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Compositions of stochastic systems with final sequence states and interdependent transitions

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Abstract - In this paper the compositions of stochastic systems with final sequence states and interdependent transitions are studied. The ordered and unordered sequential compositions and excludable and nonexcludable parallel compositions are analyzed. For these compositions the problem of determining the main probabilistic characteristics (distribution low, expectation, variance, n-order initial moments) of the evolution time is considered. The elaborated polynomial algorithms are based on the main properties of degenerated homogeneous linear recurrences.

Key words and phrases : stochastic system with final sequence states, evolution time, sequential composition, parallel composition, homogeneous linear recurrence.

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1. Introduction

In the last years various modifications of the continuous and discrete Markov processes have been intensively studied. These stochastic systems are applied for solving many important problems from some actual domains: economy, technology, medicine, industry, biology and others.

The main properties of Markov stochastic systems were described in [2], [3], [12] and [14]. Various stochastic models were analyzed and simulated in [5]. Some recent applications of discrete Markov processes are described in [4], [10], [11] and [15].

The stochastic systems with final sequence states generalize the discrete Markov processes. For these systems a stopping condition is defined. In this case the problem of determining the main probabilistic characteristics of evolution time of the system, is important to be solved. This problem was studied in [7] and [10]. The obtained algorithms have polynomial complexity and are based on the main properties of homogeneous linear recurrences, the basic properties of generating function (described in [13]) and numerical derivation of regular rational fractions.

In this paper the compositions of stochastic systems with final sequence states and interdependent transitions are studied. These systems represent an extension of stochastic systems with final sequence states, studied in [8] and [9], and generalize the results obtained in [6]. The ordered and unordered sequential compositions and the excludable and nonexcludable parallel compositions are analyzed. For these composed stochastic systems the efficient methods for determining the main probabilistic characteristics (expectation, variance, initial moments) of evolution time are elaborated.

2. Statement of the problem

In this section the problems that will be analyzed in this paper are formulated. The stochastic systems with final sequence states and interdependent transitions ($IDSSFSS$) are described. The sequential and parallel compositions of these systems are presented. For all studied models the evolution time is defined.

2.1. IDSSFSS

Let us consider a discrete stochastic system L with the set of possible states denoted by V, such that $|V| = \omega < \infty$. The state of the system at every moment of time $t \in \mathbb{N}$ is denoted by $v(t) \in V$.

Let $p^*(v)$ represents the probability that $v(0) = v$, $v \in V$, and $p(u, v)$ represents the probability of transition from the state $u \in V$ to $v \in V$.

Let $x_1, x_2, \ldots, x_m \in V$ be fixed. We say that the system stops, when it passes through all the states $x_1, x_2, \ldots, x_m$, consecutively. The moment of time when the system stops is denoted by T and it is called the stopping time of the system. Since the starting time of the system is equal to 0, the stopping time T represents the evolution time of the stochastic system L. Our goal is to characterize the evolution time T for determining its expectation, variance and initial moments.

2.2. Sequential and parallel compositions of IDSSFSS

The sequential and parallel compositions of IDSSFSS are defined in the similar way that the definition of the sequential and parallel compositions of ISSFSS (stochastic systems with final sequence states and independent transitions), introduced in [6]. We consider IDSSFSS $L^{[k]}$ with evolution time $T^{[k]}$, $k = \overline{1, s}$.

Definition 2.1. *The stochastic system $L_S^{[0]}$, whose evolution is obtained by concatenating the evolutions of the systems $L^{[k]}$, $k = \overline{1, s}$, is called sequential composition of these systems. If the concatenation is performed in fixed* *order, then the sequential composition is called ordered, otherwise is called unordered.*

Let S_s be the set of all permutations of degree s. We consider an arbitrary permutation $\delta \in S_s$. We suppose that the sequential composed system $L_S^{[0]}$ goes from final state of the component system $L^{[\delta(k)]}$ to the initial state of the successor component system $L^{[\delta(k+1)]}$ in $\tau_{\delta(k),\ \delta(k+1)}$ time units. So, for the system $L_S^{[0]}$ the matrix $\Lambda = (\tau_{j,\ k})_{j,\ k=\overline{1,\ s}}$ of transit time through intermediate systems is defined.

If the system $L_S^{[0]}$ represents an ordered sequential composition of the systems $L^{[k]}$, $k = \overline{1, s}$, generated by permutation $\delta \in S_s$, then we denote $L_S^{[0]} = \sum\limits_{k=1}^{s} L^{[\delta(k)]}[\Lambda]$.

Let $\phi : S_s \to [0, 1]$ be a probability function and $\delta[\phi]$ be a discrete random variable with distribution ϕ. If the stochastic system $L_{S,\ \phi}^{[0]}$ represents an unordered sequential composition of the systems $L^{[k]}$, $k = \overline{1, s}$, generated by distribution ϕ of order, then we write $L_{S,\ \phi}^{[0]} = \sum\limits_{k=1}^{s} L^{[\delta[\phi](k)]}[\Lambda]$.

Definition 2.2. *The stochastic system $L_P^{[0]}$, whose evolution is formed by simultaneous evolutions of the systems $L^{[k]}$, $k = \overline{1, s}$, is called parallel composition of these systems. A parallel composition is called excludable if the finishing evolution of every component system interrupts immediately the evolution of all component systems. A parallel composition is called nonexcludable if the finishing evolution of every component system does not interrupt the evolution of other component systems.*

If L_m represents an excludable parallel composition of the stochastic systems $L^{[k]}$, $k = \overline{1, s}$, then we denote $L_m = \bigcap\limits_{k=1}^{s} L^{[k]}$. If L_M represents a nonexcludable parallel composition of the stochastic systems $L^{[k]}$, $k = \overline{1, s}$, then we write $L_M = \bigcup\limits_{k=1}^{s} L^{[k]}$.

3. Homogeneous linear recurrences

The elaborated algorithms for probabilistic characterization of the evolution time are based on the theory of the homogeneous linear recurrences. In this section the main properties of these recurrences are described.

3.1. Definitions and notations

Next we remind the main definitions and notations from [6], [7] and [10]. We consider an arbitrary subfield K of the field $\mathbb{C}$.

Definition 3.1. *The sequence* $a = \{a_n\}_{n=0}^{\infty} \subseteq \mathbb{C}$ *is called homogeneous linear* $m-$*recurrent sequence on the set* K *if there exists* $q = (q_k)_{k=0}^{m-1} \in K^m$ *such that* $a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}$*, for all* $n \geq m$*. The vector* q *represents the generating vector and the vector* $I_m^{[a]} = (a_n)_{n=0}^{m-1}$ *is called initial state of the sequence* a*. If* $q_{m-1} \neq 0$ *then the sequence* a *is called non-degenerated; else it is called degenerated.*

Definition 3.2. *The sequence* a *is called homogeneous linear recurrent sequence on the set* K *if there exists* $m \in \mathbb{N}^*$ *such that the sequence* a *represents a homogeneous linear* m*-recurrent sequence on the set* K*.*

We introduce the following notations:

$Rol[K][m]$ is the set of non-degenerated homogeneous linear m-recurrent sequences on the set K;
$Rol[K]$ is the set of non-degenerated homogeneous linear recurrent sequences on the set K;
$G[K][m](a)$ is the set of generating vectors of length m of $a \in Rol[K][m]$;
$G[K](a)$ is the set of generating vectors of $a \in Rol[K]$;

$Rol^*[K][m]$ is the set of homogeneous linear m-recurrent sequences on K;
$Rol^*[K]$ is the set of homogeneous linear recurrent sequences on K;
$G^*[K][m](a)$ is the set of generating vectors of length m of $a \in Rol^*[K][m]$;
$G^*[K](a)$ is the set of generating vectors of $a \in Rol^*[K]$.

Definition 3.3. *The function* $G^{[a]}(z) = \sum_{n=0}^{\infty} a_n z^n$ *is called generating function of the sequence* $a = (a_n)_{n=0}^{\infty} \subseteq \mathbb{C}$ *and the function* $G_t^{[a]}(z) = \sum_{n=0}^{t-1} a_n z^n$ *is called partial generating function of order* t *of the sequence* $a = (a_n)_{n=0}^{\infty} \subseteq \mathbb{C}$*.*

Definition 3.4. *Let* $a \in Rol^*[K][m]$ *and* $q \in G^*[K][m](a)$*. For the sequence* a *we will consider the unit characteristic polynomial* $H_m^{[q]}(z) = 1 - zG_m^{[q]}(z)$ *and the characteristic equation* $H_m^{[q]}(z) = 0$*. For an arbitrary* $\alpha \in K^*$ *the polynomial* $H_{m,\alpha}^{[q]}(z) = \alpha H_m^{[q]}(z)$ *is called characteristic polynomial of the sequence* a *of order* m*.*

The following notations are introduced:

$H[K][m](a)$ is the set of characteristic polynomials of order m of the sequence $a \in Rol[K][m]$;
$H[K](a)$ is the set of characteristic polynomials of the sequence $a \in Rol[K]$;
$H^*[K][m](a)$ is the set of characteristic polynomials of order m of the sequence $a \in Rol^*[K][m]$;
$H^*[K](a)$ is the set of characteristic polynomials of the sequence $a \in Rol^*[K]$.

Definition 3.5. *The sequence $a \in Rol^*[K]$ is called m-minimal on the set K if $a \in Rol^*[K][m]$ and $a \notin Rol^*[K][t]$, for all $t < m$. The number m is called dimension of sequence a on the set K (denote $dim[K](a) = m$).*

3.2. Main properties of homogeneous linear recurrences

The main properties of homogeneous linear recurrences are described by the following theorems. The proofs of these theorems are based on the proofs of corresponding theorems from [6], presented also in [10].

The next theorem represents the formula for determining the generating function:

Theorem 3.1. *If $a \in Rol^*[K][m]$ and $q \in G^*[K][m](a)$, then*

$$G^{[a]}(z) = \frac{G_m^{[a]}(z) - \sum\limits_{k=0}^{m-1} q_k z^{k+1} G_{m-1-k}^{[a]}(z)}{H_m^{[q]}(z)} \, . \tag{3.1}$$

Proof. Let $a \in Rol^*[K][m]$ and $q \in G^*[K][m](a)$. Similarly than in [10],

$$\begin{aligned} G^{[a]}(z) - G_m^{[a]}(z) &= \sum_{n=m}^{\infty} a_n z^n = \sum_{n=m}^{\infty} z^n \sum_{k=0}^{m-1} q_k a_{n-1-k} \\ &= z \sum_{k=0}^{m-1} q_k z^k \sum_{n=m}^{\infty} a_{n-1-k} z^{n-1-k} \\ &= z \sum_{k=0}^{m-1} q_k z^k (G^{[a]}(z) - G_{m-1-k}^{[a]}(z)), \end{aligned}$$

that implies the formula (3.1). In this way we obtain the assertion. □

The following theorem is more important and allows to increase the order of homogeneous linear recurrence $a \in Rol^*[K][m]$:

Theorem 3.2. *If $a \in Rol^*[K][m]$ and $P(z) \in H^*[K][m](a)$ then $a \in Rol^*[K][m+1]$ and $Q(z) = (z - \alpha)P(z) \in H^*[K][m+1](a)$, $\forall \alpha \in K^*$.*

Proof. Let $a \in Rol^*[K][m]$, $P(z) \in H^*[K][m](a)$ and $r = deg(P(z))$. We consider the subsequence $b = (b_n)_{n=0}^{\infty}$ of the sequence a, where $b_n = a_{n+m-r}$, for all $n \geq 0$. It is easy to observe that $b \in Rol[K][r]$ and $P(z) \in H[K][r](b)$. Applying Theorem 2 from [6], we obtain $b \in Rol[K][r+1]$ and $Q(z) \in H[K][r+1](b)$, that implies the assertion. □

The function $L.C.M.$ means the least common multiple of respectively polynomials. The following properties hold:

Theorem 3.3. *Let $a^{(j)} \in Rol^*[K][m_j]$, $P_j(z) \in H^*[K][m_j](a^{(j)})$, $\alpha_j \in \mathbb{C}$, $j = \overline{1,t}$. Then $a = \sum_{k=1}^{t} \alpha_k a^{(k)} \in Rol^*[K][m]$ with characteristic polynomial $P(z) = L.C.M.((P_j(z))_{j=1}^{t}) \in H^*[K][m](a)$, where $m = r + s$, $r = deg(P(z))$, $r_j = deg(P_j(z))$, $j = \overline{1,t}$ and $s = \max_{j=\overline{1,t}}(m_j - r_j)$.*

Proof. Let $a^{(j)} \in Rol^*[K][m_j]$, $P_j(z) \in H^*[K][m_j](a^{(j)})$, $\alpha_j \in \mathbb{C}$, $j = \overline{1,t}$. We consider the subsequence $b^{(j)} = (b_n^{(j)})_{n=0}^{\infty}$ of the sequence $a^{(j)}$, where $b_n^{(j)} = a_{n+m_j-r_j}^{(j)}$, $j = \overline{1,t}$, for all $n \geq 0$. It is easy to observe that

$$b^{(j)} \in Rol[K][r_j] \text{ and } P_j(z) \in H[K][r_j](b^{(j)}),\ j = \overline{1,t}.$$

Next, we consider the subsequence $c^{(j)} = (c_n^{(j)})_{n=0}^{\infty}$ of the sequence $b^{(j)}$, where $c_n^{(j)} = a_{n+s}^{(j)}$, $j = \overline{1,t}$, for all $n \geq 0$. We have $c^{(j)} \in Rol[K][r_j]$ and $P_j(z) \in H[K][r_j](c^{(j)})$, $j = \overline{1,t}$. Applying Theorem 3 from [6], we obtain $c = (a_{n+s})_{n=0}^{\infty} \in Rol[K][r]$ and $P(z) \in H[K][r](c)$, that implies $a \in Rol^*[K][r+s]$ and $P(z) \in H^*[K][r+s](a)$. □

Theorem 3.4. *Let $a \in Rol^*[\mathbb{C}][m_1]$, $b \in Rol^*[\mathbb{C}][m_2]$, $u \in G^*[\mathbb{C}][m_1](a)$ and $v \in G^*[\mathbb{C}][m_2](b)$. Suppose that all distinct roots z_k of multiplicity s_k, $k = \overline{0, p-1}$, of the polynomial $H_{m_1}^{[u]}(z)$ and all distinct roots z_l^* of multiplicity s_l^*, $l = \overline{0, p^*-1}$, of the polynomial $H_{m_2}^{[v]}(z)$ are known. Then $ab \in Rol^*[\mathbb{C}][m]$ and $P(z) = L.C.M.(\{(z - z_k z_l^*)^{s_k+s_l^*-1} \mid k = \overline{0,p-1},\ l = \overline{0,p^*-1}\})$ belongs to $H^*[\mathbb{C}][m](ab)$, where $m = r + s$, $r_1 = deg(H_{m_1}^{[u]}(z))$, $r_2 = deg(H_{m_2}^{[v]}(z))$, $r = deg(P(z))$ and $s = \max\{m_1 - r_1,\ m_2 - r_2\}$.*

Proof. Let $a \in Rol^*[\mathbb{C}][m_1]$, $b \in Rol^*[\mathbb{C}][m_2]$, $u \in G^*[\mathbb{C}][m_1](a)$ and $v \in G^*[\mathbb{C}][m_2](b)$. We consider the subsequence $c^{(1)} = (c_n^{(1)})_{n=0}^{\infty}$ of the sequence a and the subsequence $c^{(2)} = (c_n^{(2)})_{n=0}^{\infty}$ of the sequence b, where $c_n^{(1)} = a_{n+m_1-r_1}$ and $c_n^{(2)} = b_{n+m_2-r_2}$, for all $n \geq 0$. It is easy to observe that $c^{(1)} \in Rol[\mathbb{C}][r_1]$ and $c^{(2)} \in Rol[\mathbb{C}][r_2]$ with generating vectors $u \in G[\mathbb{C}][r_1](c^{(1)})$ and $v \in G[\mathbb{C}][r_2](c^{(2)})$. Next, we consider the subsequence $d^{(j)} = (d_n^{(j)})_{n=0}^{\infty}$ of the sequence $c^{(j)}$, $j = \overline{1,2}$, where $d_n^{(1)} = a_{n+s}$ and $d_n^{(2)} = b_{n+s}$, for all $n \geq 0$. We have $d^{(j)} \in Rol[\mathbb{C}][r_j]$, $j = \overline{1,2}$, $u \in G[\mathbb{C}][r_1](d^{(1)})$ and $v \in G[\mathbb{C}][r_2](d^{(2)})$. Applying Theorem 4 from [6], we obtain $d = d^{(1)}d^{(2)} \in Rol[\mathbb{C}][r]$ and $P(z) \in H[\mathbb{C}][r](d)$, that implies $ab \in Rol^*[\mathbb{C}][r+s]$ and $P(z) \in H^*[\mathbb{C}][r+s](ab)$. □

Theorem 3.5. *For each polynomial $P(X) \in \mathbb{C}[X]$ with $deg(P(X)) = m$, $c = (P(n))_{n=0}^{\infty} \in Rol[\mathbb{R}][m+1]$ and $Q(z) = (1-z)^{m+1} \in H[\mathbb{R}][m+1](c)$.*

Using Definition 3.5 and Theorems 3.3 and 3.4, we obtain:

Theorem 3.6. *The main properties of the dimension are:*

1. $dim[K]\left(\sum\limits_{k=1}^{t}\alpha_k a^{(k)}\right) \le \sum\limits_{k=1}^{t} dim[K](a^{(k)}),\ \forall a^{(k)} \in Rol^*[K],\ \alpha_k \in \mathbb{C},$ $k=\overline{1,t};$

2. $dim[\mathbb{C}]\left(\prod\limits_{k=1}^{t} a^{(k)}\right) \le \prod\limits_{k=1}^{t} dim[\mathbb{C}](a^{(k)}),\ \forall a^{(k)} \in Rol^*[\mathbb{C}],\ k=\overline{1,t}.$

Proof. In this proof we use the notations from the Theorems 3.3 and 3.4.

1. For $\forall a^{(k)} \in Rol^*[K]$ and $\alpha_k \in \mathbb{C}$, $k=\overline{1,t}$, we have

$$\begin{aligned} dim[K]\left(\sum_{k=1}^{t}\alpha_k a^{(k)}\right) &\le r+s \le \sum_{k=1}^{t} r_k + \sum_{k=1}^{t}(m_k - r_k) \\ &= \sum_{k=1}^{t} m_k = \sum_{k=1}^{t} dim[K](a^{(k)}); \end{aligned}$$

2. We use the mathematical induction method. For $t=2$ and $\forall a^{(k)} \in Rol^*[\mathbb{C}]$, $k=\overline{1,2}$, we have

$$\begin{aligned} dim[\mathbb{C}](a^{(1)}a^{(2)}) &\le r+s \le r_1 r_2 + (m_1 - r_1)(m_2 - r_2) \\ &= m_1 m_2 - r_1(m_2 - r_2) - r_2(m_1 - r_1) \\ &\le m_1 m_2 = dim[\mathbb{C}](a^{(1)})dim[\mathbb{C}](a^{(2)}). \end{aligned}$$

Let us consider true the affirmation for all $2 \le t \le T$, so,

$$dim[\mathbb{C}]\left(\prod_{k=1}^{t} a^{(k)}\right) \le \prod_{k=1}^{t} dim[\mathbb{C}](a^{(k)}),\ \forall a^{(k)} \in Rol^*[\mathbb{C}],\ k=\overline{1,t}.$$

For $t=T+1$ we obtain

$$\begin{aligned} dim[\mathbb{C}]\left(\prod_{k=1}^{T+1} a^{(k)}\right) &\le dim[\mathbb{C}]\left(\prod_{k=1}^{T} a^{(k)}\right) dim[\mathbb{C}](a^{(T+1)}) \\ &\le \prod_{k=1}^{T+1} dim[\mathbb{C}](a^{(k)}). \end{aligned}$$

So, the property is true for all $t \ge 2$.

□

The dimension and the unique minimal generating vector of the sequence $a \in Rol^*[\mathbb{C}][m]$ can be determined by using the following minimization method:

Theorem 3.7. *If $a \in Rol^*[\mathbb{C}][m]$ is a not null sequence, then $dim[\mathbb{C}](a) = R$ and $q = (q_0, q_1, \ldots, q_{R-1}) \in G^*[\mathbb{C}][R](a)$, where*

$$R = rank(A_m^{[a]}),\ A_n^{[a]} = (a_{i+j})_{i,j=\overline{0,n-1}},\ f_n^{[a]} = (a_k)_{k=\overline{n,2n-1}}, \forall n \geq 1$$

and the vector $x = (q_{R-1}, q_{R-2}, \ldots, q_0)$ represents the unique solution of the system $A_R^{[a]} x^T = (f_R^{[a]})^T$.

The homogeneous linear recurrent property of distribution function is:

Theorem 3.8. *If $a \in Rol^*[K][m]$, $P(z) \in H^*[K][m](a)$ and $b_n = G_n^{[a]}(1)$, $n = \overline{0,\infty}$, then $b = (b_n)_{n=0}^{\infty} \in Rol^*[K][m+1]$ and $Q(z) = (z-1)P(z)$ belongs to $H^*[K][m+1](b)$.*

Proof. Let $P(z) = H_{m,\ \lambda}^{[q]}(z)$ and $\lambda = P(0)$. We have $Q(z) = -\lambda R(z)$, where $R(z) = 1 - z \sum\limits_{k=0}^{m} q_k^* z^k$ and the coefficients q_k^*, $k = \overline{0,m}$, are given by the relations $q_0^* = q_0 + 1$, $q_k^* = q_k - q_{k-1}$, $k = \overline{1, m-1}$, $q_m^* = -q_{m-1}$. For all $n \geq m+1$ we obtain

$$\begin{aligned}
\sum_{k=0}^{m} q_k^* b_{n-1-k} &= (q_0+1)b_{n-1} - q_{m-1}b_{n-1-m} + \sum_{k=1}^{m-1}(q_k - q_{k-1})b_{n-1-k} \\
&= b_{n-1} + \sum_{k=0}^{m-1} q_k(b_{n-1-k} - b_{n-2-k}) = b_{n-1} + \sum_{k=0}^{m-1} q_k a_{n-2-k} \\
&= b_{n-1} + a_{n-1} = b_n.
\end{aligned}$$

So, $b \in Rol^*[K][m+1]$ and $Q(z) \in H^*[K][m+1](b)$. □

3.3. Homogeneous linear recurrent distributions

In this section a new algorithm for determining the main probabilistic characteristics of natural random variables with homogeneous linear recurrent distribution is elaborated. The elaborated algorithm is based only on properties of homogeneous linear recurrences. The proof of the algorithm is similar that for the non-degenerated case from [6].

We consider a natural random variable ξ. Let $a_n = P(\xi = n)$, $n = \overline{0,\infty}$. The sequence $a = (a_n)_{n=0}^{\infty}$ is called distribution of random variable ξ and is noted $a = rep(\xi)$.

Theorem 3.9. *If $a = rep(\xi) \in Rol^*[\mathbb{C}]$, then $a \in Rol^*[\mathbb{R}]$ and the formula $dim[\mathbb{R}](a) = dim[\mathbb{C}](a)$ holds.*

Proof. Let $a = rep(\xi) \in Rol^*[\mathbb{C}][m]$, $q \in G^*[\mathbb{C}][m](a)$ and $m = dim[\mathbb{C}](a)$. Then $b = (a_n)_{n=m-r}^{\infty} \in Rol[\mathbb{C}][r]$ and $Q = (q_k)_{k=0}^{r-1} \in G[\mathbb{C}][r](b)$, where

$r = deg(H_m^{[q]}(z))$, that implies $c = (c_n)_{n=0}^{\infty} = b/(1-S) \in Rol[\mathbb{C}][r]$ and $Q \in G[\mathbb{C}][r](c)$, where $S = \sum_{k=0}^{m-r-1} a_k$. Since $\sum_{n=0}^{\infty} c_n = 1$, we can consider a random variable η with distribution $c = rep(\eta)$. Applying Theorem 9 from [6], we obtain $c \in Rol[\mathbb{R}][r]$ and $Q \in G[\mathbb{R}][r](c)$, that implies $b \in Rol[\mathbb{R}][r]$ and $Q \in G[\mathbb{R}][r](b)$. So, $a \in Rol^*[\mathbb{R}][m]$ and $q \in G^*[\mathbb{R}][m](a)$. Since $dim[\mathbb{R}](a) \geq dim[\mathbb{C}](a) = m$ and $a \in Rol^*[\mathbb{R}][m]$, we have the formula $dim[\mathbb{R}](a) = dim[\mathbb{C}](a) = m$. □

Theorem 3.10. *If $a = rep(\xi) \in Rol^*[\mathbb{C}][m]$, $q \in G^*[\mathbb{C}][m](a)$ and the relation $H_m^{[q]}(1) = 0$ holds, then $dim[\mathbb{R}](a) \neq m$ and the root $z = 1$ is fictive.*

Proof. Let $a = rep(\xi) \in Rol^*[\mathbb{C}][m]$, $q \in G^*[\mathbb{C}][m](a)$ and $H_m^{[q]}(1) = 0$. Then $b = (a_n)_{n=m-r}^{\infty} \in Rol[\mathbb{C}][r]$ and $Q = (q_k)_{k=0}^{r-1} \in G[\mathbb{C}][r](b)$, where $r = deg(H_m^{[q]}(z))$, that implies $c = (c_n)_{n=0}^{\infty} = b/(1-S) \in Rol[\mathbb{C}][r]$ and $Q \in G[\mathbb{C}][r](c)$, where $S = \sum_{k=0}^{m-r-1} a_k$. Since $\sum_{n=0}^{\infty} c_n = 1$, we can consider a random variable η with distribution $c = rep(\eta)$. Applying Theorem 10 from [6], we obtain $dim[\mathbb{R}](c) \neq r$ and the root $z = 1$ of the polynomial $H_m^{[q]}(z)$ is fictive. So, $dim[\mathbb{R}](a) \neq m$ and the root $z = 1$ of the polynomial $H_m^{[q]}(z)$ is fictive. □

Taking into account these results, we can consider only homogeneous linear recurrent distributions on $\mathbb{R}$ with minimal characteristic polynomial that do not have as a root 1. The following result allows us to calculate the initial moment $\nu_k(\xi)$ of order $k \geq 1$ of natural random variable ξ with distribution $a = rep(\xi) \in Rol^*[\mathbb{R}][m]$.

Theorem 3.11. *Let ξ be a natural random variable, $a = rep(\xi) \in Rol^*[\mathbb{R}][m]$ and $q \in G^*[\mathbb{R}][m](a)$. Then*

$$c^{(k)} = (n^k a_n)_{n=0}^{\infty} \in Rol^*[\mathbb{R}][M_k],\ q^{(k)} \in G^*[\mathbb{R}][M_k](c^{(k)})$$

and

$$\nu_k(\xi) = G^{[c^{(k)}]}(1),\ for\ all\ k \geq 1, \tag{3.2}$$

where $M_k = m(k+1)$ and

$$H_{M_k}^{[q^{(k)}]}(z) = (H_m^{[q]}(z))^{k+1} \in H^*[\mathbb{R}][M_k](c^{(k)}). \tag{3.3}$$

Proof. Let $\forall k \geq 1$. We consider $d^{(k)} = (n^k)_{n=0}^{\infty}$. We have $c^{(k)} = d^{(k)} a$. Using Theorem 3.5, we obtain

$$d^{(k)} \in Rol[\mathbb{R}][k+1]\ and\ (1-z)^{k+1} \in H[\mathbb{R}][k+1](d^{(k)}).$$

Applying Theorem 3.4, we have $c^{(k)} \in Rol^*[\mathbb{C}][m+kp]$ and

$$P_k(z) = L.C.M.(\{(z-z_t)^{s_t+k} \mid t = \overline{0,p-1}\})$$
$$= H_m^{[q]}(z)\prod_{t=0}^{p-1}(z-z_t)^k \in H^*[\mathbb{C}][m+kp](c^{(k)}),$$

where z_t are all distinct roots of respective multiplicity s_t, $t = \overline{0,p-1}$, of polynomial $H_m^{[q]}(z)$.

Since the polynomial $P_k(z) = H_m^{[q]}(z)\left(\prod_{t=0}^{p-1}(z-z_t)\right)^k$ divides the polynomial $H_{M_k}^{[q^{(k)}]}(z) = (H_m^{[q]}(z))^{k+1} \in \mathbb{R}[z]$, using the result of Theorem 3.2 and the inequality $deg(H_m^{[q]}(z)) \leq m$, we have

$$c^{(k)} \in Rol^*[\mathbb{R}][M_k],\ q^{(k)} \in G^*[\mathbb{R}][M_k](c^{(k)})\ \textit{and}\ H_{M_k}^{[q^{(k)}]}(z) \in H^*[\mathbb{R}][M_k](c^{(k)}).$$

Next, we obtain the formula $\nu_k(\xi) = \sum_{n=0}^{\infty} n^k a_n = G^{[c^{(k)}]}(1)$. These values can be determined by (3.1) and (3.3). $\square$

The expectation $E(\xi)$, the variance $V(\xi)$ and the mean square deviation $\sigma(\xi)$ are obtained by the formulas

$$E(\xi) = \nu_1(\xi);\ V(\xi) = \nu_2(\xi) - (\nu_1(\xi))^2;\ \sigma(\xi) = \sqrt{V(\xi)}. \tag{3.4}$$

Taking into account these results, we obtain a new algorithm for *probabilistic characterization of random variables with homogeneous linear recurrent distributions*:

Algorithm 1.

Input: $q \in G^*[\mathbb{R}][m](a)$, $I_m^{[a]} \in \mathbb{R}^m$, where $a = rep(\xi) \in Rol^*[\mathbb{R}][m]$.
Output: $E(\xi)$, $V(\xi)$, $\sigma(\xi)$, $\nu_k(\xi)$, $k = \overline{1,t}$, $t \geq 2$.

1. If $H_m^{[q]}(1) = 0$, go to Step 2, else go to Step 3.
2. Using Horner schema, the vector $q^* \in \mathbb{R}^{m-1}$ is determined, vector that satisfies $H_{m-1}^{[q^*]}(z) = \dfrac{H_m^{[q]}(z)}{1-z}$. Next, set $q := q^*$ and $m := m-1$ and go to Step 1.
3. The values $a_n = \sum_{k=0}^{m-1} q_k a_{n-1-k}$, $n = \overline{m, m(t+1)-1}$, are calculated.
4. For each $k = \overline{1,t}$ the next steps are executed:
   (a) The initial state $I_{M_k}^{[c^{(k)}]} = (n^k a_n)_{n=0}^{m(k+1)-1}$ is determined.

(b) The generating vector $q^{(k)}$ from the formula (3.3) is obtained.

(c) The value $\nu_k(\xi)$ using the relations (3.2) and (3.1) is calculated.

5. The values $E(\xi)$, $V(\xi)$, $\sigma(\xi)$ are obtained by using the formula (3.4).

4. Main results

In this section the elaborated methods for solving the problems introduced in Section 2 are described. All the results are theoretically grounded.

4.1. IDSSFSS

The IDSSFSS are described in Section 2.1 and the evolution time T of the system is defined. We consider the distribution $a = rep(T)$.

In [7] and [10], the properties of sequence a were studied. We obtained that $a \in Rol^*[\mathbb{R}][m\omega]$ with generating vector $q \in G^*[\mathbb{R}][m\omega](a)$ that satisfies the inequality $deg(H^{[q]}_{m\omega}(z)) \leq 2m + \omega - 2$. The dimension and the minimal generating vector of the sequence a are determined using Theorem 3.7 and the first $2m\omega$ elements of the sequence a. These elements can be calculated using formulas from [10]. The detailed description of the elaborated algorithm can be found in [10]. Using the obtained generating vector and initial state, applying Algorithm 1 from this paper, we can determine the main probabilistic characteristics of evolution time of given IDSSFSS.

4.2. Ordered and unordered sequential compositions

The sequential compositions are defined in Section 2.2. In this section the ordered and unordered sequential compositions are studied. These results are similarly with the results obtained in [6] for the sequential compositions of the stochastic systems with independent transitions.

We consider $L = \sum\limits_{k=1}^{s} L^{[\delta(k)]}[\Lambda]$, where $\delta \in S_s$ and $\Lambda = (\tau_{j,\ k})_{j,\ k=\overline{1,\ s}}$. Let T be the evolution time of composition L and $T^{[k]}$ be the evolution time of the stochastic system $L^{[k]}$, $k = \overline{1,s}$. We consider that $T^{[k]} < +\infty$, $k = \overline{1,s}$. For probabilistic characterization of the evolution time T the following lemma is necessary.

Lemma 4.1. *We consider the independent random variables $\xi^{(j)}$, $j = \overline{1,n}$. Let $\nu^*_{kj} = \nu_k(\xi^{(j)})$ be the k-order moment of the variable $\xi^{(j)}$, $j = \overline{1,n}$, for all $k \in \mathbb{N}$. Then the k-order moment of the random variable $\xi = \sum\limits_{j=1}^{n} \xi^{(j)}$ is $\nu_k(\xi) = \nu_{kn}$, where $\nu_{k1} = \nu^*_{k1}$ and $\nu_{kj} = \sum\limits_{i=0}^{k} C^i_k \nu_{i,j-1} \nu^*_{k-i,j}$, $j = \overline{2,n}$, $\forall k \in \mathbb{N}$.*

Proof. We denote $U_j = \sum\limits_{i=1}^{j} \xi^{(i)}$, $j = \overline{1, n}$. We have

$$U_1 = \xi^{(1)},\ U_j = U_{j-1} + \xi^{(j)},\ j = \overline{2, n}.$$

Let $k \geq 0$. It is easy to observe that $\nu_k(U_1) = \nu_k(\xi^{(1)})$ and, for $j = \overline{2, n}$,

$$\nu_k(U_j) = E((U_{j-1} + \xi^{(j)})^k) = \sum_{i=0}^{k} C_k^i E(U_{j-1}^i) E((\xi^{(j)})^{k-i})$$
$$= \sum_{i=0}^{k} C_k^i \nu_i(U_{j-1}) \nu_{k-i}(\xi^{(j)}).$$

Denoting $\nu_{kj}^* = \nu_k(\xi^{(j)})$, $\nu_{kj} = \nu_k(U_j)$, $j = \overline{1, n}$, we obtain the assertion. □

We observe that $T = \sum\limits_{k=1}^{s} T^{[k]} + \kappa_\Lambda(\delta)$ holds, where $\kappa_\Lambda(\delta) = \sum\limits_{k=1}^{s-1} \tau_{\delta(k), \delta(k+1)}$. The random variables $T^{[k]}$, $k = \overline{1, s}$, are independent and the value $\kappa_\Lambda(\delta)$ denotes a constant. So, we can apply Lemma 4.1 for independent random variables $\xi^{(k)} = T^{[k]}$, $k = \overline{1, s}$ and $\xi^{(s+1)} = \kappa_\Lambda(\delta)$ and obtain the moments $\nu_k(T)$, $k = \overline{0, r}$. The expectation $E(T)$, the variance $V(T)$ and the mean square deviation $\sigma(T)$ can be determined by using the relation (3.4).

We consider the unordered sequential composition $\overline{L} = \sum\limits_{k=1}^{s} L^{[\delta[\phi](k)]}[\Lambda]$ with evolution time $\overline{T}$. We have $\overline{T} = T^* + \kappa_\Lambda(\delta[\phi])$, where $T^* = \sum\limits_{k=1}^{s} T^{[k]}$. The moments $\nu_k(T^*)$, $k = \overline{0, r}$, can be determined using Lemma 4.1 for independent random variables $\xi^{(k)} = T^{[k]}$, $k = \overline{1, s}$. It is easy to observe that the random variable T^* does not depend on order generated by random permutation $\delta[\phi]$. Applying the Newton's formula, for $k = \overline{0, r}$ we have

$$\nu_k(\overline{T}) = E(E((T^* + \kappa_\Lambda(\delta[\phi]))^k \mid \delta[\phi])) = \sum_{j=0}^{k} C_k^j \nu_{k-j}(T^*) \nu_j(\kappa_\Lambda(\delta[\phi])),$$

where $\nu_j(\kappa_\Lambda(\delta[\phi])) = \sum\limits_{\delta \in S_s} \phi(\delta)(\kappa_\Lambda(\delta))^j$, $j = \overline{0, r}$.

If there exists $j \in \{1, 2, \ldots, s\}$ such that $T^{[j]} = +\infty$, then the expectation and the moments of the evolution time T are unbounded.

4.3. Excludable and nonexcludable parallel compositions

The parallel compositions are defined in Section 2.2. In this section the excludable and nonexcludable parallel compositions are studied. These results are similarly with the results obtained in [6] for the parallel compositions of the stochastic systems with independent transitions.

We consider the excludable parallel composition $L_m = \bigcap_{k=1}^{s} L^{[k]}$ and the nonexcludable parallel composition $L_M = \bigcup_{k=1}^{s} L^{[k]}$. Let T_m be the evolution time of excludable composition L_m, T_M be the evolution time of nonexcludable composition L_M and $T^{[j]}$ be the evolution time of component system $L^{[j]}$, $j = \overline{1,s}$. Suppose that $T^{[k]} < +\infty$, $k = \overline{1,s}$. It is easy to observe that $T_m = \min_{j=\overline{1,s}} T^{[j]}$ and $T_M = \max_{j=\overline{1,s}} T^{[j]}$.

Let $a^{(j)} = rep(T^{[j]})$, $j = \overline{1,s}$, $a^* = rep(T_m)$, $A^* = rep(T_M)$. In Section 4.1 we obtained that $\exists M_j \in \mathbb{N}^*$ such that $a^{(j)} \in Rol^*[\mathbb{R}][M_j]$, $j = \overline{1,s}$.

Let $F_n = P(T_m < n)$, $G_n = P(T_M < n)$, $F_n^{(j)} = P(T^{[j]} < n)$, $j = \overline{1,s}$, $n = \overline{0,\infty}$. We consider the sequences

$$F = (F_n)_{n=0}^{\infty},\ G = (G_n)_{n=0}^{\infty},\ F^{(j)} = (F_n^{(j)})_{n=0}^{\infty},\ j = \overline{1,s}.$$

We have

$$\begin{aligned}
F_n &= P(T_m < n) = 1 - P(T^{[j]} \geq n,\ j = \overline{1,s}) \\
&= 1 - \prod_{j=1}^{s} P(T^{[j]} \geq n) = 1 - \prod_{j=1}^{s} (1 - F_n^{(j)}), \\
G_n &= P(T_M \leq n-1) = P(T^{[j]} \leq n-1,\ j = \overline{1,s}) \\
&= \prod_{j=1}^{s} P(T^{[j]} \leq n-1) = \prod_{j=1}^{s} F_n^{(j)}, \\
a_n^* &= F_{n+1} - F_n,\ A_n^* = G_{n+1} - G_n, \\
F_n^{(j)} &= G_n^{[a^{(j)}]}(1) = \sum_{i=0}^{n-1} a_i^{(j)},\ j = \overline{1,s},\ n = \overline{0,\infty}.
\end{aligned}$$

Let $M^* = \prod_{j=1}^{s} (M_j + 1)$, $r^* = \prod_{j=1}^{s} (M_j + 2)$ and $m^* = r^* + 1$. Using Theorem 3.8, we obtain $F^{(j)} \in Rol^*[\mathbb{R}][M_j + 1]$, $j = \overline{1,s}$.

Applying Theorem 3.6, we obtain $G \in Rol^*[\mathbb{R}][M^*]$. Since $\overline{G} = (G_n)_{n=1}^{\infty}$ represents a subsequence of the sequence G, we have $\overline{G} \in Rol^*[\mathbb{R}][M^*]$ with the same generating vector. Using Theorem 3.3, we obtain that $A^* = \overline{G} - G$ belongs to $Rol^*[\mathbb{R}][M^*]$.

By applying three times Theorem 3.6, we obtain $F \in Rol^*[\mathbb{R}][m^*]$. Since the sequence $\overline{F} = (F_{n+1})_{n=0}^{\infty}$ is a subsequence of the sequence F, we have $\overline{F} \in Rol^*[\mathbb{R}][m^*]$. By using Theorem 3.3, we obtain that $a^* = \overline{F} - F$ belongs to $Rol^*[\mathbb{R}][m^*]$.

The values a_k^*, $k = \overline{0, 2m^* - 1}$ and A_l^*, $l = \overline{0, 2M^* - 1}$, are obtained by using the formulas described above. Using the minimization method (see Theorem 3.7), the dimensions $d^* = dim[\mathbb{R}](a^*)$ and $D^* = dim[\mathbb{R}](A^*)$

together with the minimal generating vectors $q^* \in G^*[\mathbb{R}][d^*](a^*)$ and $Q^* \in G^*[\mathbb{R}][D^*](A^*)$ are calculated. Next, we can apply Algorithm 1 to determine the probabilistic characteristics of evolution times T_m and T_M.

If there exists $j \in \{1, 2, \ldots, s\}$ such that $T^{[j]} = +\infty$, then the expectation and the moments of the evolution time T_M of nonexcludable parallel composition L_M are unbounded and $L_m = \bigcap_{k \in \{1,2,\ldots,s\} \setminus \{j\}} L^{[k]}$.

5. Numerical example

Example 5.1. We consider two IDSSFSS L_1 and L_2 with the same set of states $V = \{v_1, v_2\}$, the same initial distribution of states $p^*(v_1) = 0.3$, $p^*(v_2) = 0.7$ and the same transition probabilities $p(v_1, v_1) = p(v_2, v_1) = 0.3$, $p(v_1, v_2) = p(v_2, v_2) = 0.7$. Let $X_1 = (v_1, v_2)$ be the final sequence states of the system L_1 and $X_2 = (v_2, v_1)$ be the final sequence states of L_2. Consider the compositions $L_S = (L_1 + L_2)[\Lambda]$, $L_m = L_1 \cap L_2$ and $L_M = L_1 \cup L_2$, where $\Lambda = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. The goal is to determine the expectation, the variance, the mean square deviation and the moments of order 1 and 2 of the evolution time of stochastic systems L_1, L_2, L_S, L_m, L_M.

Solution.

Let T_k be the evolution time of the system L_k and $a^{(k)} = rep(T_k)$, $k \in \{1, 2, S, m, M\}$. Applying the method described in Section 4.1, we obtain that $a = a^{(1)} = a^{(2)} \in Rol^*[\mathbb{R}][4]$, $q^* = (1, -0.21, 0, 0) \in G^*[\mathbb{R}][4](a)$ and $I_4^{[a]} = (0, 0.21, 0.21, 0.1659)$. So, we have $a_n = a_{n-1} - 0.21a_{n-2}$, $\forall n \geq 4$. Using Theorem 3.7, we obtain $a \in Rol^*[\mathbb{R}][2]$, $q = (1, -0.21) \in G^*[\mathbb{R}][2](a)$ and $I_2^{[a]} = (0, 0.21)$.

Using the method elaborated and grounded in Section 4.3, we have that $a^{(m)} \in Rol^*[\mathbb{R}][3]$, $q^{(m)} = (0.79, -0.1659, 0.0093) \in G^*[\mathbb{R}][3](a^{(m)})$ and $I_3^{[a^{(m)}]} = (0, 0.3759, 0.2877)$. Also, we obtain that $a^{(M)} \in Rol^*[\mathbb{R}][5]$, $q^{(M)} = (1.79, -1.1659, 0.3411, -0.0441, 0.0019) \in G^*[\mathbb{R}][5](a^{(M)})$ and $I_5^{[a^{(M)}]} = (0, 0.0441, 0.1323, 0.1669, 0.1576)$.

Applying Algorithm 1 for the sequences a, $a^{(m)}$, $a^{(M)}$ and method elaborated in Section 4.2 for the random variables $T^{(1)}$, $T^{(2)}$, $T^{(S)}$, we obtain the following final results:

k	*system*	*evol. time*	$\nu_1(T_k)$	$\nu_2(T_k)$	$E(T_k)$	$V(T_k)$	$\sigma(T_k)$
1	L_1	T_1	3.7619	22.5420	3.7619	8.3901	2.8966
2	L_2	T_2	3.7619	22.5420	3.7619	8.3901	2.8966
S	L_S	T_S	9.5238	107.483	9.5238	16.7802	4.0964
m	L_m	T_m	2.3003	7.5943	2.3003	2.3029	1.5175
M	L_M	T_M	5.2236	37.4896	5.2236	10.2036	3.1943

All obtained results can be verified using Monte Carlo method described in [1]. Also, we can compare these results with the solution of Example 1 from [6], since these problems are equivalent.

References

[1] J. M. Hammersley and D. C. Handscomb, *Monte Carlo Methods*, London, Methuen & Co. Ltd., 1967.

[2] R. A. Howard, *Dynamic Probabilistic Systems*, Wiley, 1972.

[3] R.A. Howard, *Dynamic Programming and Markov Processes*, Wiley, 1960.

[4] Q. Hu and W. Yue, *Markov Decision Processes with Their Applications*, Springer-Verlag, 2008.

[5] A. M. Law and W. D. Kelton *Simulation Modelling and Analysis*, New York, McGraw-Hill, Second Edition, 1991.

[6] A. Lazari, Evolution time of composed stochastic systems with final sequence states and independent transitions, *Analele Ştiinţifice ale Universităţii "Alexandru Ioan Cuza" din Iaşi, România*, (submitted in september 2012).

[7] A. Lazari, Caracteristicile probabilistice ale timpului de evoluţie al sistemelor aleatoare discrete, *Studia Universitatis, CEP USM*, **22**, *2* (2009), 5-16.

[8] A. Lazari, *Characteristics of evolution time of random discrete systems with interdependent states and algorithm for their determination*, in Proceedings of the Conferences "Mathematics & Information Technologies: Research and Education" (MITRE 2008-2009), Chişinău, USM, pp. 23-33, 2010.

[9] A. Lazari, *Generalized and optimized algorithm for determining probabilistic characteristics of the evolution time of random discrete systems*, in Communications of the scientific conference "Actual problems of mathematics and informatics", Chişinău, pp. 115-117, 2010.

[10] A. Lazari, *Numerical methods for solving deterministic and stochastic problems in dynamic decision systems* (in romanian), PhD thesis in mathematics and physics, Chişinău, 2011.

[11] D. Lozovanu and S. Pickl, Algorithms for determining the state-time probabilities and the limit matrix in Markov chains, *Bulletin of the Academy of Science of RM*, **65**, *1* (2011), 66-82.

[12] M. Puterman, *Markov Decision Processes*, Wiley, 1993.

[13] A. N. Shiryaev, *Probability*, Springer-Verlag, 1984.

[14] W. J. Stewart *Introduction to the Numerical Solution of Markov Chains*, Princeton University Press, 1994.

[15] W. J. Stewart *Probability, Markov Chains, Queues and Simulation*, Princeton University Press, 2009.

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