents the constitutive law, where $\mathcal{A}$ is a nonlinear operator describing the purely viscous properties of the material, while $\mathcal{B}$ and $\mathcal{C}$ are the nonlinear elasticity and the linear relaxation operators, respectively which may depend explicitly on time. One-dimensional constitutive laws of the form (6.2) can be constructed by using rheological arguments, see for instance [4], Chapter 6 of [6] and [16].

Conditions (6.5) represent the frictional contact condition in which j_ν

and j_τ are given functions and the subscripts ν and τ indicate normal and tangential components of tensors and vectors. The symbol ∂j denotes the Clarke subdifferential of j with respect to the last variable. Concrete examples of frictional conditions which lead to subdifferential boundary conditions of the form (6.5) with the functions j_ν and j_τ satisfying assumptions $H(j_\nu)$ and $H(j_\tau)$ below can be found in [13, 19]. We only remark that these examples include the viscous contact and the contact with nonmonotone normal damped response, associated to a nonmonotone friction law, to Tresca's friction law or to a power-law friction.

In order to give the variational formulation of the problem (6.1)–(6.6), we recall the following notation. The inner products and the corresponding norms on $\mathbb{R}^d$ and $\mathbb{S}^d$ are defined by

$$u \cdot v = u_i v_i, \qquad \|v\|_{\mathbb{R}^d} = (v \cdot v)^{1/2} \qquad \text{for all } u, v \in \mathbb{R}^d,$$

$$\sigma : \tau = \sigma_{ij}\, \tau_{ij}, \qquad \|\tau\|_{\mathbb{S}^d} = (\tau : \tau)^{1/2} \qquad \text{for all } \sigma, \tau \in \mathbb{S}^d.$$

Summation convention over repeated indices running from 1 to d is adopted and the index that follows a comma indicates a partial derivative. We need the spaces $H = L^2(\Omega;\mathbb{R}^d)$, $\mathcal{H} = L^2(\Omega;\mathbb{S}^d)$, $H_1 = \{u \in H \mid \varepsilon(u) \in \mathcal{H}\}$, $\mathcal{H}_1 = \{\tau \in \mathcal{H} \mid \operatorname{div}\tau \in H\}$, where $\varepsilon\colon H^1(\Omega;\mathbb{R}^d) \to L^2(\Omega;\mathbb{S}^d)$ and $\operatorname{div}\colon \mathcal{H}_1 \to L^2(\Omega;\mathbb{R}^d)$ denote the deformation and the divergence operators, respectively, given by

$$\varepsilon(u) = \{\varepsilon_{ij}(u)\}, \quad \varepsilon_{ij}(u) = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \operatorname{div}\sigma = \{\sigma_{ij,j}\}.$$

Given $v \in H^{1/2}(\Gamma;\mathbb{R}^d)$ we denote by v_ν and v_τ the usual normal and the tangential components of v on the boundary Γ, $v_\nu = v \cdot \nu$, $v_\tau = v - v_\nu \nu$. Similarly, for a smooth tensor field $\sigma\colon \Omega \to \mathbb{S}^d$, we define its normal and tangential components by $\sigma_\nu = (\sigma\nu)\cdot\nu$ and $\sigma_\tau = \sigma\nu - \sigma_\nu\nu$. Let V be the closed subspace of $H^1(\Omega;\mathbb{R}^d)$ given by

$$V = \{\, v \in H^1(\Omega;\mathbb{R}^d) \mid v = 0 \ \text{ on } \Gamma_D \,\}.$$

On the space V we consider the inner product and the corresponding norm defined by

$$\langle u, v\rangle = \langle \varepsilon(u), \varepsilon(v)\rangle_{\mathcal{H}}, \quad \|v\| = \|\varepsilon(v)\|_{\mathcal{H}} \ \text{ for } u, v \in V.$$

It follows from Korn's inequality that $\|\cdot\|_{H^1(\Omega;\mathbb{R}^d)}$ and $\|\cdot\|$ are the equivalent norms on V.

In the study of problem (6.1)–(6.6) we consider the following assumptions on the viscosity operator $\mathcal{A}$, on the elasticity operator $\mathcal{B}$ and on the relaxation operator $\mathcal{C}$.

$\underline{H(\mathcal{A})}$: $\mathcal{A}\colon Q\times\mathbb{S}^d\to\mathbb{S}^d$ is such that

(a) $\mathcal{A}(\cdot,\cdot,\varepsilon)$ is measurable on Q for all $\varepsilon\in\mathbb{S}^d$.

(b) $\mathcal{A}(x,t,\cdot)$ is continuous on $\mathbb{S}^d$ for a.e. $(x,t)\in Q$.

(c) $\|\mathcal{A}(x,t,\varepsilon)\|_{\mathbb{S}^d}\le c_1\,(b(x,t)+\|\varepsilon\|_{\mathbb{S}^d})$ for all $\varepsilon\in\mathbb{S}^d$, a.e. $(x,t)\in Q$ with $b\in L^2(Q)$, $b\ge 0$ and $c_1>0$.

(d) $(\mathcal{A}(x,t,\varepsilon_1)-\mathcal{A}(x,t,\varepsilon_2)):(\varepsilon_1-\varepsilon_2)\ge m_1\|\varepsilon_1-\varepsilon_2\|^2_{\mathbb{S}^d}$ for all ε_1, $\varepsilon_2\in\mathbb{S}^d$, a.e. $(x,t)\in Q$ with $m_1>0$.

(e) $\mathcal{A}(x,t,\varepsilon):\varepsilon\ge c_2\|\varepsilon\|^2_{\mathbb{S}^d}$ for all $\varepsilon\in\mathbb{S}^d$, a.e. $(x,t)\in Q$ with $c_2>0$.

$\underline{H(\mathcal{B})}$: $\mathcal{B}\colon Q\times\mathbb{S}^d\to\mathbb{S}^d$ is such that

(a) $\mathcal{B}(\cdot,\cdot,\varepsilon)$ is measurable on Q for all $\varepsilon\in\mathbb{S}^d$.

(b) $\|\mathcal{B}(x,t,\varepsilon)\|_{\mathbb{S}^d}\le\widetilde{b_1}(x,t)+\widetilde{b_2}\,\|\varepsilon\|_{\mathbb{S}^d}$ for all $\varepsilon\in\mathbb{S}^d$, a.e. $(x,t)\in Q$ with $\widetilde{b_1}\in L^2(Q)$, $\widetilde{b_1}$, $\widetilde{b_2}\ge 0$.

(c) $\|\mathcal{B}(x,t,\varepsilon_1)-\mathcal{B}(x,t,\varepsilon_2)\|_{\mathbb{S}^d}\le L_{\mathcal{B}}\|\varepsilon_1-\varepsilon_2\|_{\mathbb{S}^d}$ for all ε_1, $\varepsilon_2\in\mathbb{S}^d$, a.e. $(x,t)\in Q$ with $L_{\mathcal{B}}>0$.

$\underline{H(\mathcal{C})}$: $\mathcal{C}\colon Q\times\mathbb{S}^d\to\mathbb{S}^d$ is such that $\mathcal{C}(x,t,\varepsilon)=c(x,t)\varepsilon$ and $c(x,t)=\{c_{ijkl}(x,t)\}$ with $c_{ijkl}=c_{jikl}=c_{lkij}\in L^2(0,T;L^\infty(\Omega))$.

The contact and frictional potentials j_ν and j_τ satisfy the following hypotheses.

$\underline{H(j_\nu)}$: $j_\nu\colon\Sigma_C\times\mathbb{R}\to\mathbb{R}$ satisfies

(a) $j_\nu(\cdot,\cdot,r)$ is measurable for all $r\in\mathbb{R}$ and $j_\nu(\cdot,\cdot,0)\in L^1(\Sigma_C)$.

(b) $j_\nu(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t)\in\Sigma_C$.

(c) $|\partial j_\nu(x,t,r)|\le c_\nu\,(1+|r|)$ for a.e. $(x,t)\in\Sigma_C$, all $r\in\mathbb{R}$ with $c_\nu>0$.

(d) $(\eta_1-\eta_2)(r_1-r_2)\ge -m_\nu|r_1-r_2|^2$ for all $\eta_i\in\partial j_\nu(x,t,r_i)$, $r_i\in\mathbb{R}$, $i=1$, 2, a.e. $(x,t)\in\Sigma_C$ with $m_\nu\ge 0$.

(e) $j^0_\nu(x,t,r;-r)\le d_\nu(1+|r|)$ for all $r\in\mathbb{R}$, a.e. $(x,t)\in\Sigma_C$ with $d_\nu\ge 0$.

$\underline{H(j_\tau)}$: $j_\tau\colon\Sigma_C\times\mathbb{R}^d\to\mathbb{R}$ satisfies

(a) $j_\tau(\cdot,\cdot,\xi)$ is measurable for all $\xi\in\mathbb{R}^d$ and $j_\tau(\cdot,\cdot,0)\in L^1(\Sigma_C)$.

(b) $j_\tau(x,t,\cdot)$ is locally Lipschitz for a.e. $(x,t)\in\Sigma_C$.

(c) $\|\partial j_\tau(x,t,\xi)\|_{\mathbb{R}^d}\le c_\tau\,(1+\|\xi\|_{\mathbb{R}^d})$ for a.e. $(x,t)\in\Sigma_C$, all $\xi\in\mathbb{R}^d$ with $c_\tau>0$.

(d) $(\eta_1-\eta_2)\cdot(\xi_1-\xi_2)\ge -m_\tau\|\xi_1-\xi_2\|^2_{\mathbb{R}^d}$ for all $\eta_i\in\partial j_\tau(x,t,\xi_i)$, $\xi_i\in\mathbb{R}^d$, $i=1$, 2, a.e. $(x,t)\in\Sigma_C$ with $m_\tau\ge 0$.

(e) $j^0_\tau(x,t,\xi;-\xi)\le d_\tau\,(1+\|\xi\|_{\mathbb{R}^d})$ for all $\xi\in\mathbb{R}^d$, a.e. $(x,t)\in\Sigma_C$ with $d_\tau\ge 0$.

The volume force and traction densities satisfy

$\underline{H(f)}$: $f_0 \in L^2(0,T;H)$, $f_N \in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^d))$

and the initial data have the regularity

$\underline{H(0)}$: $u_0 \in V$, $v_0 \in H$.

We introduce the operators $A\colon (0,T) \times V \to V^*$, $B\colon (0,T) \times V \to V^*$ and $C\colon (0,T) \times V \to V^*$ defined by

$$\langle A(t,u), v\rangle = \langle \mathcal{A}(t,\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}}, \tag{6.7}$$

$$\langle B(t,u), v\rangle = \langle \mathcal{B}(t,\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}}, \tag{6.8}$$

$$\langle C(t)u, v\rangle = \langle \mathcal{C}(t,\varepsilon(u)), \varepsilon(v)\rangle_{\mathcal{H}}, \tag{6.9}$$

for u, $v \in V$ and $t \in (0,T)$. We also consider the function $f\colon (0,T) \to V^*$ given by

$$\langle f(t), v\rangle = \langle f_0(t), v\rangle_H + (f_N(t), v)_{L^2(\Gamma_N;\mathbb{R}^d)} \quad \text{for } v \in V, \text{ a.e. } t \in (0,T). \tag{6.10}$$

Then, the variational formulation of the contact problem (6.1)–(6.6), in terms of displacement, is the following.

Problem 6.1. Find a displacement field $u \in \mathcal{V}$ such that $u' \in \mathcal{W}$ and

$$\begin{aligned} &\langle u''(t) + A(t,u'(t)) + B(t,u(t)) + \int_0^t C(t-s)u(s)\,ds, v\rangle + \\ &+ \int_{\Gamma_C} \big(j_\nu^0(x,t,u_\nu'(x,t);v_\nu(x)) + j_\tau^0(x,t,u_\tau'(x,t);v_\tau(x))\big)\,d\Gamma \geq \\ &\geq \langle f(t), v\rangle \quad \text{for all } \ v \in V, \ \text{a.e. } t \in (0,T), \\ &u(0) = u_0, \quad u'(0) = v_0. \end{aligned}$$

The unique solvability of Problem 6.1 is given by the following result, which represents an extension of Theorem 5.1 in [16].

Theorem 6.1. *Assume that* $H(\mathcal{A})$, $H(\mathcal{B})$, $H(\mathcal{C})$, $H(j_\nu)$, $H(j_\tau)$, $H(f)$, $H(0)$ *hold,* $c_2 > 2\sqrt{3}\max\{c_\nu, c_\tau\}\, c_e^2\,\|\gamma\|^2$ *and* $m_1 > (m_\nu + m_\tau)\, c_e^2\,\|\gamma\|^2$*. Then Problem* 6.1 *admits at least one solution. If, in addition,*

$$\begin{cases} \textit{either } \ j_\nu(x,t,\cdot) \textit{ and } j_\tau(x,t,\cdot) \textit{ are regular} \\ \textit{or } \ -j_\nu(x,t,\cdot) \textit{ and } -j_\tau(x,t,\cdot) \textit{ are regular} \\ \textit{for a.e. } (x,t) \in \Sigma_C, \end{cases} \tag{6.11}$$

then Problem 6.1 *has a unique solution.*

Proof. The main steps of the proof are the following.

a) Under the assumptions $H(\mathcal{A})$, $H(\mathcal{B})$ and $H(\mathcal{C})$, the operators A, B and C defined by (6.7), (6.8) and (6.9) satisfy condition $H(A)$, $H(B)$ and $H(C)$ in Section 4, respectively.

b) Let $j\colon \Sigma_C \times \mathbb{R}^d \to \mathbb{R}$ be the function defined by

$$j(x,t,\xi) = j_\nu(x,t,\xi_\nu) + j_\tau(x,t,\xi_\tau) \quad \text{a.e. } (x,t) \in \Sigma_C, \text{ all } \xi \in \mathbb{R}^d.$$

It can be shown that, under the assumptions $H(j_\nu)$ and $H(j_\tau)$, the function j satisfies condition $H(j)$ in Section 4 with $\widetilde{c} = \max\{c_\nu, c_\tau\}$ and $\widetilde{m}_2 = m_\nu + m_\tau$.

c) The assumptions $H(f)$ and $H(0)$ combined with (6.10) imply that (H_0) holds. It is clear that (H_1) also is satisfied.

The steps above allow us to apply Theorem 4.1 to obtain the existence of a solution to the hemivariational inequality in Problem 6.1. It can be easily observed that the regularity hypotheses on j_ν, j_τ or $-j_\nu$, $-j_\tau$ imply the regularity of j or $-j$, respectively. In this case by Corollary 4.1, we deduce the uniqueness of a solution to Problem 6.1. □

Denote by σ the function defined by (6.2). Then, the couple (u,σ) is called a *weak solution* of the frictional contact problem (6.1)–(6.6). We conclude, under the hypotheses of Theorem 6.1, that the frictional contact problem (6.1)–(6.6) has at least one weak solution which satisfies $u \in H^1(0,T;V) \cap C(0,T;V)$, $u' \in C(0,T;H)$, $u'' \in L^2(0,T;V^*)$, $\sigma \in C(0,T;\mathcal{H})$, $\operatorname{div}\sigma \in L^2(0,T;V^*)$. If, in addition, (6.11) holds, then the weak solution is unique.

The second problem of contact we consider is quasistatic. Following [19], its classical formulation is as follows: find the displacement field $u\colon Q \to \mathbb{R}^d$ and the stress field $\sigma\colon Q \to \mathbb{S}^d$ such that

$$-\operatorname{div}\sigma(t) = f_0(t) \ \text{ in } \ Q \tag{6.12}$$

$$\sigma(t) = \mathcal{A}(t,\varepsilon(u'(t))) + \mathcal{B}(t,\varepsilon(u(t))) \ \text{ in } \ Q \tag{6.13}$$

$$u(t) = 0 \ \text{ on } \ \Sigma_D \tag{6.14}$$

$$\sigma(t)\nu = f_N(t) \ \text{ on } \ \Sigma_N \tag{6.15}$$

$$-\sigma_\nu(t) \in \partial j_\nu(t, u'_\nu(t)), \quad -\sigma_\tau(t) \in \partial j_\tau(t, u'_\tau(t)) \ \text{ on } \ \Sigma_C \tag{6.16}$$

$$u(0) = u_0 \ \text{ in } \ \Omega. \tag{6.17}$$

The variational formulation of problem (6.12)–(6.17), in terms of velocity, is the following.

Problem 6.2. Find a velocity field $w \in \mathcal{V}$ such that

$$\langle \mathcal{A}(t, \varepsilon(w(t))), \varepsilon(v)\rangle_{\mathcal{H}} + \Big\langle \mathcal{B}\big(t, \varepsilon(\int_0^t w(s)\, ds + u_0)\big), \boldsymbol{\varepsilon}(v)\Big\rangle_{\mathcal{H}} + \\ + \int_{\Gamma_C} \big(j_\nu^0(t, w_\nu(t); v_\nu) + j_\tau^0(t, w_\tau(t); v_\tau)\big)\, d\Gamma \geq \langle f(t), v\rangle$$

for all $v \in V$ and a.e. $t \in (0,T)$.

Note that Problem 6.2 represents a stationary history-dependent hemivariational inequality. Therefore, using Theorem 5.2, we obtain the following result.

Theorem 6.2. *Assume that $H(\mathcal{A})$, $H(\mathcal{B})$ and $H(f)$ hold, and $u_0 \in V$. If one of the following hypotheses*

i) $H(j_\nu)$(a)–(d), $H(j_\tau)$(a)–(d) *and*

$$m_1 > \max\left\{\sqrt{3}(c_\nu + c_\tau), m_\nu, m_\tau\right\}\, c_e^2\, \|\gamma\|^2$$

ii) $H(j_\nu), H(j_\tau)$ *and* $m_1 > \max\{m_\nu, m_\tau\}\, c_e^2\, \|\gamma\|^2$

is satisfied, then Problem 6.2 *has at least one solution. If, in addition,* (6.11) *holds, then the solution of Problem* 6.2 *is unique.*

Let w be a solution of Problem 6.2. Since $w = u'$, by using the initial condition (6.17), it follows that

$$u(t) = \int_0^t w(s)\, ds + u_0 \quad \text{for all} \quad t \in [0,T].$$

Denote by σ the function defined by (6.13). Then, the couple (u, σ) is called a *weak solution* of the frictional contact problem (6.12)–(6.17). We conclude, under the hypotheses of Theorem 6.2, that the frictional contact problem (6.12)–(6.17) has at least one weak solution with the regularity $u \in W^{1,2}(0,T;V)$, $\sigma \in L^2(0,T;\mathcal{H})$, $\operatorname{div} \sigma \in L^2(0,T;V^*)$. If, in addition, the regularity condition (6.11) holds, then the weak solution of is unique.

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References

[1] F.H. Clarke, *Optimization and Nonsmooth Analysis,* Wiley, Interscience, New York, 1983.

[2] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Theory*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[3] Z. Denkowski, S. Migórski and N.S. Papageorgiou, *An Introduction to Nonlinear Analysis: Applications*, Kluwer Academic/Plenum Publishers, Boston, Dordrecht, London, New York, 2003.

[4] A.D. Drozdov, *Finite Elasticity and Viscoelasticity, A Course in the Nonlinear Mechanics of Solids*, World Scientific, Singapore, 1996.

[5] C. Eck, J. Jarušek and M. Krbeč, *Unilateral Contact Problems: Variational Methods and Existence Theorems*, Pure and Applied Mathematics **270**, Chapman/CRC Press, New York, 2005.

[6] W. Han and M. Sofonea, *Quasistatic Contact Problems in Viscoelasticity and Viscoplasticity*, American Mathematical Society/International Press, Providence, RI, 2002.

[7] A. Kulig, Hyperbolic hemivariational inequalities for dynamic viscoelastic contact problems, *Journal of Elasticity*, **110** (2013), 1-31.

[8] A. Kulig and S. Migórski, Solvability and continuous dependence results for second order nonlinear evolution inclusions with a Volterra-type operator, *Nonlinear Analysis*, **75** (2012), 4729-4746.

[9] S. Migórski, Evolution hemivariational inequalities in infinite dimension and their control, *Nonlinear Analysis*, **47** (2001), 101-112.

[10] S. Migórski, Dynamic hemivariational inequality modeling viscoelastic contact problem with normal damped response and friction, *Applicable Analysis*, **84** (2005), 669-699.

[11] S. Migórski, Existence of solutions to nonlinear second order evolution inclusions without and with impulses, *Dynamics of Continuous, Discrete and Impulsive Systems Series B*, **18** (2011), 493-520.

[12] S. Migórski and A. Ochal, Hemivariational inequality for viscoelastic contact problem with slip dependent friction, *Nonlinear Analysis*, **61** (2005), 135-161.

[13] S. Migórski and A. Ochal, A unified approach to dynamic contact problems in viscoelasticity, *J. Elasticity*, **83** (2006), 247-275.

[14] S. Migórski and A. Ochal, Existence of solutions for second order evolution inclusions and their applications to mechanical contact problems, *Optimization*, **55** (2006), 101-120.

[15] S. MIGÓRSKI and A. OCHAL, Quasistatic hemivariational inequality via vanishing acceleration approach, *SIAM J. Math. Anal.*, **41** (2009), 1415-1435.

[16] S. MIGÓRSKI, A. OCHAL and M. SOFONEA, Integrodifferential hemivariational inequalities with applications to viscoelastic frictional contact, *Mathematical Models and Methods in Applied Sciences*, **18** (2008), 271-290.

[17] S. MIGÓRSKI, A. OCHAL and M. SOFONEA, Analysis of a dynamic contact problem for electro-viscoelastic cylinders, *Nonlinear Anal.*, **73** (2010), 1221-1238.

[18] S. MIGÓRSKI, A. OCHAL and M. SOFONEA, History-dependent subdifferential inclusions and hemivariational inequalities in contact mechanics, *Nonlinear Anal. Real World Appl.*, **12** (2011), 3384-3396.

[19] S. MIGÓRSKI, A. OCHAL and M. SOFONEA, *Nonlinear Inclusions and Hemivariational Inequalities. Models and Analysis of Contact Problems*, Advances in Mechanics and Mathematics **26**, Springer, New York, 2013.

[20] Z. NANIEWICZ and P.D. PANAGIOTOPOULOS, *Mathematical Theory of Hemivariational Inequalities and Applications,* Marcel Dekker, Inc., New York, Basel, Hong Kong, 1995.

[21] P.D. PANAGIOTOPOULOS, *Hemivariational Inequalities, Applications in Mechanics and Engineering*, Springer-Verlag, Berlin, 1993.

[22] M. SHILLOR, M. SOFONEA and J.J. TELEGA, *Models and Analysis of Quasistatic Contact*, Lecture Notes in Physics **655**, Springer, Berlin, 2004.

[23] M. SOFONEA and A. MATEI, *Mathematical Models in Contact Mechanics*, London Mathematical Society, Lecture Note Series **398**, Cambridge University Press, 2012.

[24] E. ZEIDLER, *Nonlinear Functional Analysis and Applications II A/B,* Springer, New York, 1990.

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