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Weak solvability via Lagrange multipliers for two frictional contact models

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Abstract - We consider two frictional contact models, for nonlinearly elastic materials. For every model, we deliver a weak formulation as a generalized saddle point problem, and then we prove the existence, uniqueness and stability of weak solution. The proofs rely on abstract results in the study of a class of abstract generalized saddle point problems.

Key words and phrases : contact model, nonlinearly elastic material, friction laws, prescribed normal stress, bilateral contact, Lagrange multipliers, saddle point, fixed point, weak solution.

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1. Introduction

The purpose of the present paper is to solve two frictional contact models which involves nonlinear elastic operators, possibly multi-valued. The envisaged approach is a variational approach which involves dual Lagrange multipliers. In this approach the unknown is a pair of the displacement field and a Lagrange multiplier related to the tangential component of the Cauchy vector on the area of contact. After giving mixed weak formulations, the existence, uniqueness and stability of the solutions will be discussed. The models we analyze herein were studied in [18] for a class of single-valued nonlinear elastic operators; there, a primal variational formulation (in terms of displacement) and a dual variational formulation (in terms of stress) were delivered, together with their analysis.

The present paper follows [12] and [13]. In [12] it was studied the weak solvability via Lagrange multipliers of frictionless unilateral and frictional bilateral contact problems involving single-valued nonlinear elastic operators. In [13] it was studied the weak solvability via Lagrange multipliers of a class of frictionless unilateral contact problem involving nonlinear operators, possibly multi-valued. In contrast to [12] we treat here frictional contact models involving nonlinear elastic operators, possibly multi-valued, the constitutive law being described by a subdifferential inclusion. Besides, the present study envisages not only the frictional bilateral contact condition, but also the frictional contact condition with prescribed normal stress.

In contrast to [13], the present paper focuses on frictional models. The analysis me make here relies on abstract results on generalized saddle point problems, connected to the abstract results in [12] and [13]. For other papers devoted in recent years to mixed variational formulations in mechanics we refer e.g. to [2, 10, 14, 15, 16, 17]. The mixed variational formulations in non-smooth mechanics are suitable to efficiently approximate the weak solutions and this motivates the research in this direction; for modern numerical techniques involving saddle point problems we refer, e.g., to [1, 8, 9, 11].

Since the saddle point of a functional is the key of the variational approaches via Lagrange multipliers, it is worth to recall here its definition.

Definition 1.1. *Let A and B be two non-empty sets. A pair $(u, \lambda) \in A \times B$ is said to be a saddle point of a functional $\mathcal{L} : A \times B \to \mathbb{R}$ if and only if*

$$\mathcal{L}(u, \mu) \le \mathcal{L}(u, \lambda) \le \mathcal{L}(v, \lambda) \quad \text{for all } v \in A,\ \mu \in B.$$

Also, a basic tool of the variational approaches via Lagrange multipliers is the following existence result.

Theorem 1.1. *Let $(X, (\cdot,\cdot)_X, \|\cdot\|_X)$, $(Y, (\cdot,\cdot)_Y, \|\cdot\|_Y)$ be two real Hilbert spaces and let $A \subseteq X$, $B \subseteq Y$ be non-empty, closed, convex subsets. Assume that a real functional $\mathcal{L} : A \times B \to \mathbb{R}$ satisfies the following conditions*

$$\begin{aligned} &v \to \mathcal{L}(v, \mu) \quad \text{is convex and lower semicontinuous} \quad \text{for all } \mu \in B,\\ &\mu \to \mathcal{L}(v, \mu) \quad \text{is concave and upper semicontinuous} \quad \text{for all } v \in A. \end{aligned}$$

Moreover, assume that

$$A \text{ is bounded or } \lim_{\|v\|_X \to \infty, v \in A} \mathcal{L}(v, \mu_0) = \infty \text{ for some } \mu_0 \in B$$

$$\text{and}$$

$$B \text{ is bounded or } \lim_{\|\mu\|_Y \to \infty, \mu \in B} \inf_{v \in A} \mathcal{L}(v, \mu) = -\infty.$$

Then, the functional $\mathcal{L}$ has at least one saddle point.

Details on the saddle point theory and its applications can be found for instance in [3, 4, 5, 7].

2. Abstract auxiliary results

Let $(X, (\cdot, \cdot)_X, \|\cdot\|_X)$ and $(Y, (\cdot, \cdot)_Y, \|\cdot\|_Y)$ be two Hilbert spaces. We make the following assumptions.

Assumption 2.1. $A : X \to X$ is a nonlinear operator such that:

*i*1) $\exists\, m_A > 0 : (Au - Av,\, u - v)_X \ge m_A \|u - v\|_X^2$ for all $u, v \in X$,

*i*2) $\exists\, L_A > 0 : \|Au - Av\|_X \le L_A \|u - v\|_X$ for all $u, v \in X$.

Assumption 2.2. $b : X \times Y \to R$ is a bilinear form such that:

$j1)$ $\exists\, M_b > 0 : |b(v,\mu)| \leq M_b \|v\|_X \|\mu\|_Y$ for all $v \in X$, $\mu \in Y$,

$j2)$ $\exists\, \alpha > 0 : \inf_{\mu \in Y, \mu \neq 0_Y} \sup_{v \in X, v \neq 0_X} \dfrac{b(v,\mu)}{\|v\|_X \|\mu\|_Y} \geq \alpha.$

Assumption 2.3. $\Lambda \subset Y$ is a closed convex bounded set such that $0_Y \in Y$.

Let us consider the following problem.

Problem 2.1. For given $f,\, h \in X$, find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{array}{rcll} (Au,v)_X + b(v,\lambda) & = & (f,\, v)_X & \text{for all } v \in X, \\ b(u,\mu-\lambda) & \leq & b(h,\mu-\lambda) & \text{for all } \mu \in \Lambda. \end{array}$$

The existence, uniqueness and stability of solution for this mixed variational problem was proved in [12] Section 5 for the case Λ unbounded. For the convenience of the reader, we shall justify below the existence, uniqueness and stability of solution of Problem 2.1 in the case Λ bounded. However, in order to avoid repetitions, we shall pick up from Section 5 of [12] the results which can be proved at the some manner.

Theorem 2.1. *[An existence and uniqueness result] If Assumptions* 2.1-2.3 *hold true, then Problem* 2.1 *has a unique solution* $(u,\lambda) \in X \times \Lambda$.

Proof. Let $\eta \in X$ be arbitrarily fixed. We consider the following intermediate problem: for given $f,\, h \in X$, find $u_\eta \in X$ and $\lambda_\eta \in \Lambda$ such that

$$(u_\eta, v)_X + \frac{m_A}{2L_A^2}\, b(v,\lambda_\eta) = (\frac{m_A}{2L_A^2}\, f - \frac{m_A}{2L_A^2}\, A\eta + \eta,\, v)_X \quad v \in X, \tag{2.1}$$

$$b(u_\eta, \mu-\lambda_\eta) \leq b(h,\mu-\lambda_\eta) \quad \mu \in \Lambda. \tag{2.2}$$

As in [12], it can be proved that a pair $(u_\eta,\lambda_\eta) \in X \times \Lambda$ verifies (2.1)-(2.2) if and only if is a saddle point of the functional $\mathcal{L}_\eta : X \times \Lambda \to \mathbb{R}$,

$$\mathcal{L}_\eta(v,\mu) = \frac{1}{2}(v,v)_X - (\frac{m_A}{2L_A^2}\, f - \frac{m_A}{2L_A^2}\, A\eta + \eta, v)_X + \frac{m_A}{2L_A^2} b(v-h,\mu). \tag{2.3}$$

On the other hand, this functional has at least one saddle point. Indeed, $v \to \mathcal{L}_\eta(v,\, \mu)$ is a convex and lower semi-continuous map on X, for all $\mu \in \Lambda$ and $\mu \to \mathcal{L}_\eta(v,\, \mu)$ is a concave and upper semi-continuous map on Λ for all $v \in X$. Besides,

$$\lim_{\|v\|_X \to \infty, v \in X} \mathcal{L}_\eta(v, 0_Y) = \infty.$$

Since Λ is bounded, then Theorem 1.1 ensures us that $\mathcal{L}$ has at least one saddle point.

Consequently, the intermediate problem has at least one solution. Moreover, using the techniques in [12], we can prove that the solution is unique.

Using the unique solution of the intermediate problem we can define a contraction as follows,

$$T : X \to X, \qquad T(\eta) = u_\eta,$$

where u_η is the first component of the pair solution $(u_\eta, \lambda_\eta) \in X \times \Lambda$.

If we denote by η^* the unique fixed point of the operator T, the unique solution of the intermediate problem (2.1)-(2.2) with $\eta = \eta^*$, $(u_{\eta*}, \lambda_{\eta*})$, is the unique solution of Problem 2.1; for details, see Section 5 in [12]. □

We also have a stability result.

Theorem 2.2. *[A stability result] Assumptions* 2.1-2.3 *hold true.*

i) If $h = 0_X$, then there exists $C = C(\alpha, L_A, m_A) > 0$ such that

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \le C\|f_1 - f_2\|_X, \tag{2.4}$$

where (u_1, λ_1), (u_2, λ_2) are two solutions of Problem 2.1 *corresponding to the data f_1, $f_2 \in X$.*

ii) If $h \neq 0_X$, there exists $C = C(\alpha, L_A, m_A, M_b) > 0$ such that

$$\|u_1 - u_2\|_X + \|\lambda_1 - \lambda_2\|_Y \le C(\|f_1 - f_2\|_X + \|h_1 - h_2\|_X), \tag{2.5}$$

where (u_1, λ_1), (u_2, λ_2) are two solutions of Problem 2.1 *corresponding to the data f_1, $h_1 \in X$ and f_2, $h_2 \in X$, $(h_i \neq 0_X$, $i \in \{1, 2\})$.*

This stability result relies on the techniques used in order to prove Theorems 5.7 and 5.8 in [12].

Next, we focus on the following problem.

Problem 2.2. Find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} J(v) - J(u) + b(v - u, \lambda) &\ge (f, v - u)_X \quad &&\text{for all } v \in X \\ b(u, \mu - \lambda) &\le 0 &&\text{for all } \mu \in \Lambda. \end{aligned}$$

Herein J is a functional which fulfills the following assumption.

Assumption 2.4. $J : X \to [0, \infty)$ is a convex lower semicontinuous functional. In addition, there exist m_1, $m_2 > 0$ such that, for all $v \in X$, we have $m_1\|v\|_X^2 \ge J(v) \ge m_2\|v\|_X^2$.

Theorem 2.3. *If Assumptions* 2.4, 2.2 *and* 2.3 *hold true, then Problem* 2.2 *has at least one solution.*

Proof. Let us define

$$\mathcal{L} : X \times \Lambda \to \mathbb{R}\ , \quad \mathcal{L}(v, \mu) = J(v) - (f, v)_X + b(v, \mu).$$

By standard arguments it can be proved that a pair (u, λ) is a solution of Problem 2.2 if and only if it is a saddle point of the functional $\mathcal{L}$, i. e.

$$\mathcal{L}(u, \mu) \leq \mathcal{L}(u, \lambda) \leq \mathcal{L}(v, \lambda), \qquad \forall v \in X, \forall \mu \in \Lambda. \tag{2.6}$$

Besides, the functional $\mathcal{L}$ has at least one saddle point. Indeed, keeping in mind the definition of the functional $\mathcal{L}$, as J is convex and lower semicontinuous and the functional b is bilinear and continuous, it is straightforward to deduce that, for all $\mu \in \Lambda$, $v \to \mathcal{L}(v,\, \mu)$ is convex and lower semicontinuous, and, for all $v \in X$, $\mu \to \mathcal{L}(v,\, \mu)$ is concave and upper semicontinuous. In addition, we note that

$$\mathcal{L}(v, 0_Y) = J(v) + b(v, 0_Y) - (f, v)_X \geq m_2 \|v\|_X^2 - \|f\|_X \|v\|_X,$$

which allows us to say that

$$\lim_{\|v\|_X \to \infty} \mathcal{L}(v, 0_Y) = \infty.$$

As Λ is a bounded subset of the space Y, we can apply Theorem 1.1 to deduce that the functional $\mathcal{L}$ has at least one saddle point. □

Let us make an additional assumption.

Assumption 2.5. $J : X \to [0, \infty)$ is a Gâteaux differentiable functional.

Denoting by ∇J the Gâteaux differential of J then Problem 2.2 is equivalent to the following problem: find $u \in X$ and $\lambda \in \Lambda$ such that

$$\begin{aligned} (\nabla J(u), v)_X + b(v, \lambda) = (f, v)_X \qquad & \text{for all } v \in X, \\ b(u, \mu - \lambda) \leq 0 \qquad & \text{for all } \mu \in \Lambda. \end{aligned}$$

To continue, we introduce a new assumption as follows.

Assumption 2.6.
h_1) $\exists\, m > 0 : (\nabla J(u) - \nabla J(v), u - v)_X \geq m \|u - v\|_X^2 \quad u, v \in X.$
h_2) $\exists\, L > 0 : \|\nabla J(u) - \nabla J(v)\|_X \leq L \|u - v\|_X \quad u, v \in X.$

Denoting ∇J by A, the following theorem is a straightforward consequence of Theorems 2.1 and 2.2.

Theorem 2.4. *If Assumptions* 2.4, 2.2, 2.3, 2.5 *and* 2.6 *hold true, then Problem* 2.2 *has a unique solution, and in addition, there exists* $C > 0$ *such that*

$$\|u - u^*\|_X + \|\lambda - \lambda^*\|_Y \leq C \|f - f^*\|_X$$

where (u, λ) *and* (u^*, λ^*) *are two solutions of Problem* 2.2 *corresponding to the data* $f \in X$ *and* $f^* \in X$.

Remark 2.1. This section focuses on the case Λ bounded because this case fits to the mechanical framework we investigate below. Nevertheless, it is worth to emphasize that replacing Assumption 2.3 with the following one

$$\Lambda \text{ is a closed convex subset of } Y \text{ such that } 0_Y \in \Lambda, \tag{2.7}$$

which is more general, covering the "bounded case" as well as the "unbounded case", then Theorem 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4 are also valid. For details on the case Λ unbounded see Section 5 of [12] and Section 4 in [13].

3. The models and their weak solvability

We consider a body that occupies the bounded domain $\Omega \subset \mathbb{R}^3$, with the boundary partitioned into three measurable parts, Γ_1, Γ_2 and Γ_3, such that $meas(\Gamma_1) > 0$. The unit outward normal vector to Γ is denoted by $\boldsymbol{n}$ and is defined almost everywhere. The body Ω is clamped on Γ_1, body forces of density $\boldsymbol{f}_0$ act on Ω and surface traction of density $\boldsymbol{f}_2$ acts on Γ_2. On Γ_3 the body is in frictional contact with a foundation. We denote by $\boldsymbol{u} = (u_i)$ the displacement field, by $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon}(\boldsymbol{u})$ the infinitesimal strain tensor and by $\boldsymbol{\sigma} = (\sigma_{ij})$ the Cauchy stress tensor. Everywhere below $\overline{\Omega}$ denotes $\Omega \cup \partial\Omega$.

According to the previous physical setting we can state the following boundary value problem.

Problem 3.1. Find $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{S}^3$ such that

$$\mathrm{Div}\, \boldsymbol{\sigma} + \boldsymbol{f}_0 = \boldsymbol{0} \qquad \text{in } \Omega, \tag{3.1}$$

$$\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \qquad \text{in } \Omega, \tag{3.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \qquad \text{on } \Gamma_1, \tag{3.3}$$

$$\boldsymbol{\sigma}\boldsymbol{\nu} = \boldsymbol{f}_2 \qquad \text{on } \Gamma_2, \tag{3.4}$$

$$\sigma_\nu = F,\ \|\boldsymbol{\sigma}_\tau\| \le k|\sigma_\nu|,\ \boldsymbol{\sigma}_\tau = -k|\sigma_\nu|\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \neq \boldsymbol{0} \qquad \text{on } \Gamma_3, \tag{3.5}$$

where $\mathbb{S}^3$ is the space of second-order symmetric tensors on $\mathbb{R}^3$, $\omega : \mathbb{S}^3 \to [0, \infty)$ is a constitutive function, $F : \Gamma_3 \to \mathbb{R}_+$ is the prescribed normal stress and $k : \Gamma_3 \to \mathbb{R}_+$ is the coefficient of friction. We recall that the normal and the tangential components of the Cauchy vector $\boldsymbol{\sigma\nu}$ are given by the formulas $\sigma_\nu = (\boldsymbol{\sigma\nu}) \cdot \boldsymbol{\nu}$, $\boldsymbol{\sigma}_\tau = \boldsymbol{\sigma\nu} - \sigma_\nu \boldsymbol{\nu}$. Besides, the normal and the tangential components on the boundary of the displacement vector are defined as follows $u_\nu = \boldsymbol{u} \cdot \boldsymbol{\nu}$, $\boldsymbol{u}_\tau = \boldsymbol{u} - u_\nu \boldsymbol{\nu}$. Everywhere in this paper, $\|\cdot\|$ denotes the Euclidean norm on $\mathbb{R}^3$ and $\mathbb{S}^3$, and $|\cdot|$ denotes the absolute value of a real number.

Problem 3.1 has the following structure: (3.1) represents the equilibrium equation, (3.2) represents the constitutive law, (3.3) represents the displacements boundary condition, (3.4) represents the traction boundary condition and (3.5) models the frictional contact with prescribed normal stress. For details on this model we send the reader to, e.g., [18].

We shall study the weak solvability of this model under the following assumptions.

Assumption 3.1. $\omega : \mathbb{S}^3 \to [0,\infty)$ is a convex, lower semicontinuous functional. In addition, there exist $\alpha_1, \alpha_2 > 0$ such that $\alpha_1\|\boldsymbol{\varepsilon}\|^2 \geq \omega(\boldsymbol{\varepsilon}) \geq \alpha_2\|\boldsymbol{\varepsilon}\|^2$ for all $\boldsymbol{\varepsilon} \in \mathbb{S}^3$.

Assumption 3.2. ω is Gâteaux differentiable, and in addition

o_1) $\exists\, L > 0 : \|\nabla\omega(\boldsymbol{\varepsilon}) - \nabla\omega(\boldsymbol{\tau})\| \leq L\|\boldsymbol{\varepsilon} - \boldsymbol{\tau}\|$ for all $\boldsymbol{\varepsilon}, \boldsymbol{\tau} \in \mathbb{S}^3$,

o_2) $\exists\, m > 0 : (\nabla\omega(\boldsymbol{\varepsilon}) - \nabla\omega(\boldsymbol{\tau})) \cdot (\boldsymbol{\varepsilon} - \boldsymbol{\tau}) \geq m\|\boldsymbol{\varepsilon} - \boldsymbol{\tau}\|^2$ for all $\boldsymbol{\varepsilon}, \boldsymbol{\tau} \in \mathbb{S}^3$.

An example of such a function is the following one:

$$\omega : \mathbb{S}^3 \to [0,\infty), \quad \omega(\boldsymbol{\varepsilon}) = \frac{1}{2}\mathcal{E}\boldsymbol{\varepsilon} \cdot \boldsymbol{\varepsilon} + \frac{\beta}{2}\|\boldsymbol{\varepsilon} - P_K\boldsymbol{\varepsilon}\|^2$$

where $\mathcal{E}$ is a fourth order symmetric tensor satisfying the ellipticity condition, β is a strictly positive constant, $K \subset \mathbb{S}^3$ denotes a closed convex set which contains the zero element of $\mathbb{S}^3$ and $P_K : \mathbb{S}^3 \to K$ is the projection operator.

Assumption 3.3. The density of the volume forces verifies $\boldsymbol{f}_0 \in L^2(\Omega)^3$ and the density of the tractions verifies $\boldsymbol{f}_2 \in L^2(\Gamma_2)^3$.

Assumption 3.4. The prescribed normal stress verifies $F \in L^2(\Gamma_3)$ and $F(\boldsymbol{x}) \geq 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.

Assumption 3.5. The coefficient of friction verifies $k \in L^\infty(\Gamma_3)$ and $k(\boldsymbol{x}) \geq 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.

Let us replace now (3.5) with the following condition

$$u_\nu = 0,\ \|\boldsymbol{\sigma}_\tau\| \leq \zeta,\ \boldsymbol{\sigma}_\tau = -\zeta\,\frac{\boldsymbol{u}_\tau}{\|\boldsymbol{u}_\tau\|} \text{ if } \boldsymbol{u}_\tau \neq \boldsymbol{0}. \tag{3.6}$$

This condition is a frictional bilateral contact condition where $\zeta : \Gamma_3 \to \mathbb{R}_+$ denotes the friction bound.

Now, a second model can be formulated as follows.

Problem 3.2. Find $\boldsymbol{u} : \overline{\Omega} \to \mathbb{R}^3$ and $\boldsymbol{\sigma} : \overline{\Omega} \to \mathbb{S}^3$ such that (3.1)-(3.4) and (3.6) hold true.

We shall study Problem 3.2 under Assumptions 3.1-3.3, and in addition we shall make the following assumption.

Assumption 3.6. The friction bound verifies $\zeta \in L^2(\Gamma_3)$ and $\zeta(\boldsymbol{x}) \geq 0$ a.e. $\boldsymbol{x} \in \Gamma_3$.

3.1. Weak solvability of Problem 3.1

In order to deliver a weak formulation we assume that $\boldsymbol{u}$ and $\boldsymbol{\sigma}$ are smooth enough functions which verify (3.1)-(3.5). Using a Green's formula, for all $\boldsymbol{v} \in H^1(\Omega)^3$ we have

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega)^3} + \int_{\Gamma} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) \cdot \boldsymbol{\gamma}\, \boldsymbol{v}(\boldsymbol{x})\, d\Gamma.$$

Let us introduce the space

$$V = \{\, \boldsymbol{v} \in H^1(\Omega)^3 \ : \ \boldsymbol{\gamma}\, \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_1\}, \tag{3.7}$$

where $\boldsymbol{\gamma}$ denotes the Sobolev trace operator $\boldsymbol{\gamma} : H^1(\Omega)^3 \to L^2(\Gamma)^3$. We recall that $\boldsymbol{\gamma}$ is a linear, continuous and compact operator. The space V is a Hilbert space endowed with the inner product

$$(\cdot,\cdot)_V : V \times V \to \mathbb{R} \quad (\boldsymbol{u}, \boldsymbol{v})_V = (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}}. \tag{3.8}$$

For all $\boldsymbol{v} \in V$ we have

$$(\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}} = (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega)^3} + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma}\, \boldsymbol{v}(\boldsymbol{x})\, d\Gamma$$
$$+ \int_{\Gamma_3} \boldsymbol{\sigma}(\boldsymbol{x})\boldsymbol{\nu}(\boldsymbol{x}) \cdot \boldsymbol{\gamma}\, \boldsymbol{v}(\boldsymbol{x})\, d\Gamma.$$

Let us introduce the space

$$L^2_s(\Omega)^{3\times 3} = \{\boldsymbol{\mu} = (\mu_{ij}) \ : \ \mu_{ij} \in L^2(\Omega),\ \mu_{ij} = \mu_{ji} \text{ for all } i, j \in \{1, 2, 3\}\}$$

which is a Hilbert space endowed with the inner product

$$(\boldsymbol{\mu}, \boldsymbol{\tau})_{L^2(\Omega)^{3\times 3}} = \int_{\Omega} \mu_{ij}(\boldsymbol{x})\tau_{ij}(\boldsymbol{x})\, dx.$$

We define a functional as follows,

$$W : L^2_s(\Omega)^{3\times 3} \to [0, \infty), \quad W(\boldsymbol{\tau}) = \int_{\Omega} \omega(\boldsymbol{\tau}(\boldsymbol{x}))\, dx. \tag{3.9}$$

Since $\boldsymbol{\sigma}(\boldsymbol{x}) \in \partial\omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))$ almost everywhere in Ω, for all $\boldsymbol{v} \in H^1(\Omega)^3$ we have

$$\omega(\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x}))) - \omega(\boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))) \ \geq \ \boldsymbol{\sigma}(\boldsymbol{x}) \cdot (\boldsymbol{\varepsilon}(\boldsymbol{v}(\boldsymbol{x})) - \boldsymbol{\varepsilon}(\boldsymbol{u}(\boldsymbol{x}))).$$

Hence, we can write

$$W(\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(\boldsymbol{\varepsilon}(\boldsymbol{u})) \geq \int_{\Gamma} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})\, d\Gamma$$
$$-(\text{Div}\ \boldsymbol{\sigma}, \boldsymbol{v} - \boldsymbol{u})_{L^2(\Omega)^3} \qquad \forall \boldsymbol{v} \in H^1(\Omega)^3,$$

and from this

$$W(\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(\boldsymbol{\varepsilon}(\boldsymbol{u})) \geq \int_{\Gamma_2} \boldsymbol{f}_2 \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})\, d\Gamma$$

$$+ \int_{\Gamma_3} \boldsymbol{\sigma\nu} \cdot (\boldsymbol{\gamma v} - \boldsymbol{\gamma u})\, d\Gamma + (\boldsymbol{f}_0, \boldsymbol{v} - \boldsymbol{u})_{L^2(\Omega)^3} \quad \forall \boldsymbol{v} \in V.$$

Using Riesz's representation Theorem, we define $\boldsymbol{f} \in V$ such that, for all $\boldsymbol{v} \in V$,

$$(\boldsymbol{f}, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma - \int_{\Gamma_3} F(\boldsymbol{x}) v_\nu(\boldsymbol{x})\, d\Gamma.$$

Therefore, for all $\boldsymbol{v} \in V$,

$$W(\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(\boldsymbol{\varepsilon}(\boldsymbol{u})) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V + + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, d\Gamma.$$

Next, we define the functional

$$J : V \to [0, \infty), \quad J(\boldsymbol{v}) = W(\boldsymbol{\varepsilon}(\boldsymbol{v})). \tag{3.10}$$

Let D be the dual of the Hilbert space

$$M = \{\widetilde{\boldsymbol{v}} = \boldsymbol{\gamma v}_{|\Gamma_3} \quad \boldsymbol{v} \in V\}.$$

We define $\boldsymbol{\lambda} \in D$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{w}_\tau(\boldsymbol{x})\, d\Gamma \qquad \text{for all } \boldsymbol{w} \in M,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between D and M, and $\boldsymbol{w}_\tau = \boldsymbol{w} - (\boldsymbol{w} \cdot \boldsymbol{\nu}|_{\Gamma_3})\boldsymbol{\nu}|_{\Gamma_3}$. Furthermore, we define a bilinear form as follows,

$$b : V \times D \to \mathbb{R}, \quad b(\boldsymbol{v}, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}_{|\Gamma_3} \rangle, \quad \text{for all } \boldsymbol{v} \in V,\ \boldsymbol{\mu} \in D.$$

Let us introduce the following subset of D,

$$\Lambda = \Big\{ \boldsymbol{\mu} \in D \,:\, \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}|_{\Gamma_3} \rangle \leq \int_{\Gamma_3} k\, F\, \|\boldsymbol{\gamma v}_\tau\|\, d\Gamma \quad \boldsymbol{v} \in V \Big\}. \tag{3.11}$$

It is easy to observe that $\boldsymbol{\lambda} \in \Lambda$. Moreover, by (3.5) it follows that

$$b(\boldsymbol{u}, \boldsymbol{\lambda}) \;=\; \int_{\Gamma_3} k(\boldsymbol{x}) F(\boldsymbol{x}) \|\boldsymbol{\gamma u}_\tau(\boldsymbol{x})\|\, d\Gamma,$$

and, by (3.11),

$$b(\boldsymbol{u}, \boldsymbol{\mu}) \;\leq\; \int_{\Gamma_3} k(\boldsymbol{x}) F(\boldsymbol{x}) \|\boldsymbol{\gamma u}_\tau(\boldsymbol{x})\|\, d\Gamma \qquad \text{for all } \boldsymbol{\mu} \in \Lambda.$$

Consequently, we are led to the following weak formulation of Problem 3.1.

Problem 3.3. Find $\boldsymbol{u} \in V$ and $\boldsymbol{\lambda} \in \Lambda$, such that

$$\begin{aligned} J(\boldsymbol{v}) - J(\boldsymbol{u}) + b(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{\lambda}) &\geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V && \text{for all } \boldsymbol{v} \in V, \\ b(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq 0 && \text{for all } \boldsymbol{\mu} \in \Lambda. \end{aligned}$$

A solution of Problem 3.3 is called a *weak solution* of Problem 3.1.

Theorem 3.1. *If Assumptions* 3.1, 3.3-3.5 *hold true, then Problem* 3.3 *has at least one solution* $(\boldsymbol{u}, \boldsymbol{\lambda}) \in V \times \Lambda$. *If, in addition, Assumption* 3.2 *is fulfilled, then Problem* 3.3 *has a unique solution; moreover, there exists* $C > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_V + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_D \leq C \|\boldsymbol{f} - \boldsymbol{f}^*\|_V, \tag{3.12}$$

where $(\boldsymbol{u}, \boldsymbol{\lambda})$ *and* $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ *are two solutions of Problem* 3.3 *corresponding to the data* $\boldsymbol{f} \in V$ *and* $\boldsymbol{f}^* \in V$.

Proof. Let us set $X = V$, $Y = D$ and $f = \boldsymbol{f}$. Due to Assumption 3.1, the functional J defined by (3.10) fulfills Assumption 2.4. On the other hand, by standard arguments, see e.g. [12], it can be proved that the form b fulfills Assumption 2.2. Also, the set Λ defined by (3.11) fulfills Assumption 2.3. Then, the first assertion of Theorem 3.1 is a straightforward consequence of Theorem 2.3. In addition, due to Assumption 3.2, the functional J fulfills also Assumptions 2.5-2.6. Hence, we conclude Theorem 3.1 applying Theorem 2.4. □

3.2. Weak solvability of Problem 3.2

In order to weakly solve Problem 3.2, we use a technique similar to that used in the previous section. However, the functional frame is different.

In this section we need to introduce the space

$$V_1 = \Big\{ \boldsymbol{v} \in V \mid v_\nu = 0 \text{ a.e. on } \Gamma_3 \Big\}$$

which is a closed subspace of the space V defined in (3.7). Obviously, $(V_1, (\cdot,\cdot)_{V_1}, \|\cdot\|_{V_1})$ is a Hilbert space, where

$$(\cdot,\cdot)_{V_1} : V_1 \times V_1 \to \mathbb{R} \quad (\boldsymbol{u}, \boldsymbol{v})_{V_1} = (\boldsymbol{u}, \boldsymbol{v})_V \quad \text{for all } \boldsymbol{u}, \boldsymbol{v} \in V_1.$$

For all $\boldsymbol{v} \in V_1$ we have

$$\begin{aligned} (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v}))_{L^2(\Omega)^{3\times 3}} &= (\boldsymbol{f}_0, \boldsymbol{v})_{L^2(\Omega)^3} + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma}\, \boldsymbol{v}(\boldsymbol{x})\, d\Gamma \\ &\quad + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{v}_\tau(\boldsymbol{x})\, d\Gamma. \end{aligned}$$

Using Riesz's representation Theorem we define $\boldsymbol{f}_1 \in V_1$ such that, for all $\boldsymbol{v} \in V_1$,

$$(\boldsymbol{f}_1,\, \boldsymbol{v})_V = \int_\Omega \boldsymbol{f}_0(\boldsymbol{x}) \cdot \boldsymbol{v}(\boldsymbol{x})\, dx + \int_{\Gamma_2} \boldsymbol{f}_2(\boldsymbol{x}) \cdot \boldsymbol{\gamma v}(\boldsymbol{x})\, d\Gamma.$$

Therefore, for all $\boldsymbol{v} \in V_1$,

$$W(\boldsymbol{\varepsilon}(\boldsymbol{v})) - W(\boldsymbol{\varepsilon}(\boldsymbol{u})) \geq (\boldsymbol{f}, \boldsymbol{v} - \boldsymbol{u})_V + \int_{\Gamma_3} \boldsymbol{\sigma}_\tau \cdot (\boldsymbol{v}_\tau - \boldsymbol{u}_\tau)\, d\Gamma,$$

with W defined by (3.9). Next, we define the functional

$$J_1 : V_1 \to [0, \infty), \quad J_1(\boldsymbol{v}) = W(\boldsymbol{\varepsilon}(\boldsymbol{v})).$$

Let D_1 be the dual of the Hilbert space

$$M_1 = \{\widetilde{\boldsymbol{v}} = \boldsymbol{\gamma v}_{|\Gamma_3} \quad \boldsymbol{v} \in V_1\}.$$

We define $\boldsymbol{\lambda} \in D_1$ such that

$$\langle \boldsymbol{\lambda}, \boldsymbol{w} \rangle = -\int_{\Gamma_3} \boldsymbol{\sigma}_\tau(\boldsymbol{x}) \cdot \boldsymbol{w}_\tau(\boldsymbol{x})\, d\Gamma, \qquad \text{for all } \boldsymbol{w} \in M_1,$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between D_1 and M_1, and as usual, $\boldsymbol{w}_\tau = \boldsymbol{w} - (\boldsymbol{w} \cdot \boldsymbol{\nu}|_{\Gamma_3})\boldsymbol{\nu}|_{\Gamma_3}$. Furthermore, we define a bilinear form as follows,

$$b : V_1 \times D_1 \to \mathbb{R}, \quad b_1(\boldsymbol{v},\, \boldsymbol{\mu}) = \langle \boldsymbol{\mu}, \boldsymbol{\gamma v}_{|\Gamma_3} \rangle, \quad \text{for all } \boldsymbol{v} \in V_1,\, \boldsymbol{\mu} \in D_1.$$

Let us introduce the following subset of D_1,

$$\Lambda_1 = \left\{ \boldsymbol{\mu} \in D_1 \,:\, \langle \boldsymbol{\mu},\, \boldsymbol{\gamma v}|_{\Gamma_3} \rangle \leq \int_{\Gamma_3} \zeta(\boldsymbol{x})\, \|\boldsymbol{\gamma v}_\tau\|\, d\Gamma \quad \boldsymbol{v} \in V_1 \right\}.$$

Clearly, $\boldsymbol{\lambda} \in \Lambda_1$. Furthermore,

$$\begin{aligned} b_1(\boldsymbol{u}, \boldsymbol{\lambda}) &= \int_{\Gamma_3} \zeta(\boldsymbol{x}) \|\boldsymbol{\gamma u}_\tau(\boldsymbol{x})\|\, d\Gamma, \\ b_1(\boldsymbol{u}, \boldsymbol{\mu}) &\leq \int_{\Gamma_3} \zeta(\boldsymbol{x}) \|\boldsymbol{\gamma u}_\tau(\boldsymbol{x})\|\, d\Gamma \qquad \text{for all } \boldsymbol{\mu} \in \Lambda_1. \end{aligned}$$

Consequently, we are led to the following weak formulation of Problem 3.2.

Problem 3.4. Find $\boldsymbol{u} \in V_1$ and $\boldsymbol{\lambda} \in \Lambda_1$, such that

$$\begin{aligned} J_1(\boldsymbol{v}) - J_1(\boldsymbol{u}) + b_1(\boldsymbol{v} - \boldsymbol{u}, \boldsymbol{\lambda}) &\geq (\boldsymbol{f}_1,\, \boldsymbol{v} - \boldsymbol{u})_V && \text{for all } \boldsymbol{v} \in V_1, \\ b_1(\boldsymbol{u}, \boldsymbol{\mu} - \boldsymbol{\lambda}) &\leq 0 && \text{for all } \boldsymbol{\mu} \in \Lambda_1. \end{aligned}$$

Setting $X = V_1$, $Y = D_1$, $\Lambda = \Lambda_1$ and $f = \boldsymbol{f}_1$, the following theorem is a straightforward consequence of Theorem 2.4 in Section 2.

Theorem 3.2. *If Assumptions* 3.1-3.3 *and* 3.6 *hold true, then Problem* 3.4 *has a unique solution; moreover, there exists* $C > 0$ *such that*

$$\|\boldsymbol{u} - \boldsymbol{u}^*\|_{V_1} + \|\boldsymbol{\lambda} - \boldsymbol{\lambda}^*\|_{D_1} \leq C\|\boldsymbol{f} - \boldsymbol{f}^*\|_{V_1}$$

where $(\boldsymbol{u}, \boldsymbol{\lambda})$ *and* $(\boldsymbol{u}^*, \boldsymbol{\lambda}^*)$ *are two solutions of Problem* 3.4 *corresponding to the data* $\boldsymbol{f} \in V_1$ *and* $\boldsymbol{f}^* \in V_1$.

A solution of Problem 3.4 is called a *weak solution* of Problem 3.2.

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