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A class of implicit evolution inequalities and applications to dynamic contact problems

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Abstract - This paper deals with the analysis of a class of implicit variational inequalities which generalizes some dynamic contact problems coupling adhesion and friction between two viscoelastic bodies of Kelvin-Voigt type. Existence and uniqueness results are proved for a general system of evolution inequalities that constitutes a unified approach to study some complex dynamic surface interactions, including rebonding, debonding and friction conditions. The proofs are based on incremental formulations, several estimates, compactness arguments and a fixed point technique. These results are applied to dynamic frictional contact conditions with reversible adhesion and the coefficient of friction depending on the slip velocity.

Key words and phrases : implicit inequalities, dynamic problems, variational solutions, adhesion, friction, viscoelasticity.

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1. Introduction

The aim of this paper is to study a system of evolution inequalities that generalizes some interaction laws including dynamic contact, recoverable adhesion and friction between two viscoelastic bodies, when the coefficient of friction depends on the slip velocity, which represents a more realistic model than the one described in [11].

An interface model coupling unilateral contact, irreversible adhesion and local friction for elastic bodies was considered in the quasistatic case in [26] and its mathematical analysis has been provided in [9].

Nonlocal friction laws, given by appropriate regularizations of the normal component of the stress vector which occurs in the Coulomb friction conditions, and normal compliance models have been considered by several authors in the (quasi)static case, see, e.g. [17, 27] and references therein.

Dynamic frictional contact problems with normal compliance laws for a viscoelastic body have been studied in [23, 17, 18, 5] and dynamic unilateral or bilateral contact problems with friction for viscoelastic bodies have been considered in [16, 12, 19, 20, 10].

More recently, dynamic frictionless problems with adhesion were studied in [6, 21, 29] and a dynamic viscoelastic problem coupling unilateral contact, recoverable adhesion and nonlocal, mathematically consistent, friction was analyzed in [11].

Some interesting relaxed unilateral contact condition with pointwise friction have been proposed in [25] in the static case and extended in [8] to the quasistatic case.

This paper is organized as follows. First, existence and uniqueness results are proved for a general system of evolution inequalities that constitutes a unified approach to study some complex dynamic surface interactions, including rebonding, debonding and friction conditions.

Second, the classical formulation of a dynamic contact problem with adhesion and friction is presented and a corresponding variational formulation is given as a system which contains an implicit evolution variational inequality coupled with a parabolic variational inequality.

Finally, based on the previous abstract results, the existence and uniqueness of variational solutions are analyzed.

2. Analysis of a system of evolution inequalities

Let $(H_0, |.|, (.\,,.))$, $(V_0, \|.\|, \langle .\,,.\rangle)$, $(U_0, \|.\|_{U_0})$ and $(\Pi_0, |.|_{\Pi_0}, (.\,,.)_{\Pi_0})$ be four Hilbert spaces such that $V_0 \subset U_0 \subseteq H_0$, the imbedding from V_0 into U_0 is compact and V_0 dense in H_0.

Let Λ_0 be a closed convex set in Π_0 such that $0 \in \Lambda_0$. Suppose also that Λ_0 is bounded, to simplify the estimates.

Define two bilinear and symmetric forms, $a_0, b_0 : V_0 \times V_0 \to \mathbb{R}$ and the mapping $\gamma_0 : V_0 \times \Pi_0 \times \Pi_0 \to \mathbb{R}$ such that

$$\exists\, m_a, m_b > 0 \quad a_0(u,v) \le m_a \|u\| \, \|v\|, \quad b_0(u,v) \le m_b \|u\| \, \|v\|, \tag{2.1}$$

$$\exists\, A, B > 0 \quad a_0(v,v) \ge A \|v\|^2, \quad b_0(v,v) \ge B \|v\|^2 \quad \forall\, u, v \in V_0, \tag{2.2}$$

$$\forall\, u \in V_0, \quad \gamma_0(u,\cdot,\cdot) \text{ is a bilinear and symmetric form,} \tag{2.3}$$

$$\exists\, m_\gamma > 0 \text{ such that } \forall\, u_{1,2} \in V_0, \forall\, \delta_{1,2} \in \Lambda_0, \forall\, \eta \in \Pi_0,$$

$$|\gamma_0(u_1,\delta_1,\eta) - \gamma_0(u_2,\delta_2,\eta)| \le m_\gamma (\|u_1 - u_2\| + |\delta_1 - \delta_2|_{\Pi_0}) \, |\eta|_{\Pi_0}, \tag{2.4}$$

$$\gamma_0(u,\eta,\eta) \ge 0 \quad \forall\, u \in V_0, \ \forall\, \eta \in \Pi_0. \tag{2.5}$$

Let $\phi_0 : [0,T] \times \Lambda_0 \times V_0^3 \to \mathbb{R}$ be a mapping such that

$$\phi_0(t,\eta,\cdot,\cdot,\cdot) \text{ is sequentially weakly continuous,} \tag{2.6}$$

$$\phi_0(t,\eta,u,v,w_1+w_2) \le \phi_0(t,\eta,u,v,w_1) + \phi_0(t,\eta,u,v,w_2), \tag{2.7}$$

$$\phi_0(t,\eta,u,v,\theta w) = \theta\, \phi_0(t,\eta,u,v,w), \tag{2.8}$$

$$\phi_0(0,0,0,0,w) = 0, \tag{2.9}$$

$$\forall\, t \in [0,T],\ \forall\, \eta \in \Lambda_0,\ \forall\, u,\ v,\ w,\ w_{1,2} \in V_0,\ \forall\, \theta \ge 0,$$

$$\begin{aligned} &\exists\, m_\phi > 0 \text{ such that } \forall\, t_{1,2} \in [0,T],\ \forall\, \eta_{1,2} \in \Lambda_0,\ \forall\ u_{1,2}, v_{1,2}, w_{1,2} \in V_0,\\ &\quad |\phi_0(t_1,\eta_1,u_1,v_1,w_1) - \phi_0(t_1,\eta_1,u_1,v_1,w_2)\\ &\quad + \ \phi_0(t_2,\eta_2,u_2,v_2,w_2) - \phi_0(t_2,\eta_2,u_2,v_2,w_1)|\\ &\quad \le m_\phi\, (|t_1-t_2| + |\eta_1-\eta_2|_{\Pi_0} + \|u_1-u_2\|_{U_0} + \|v_1-v_2\|_{U_0})\, \|w_1-w_2\|. \end{aligned} \tag{2.10}$$

Assume that $L_0 \in W^{1,\infty}(0,T;V_0)$, $u_0, u_1 \in V_0$, $\beta_0 \in \Lambda_0$ and that the following compatibility condition holds: $\exists\, l_0 \in H_0$ such that $\forall\, w \in V_0$

$$(l_0,w) + a_0(u_0,w) + b_0(u_1,w) + \phi_0(0,\beta_0,u_0,u_1,w) = \langle L_0(0), w\rangle. \tag{2.11}$$

Consider the following problem.

Problem $\boldsymbol{Q}$: Find $u \in W^{2,2}(0,T;H_0) \cap W^{1,2}(0,T;V_0)$, $\beta \in W^{1,\infty}(0,T;\Pi_0)$ such that $u(0) = u_0$, $\dot u(0) = u_1$, $\beta(0) = \beta_0$, $\beta(\tau) \in \Lambda_0$ for all $\tau \in]0,T[$, and a.e. $t \in]0,T[$

$$\begin{aligned} &(\ddot u, v-\dot u) + a_0(u, v-\dot u) + b_0(\dot u, v-\dot u)\\ &\qquad +\phi_0(t,\beta,u,\dot u,v) - \phi_0(t,\beta,u,\dot u,\dot u) \ge \langle L_0, v-\dot u\rangle \quad \forall\, v \in V_0, \end{aligned} \tag{2.12}$$

$$(\dot\beta, \eta-\beta)_{\Pi_0} + \gamma_0(u,\beta,\eta-\beta) \ge 0 \quad \forall\, \eta \in \Lambda_0. \tag{2.13}$$

Define the set

$$X_0 = \{\eta \in C^0([0,T];\Pi_0)\,;\, \eta(0) = \beta_0,\ \ \eta(t) \in \Lambda_0\ \forall\, t \in]0,T]\},$$

where the Banach space $C^0([0,T];\Pi_0)$ is endowed with the norm

$$\|\eta\|_k = \max_{t\in[0,T]} [\exp(-kt)\, |\eta(t)|_{\Pi_0}] \quad \text{for all } \eta \in C^0([0,T];\Pi_0),\ k \ge 0.$$

The existence and uniqueness of the solution of the problem Q will be proved by using the following lemmas and a fixed point argument, see also [11] for the particular case when the coefficient of friction is slip rate independent.

Lemma 2.1. *For each $\beta \in X_0$ there exists a unique $u_\beta \in W^{2,2}(0,T;H_0)\cap W^{1,2}(0,T;V_0)$, solution of the inequality* (2.12) *such that* $u_\beta(0) = u_0$, $\dot{u}_\beta(0) = u_1$.

The proof is based on some incremental formulations, see, e.g. [10], and on a useful estimate, see [22] or [28], which, when applied to the spaces $V_0 \subset U_0 \subseteq H_0$, implies the following result: for every $\epsilon > 0$ there exists $C_\epsilon > 0$ such that

$$\|u\|_{U_0} \leq \epsilon\,\|u\| + C_\epsilon\,|u| \quad \forall\, u \in V_0. \tag{2.14}$$

The full proof will be presented in a forthcoming paper.

Lemma 2.2. *Let $\beta_1,\, \beta_2 \in X_0$ and let $u_{\beta_1},\, u_{\beta_2}$ be the corresponding solutions of (2.12) with the same initial conditions $u_0,\, u_1$, respectively. Then there exists a constant $C_1 > 0$, independent of $\beta_1,\, \beta_2,\, u_{\beta_1},\, u_{\beta_2}$, such that for all $t \in [0,T]$*

$$|\dot{u}_{\beta_1}(t) - \dot{u}_{\beta_2}(t)|^2 + \|u_{\beta_1}(t) - u_{\beta_2}(t)\|^2 \leq C_1 \int_0^t |\beta_1(s) - \beta_2(s)|^2_{\Pi_0}\, ds. \tag{2.15}$$

Proof. Let $u_{\beta_1},\, u_{\beta_2}$ be the solutions of (2.12) corresponding to $\beta_1,\, \beta_2 \in X_0$. Taking in each inequality $v = \dot{u}_{\beta_2}$ and $v = \dot{u}_{\beta_1}$, respectively, for a.e. $s \in]0,T[$ it follows that

$$\begin{aligned}
&(\ddot{u}_{\beta_1} - \ddot{u}_{\beta_2}, \dot{u}_{\beta_1} - \dot{u}_{\beta_2}) + a_0(u_{\beta_1} - u_{\beta_2}, \dot{u}_{\beta_1} - \dot{u}_{\beta_2}) + b_0(\dot{u}_{\beta_1} - \dot{u}_{\beta_2}, \dot{u}_{\beta_1} - \dot{u}_{\beta_2})\\
&\quad \leq \phi_0(s, \beta_1, u_{\beta_1}, \dot{u}_{\beta_1}, \dot{u}_{\beta_2}) - \phi_0(s, \beta_1, u_{\beta_1}, \dot{u}_{\beta_1}, \dot{u}_{\beta_1})\\
&\quad\quad + \phi_0(s, \beta_2, u_{\beta_2}, \dot{u}_{\beta_2}, \dot{u}_{\beta_1}) - \phi_0(s, \beta_2, u_{\beta_2}, \dot{u}_{\beta_2}, \dot{u}_{\beta_2})\\
&\quad \leq m_\phi(\,|\beta_1 - \beta_2|_{\Pi_0} + \|u_{\beta_1} - u_{\beta_2}\|_{U_0} + \|\dot{u}_{\beta_1} - \dot{u}_{\beta_2}\|_{U_0}\,)\|\dot{u}_{\beta_1} - \dot{u}_{\beta_2}\|,
\end{aligned}$$

where the second inequality follows by (2.10).

For all $t \in [0,T]$, as the solutions $u_{\beta_1},\, u_{\beta_2}$ verify the same initial conditions, integrating between 0 and t yields

$$\begin{aligned}
&\frac{1}{2}|\dot{u}_{\beta_1}(t) - \dot{u}_{\beta_2}(t)|^2 + \frac{1}{2}a_0(u_{\beta_1}(t) - u_{\beta_2}(t), u_{\beta_1}(t) - u_{\beta_2}(t))\\
&+ \int_0^t b_0(\dot{u}_{\beta_1} - \dot{u}_{\beta_2}, \dot{u}_{\beta_1} - \dot{u}_{\beta_2})\, ds \leq m_\phi \int_0^t\, |\beta_1 - \beta_2|_{\Pi_0}\, \|\dot{u}_{\beta_1} - \dot{u}_{\beta_2}\|\, ds\\
&+ m_\phi \int_0^t (\, \|u_{\beta_1} - u_{\beta_2}\|_{U_0}\, \|\dot{u}_{\beta_1} - \dot{u}_{\beta_2}\| + \|\dot{u}_{\beta_1} - \dot{u}_{\beta_2}\|_{U_0}\, \|\dot{u}_{\beta_1} - \dot{u}_{\beta_2}\|\,)\, ds.
\end{aligned}$$

Using (2.14), Young's inequality for the last three terms with appropriate constants, V_0 - ellipticity of a_0, b_0 and Gronwall's inequality, the estimate (2.15) follows. □

Now, for every element $u \in W^{1,2}(0,T;V_0)$, consider the inequality (2.13) with the initial condition β_0, the solution of which is denoted by β_u. The existence and uniqueness results for this parabolic inequality follow by classical references, see, e.g. [4], [1], [2], and a direct proof is presented in [11].

Lemma 2.3. *For each $u \in W^{1,2}(0,T;V_0)$ there exists a unique solution $\beta_u \in X_0 \cap W^{1,\infty}(0,T;\Pi_0)$ of the inequality* (2.13)*.*

Lemma 2.4. *Let u_1, $u_2 \in W^{1,2}(0,T;V_0)$ and let β_{u_1}, $\beta_{u_2} \in X_0$ be the corresponding solutions of* (2.13) *with the same initial condition β_0, respectively. Then there exists a constant $C_2 > 0$, independent of u_1, u_2, β_{u_1}, β_{u_2}, such that for all $t \in [0,T]$*

$$|\beta_{u_1}(t) - \beta_{u_2}(t)|^2_{\Pi_0} \leq C_2 \int_0^t \|u_1(s) - u_2(s)\|^2 \, ds. \tag{2.16}$$

Proof. Let β_{u_1}, β_{u_2} be the solutions of (2.13) corresponding to u_1, u_2. Taking in each inequality $\eta = \beta_{u_2}$, $\eta = \beta_{u_1}$, respectively, for all $t \in]0,T[$, integrating over $[0,t]$, using (2.4) and some elementary inequality yield

$$\begin{aligned}
&\frac{1}{2}|\beta_{u_1}(t) - \beta_{u_2}(t)|^2_{\Pi_0} \\
&\qquad \leq \int_0^t [\gamma_0(u_2, \beta_{u_2}, \beta_{u_1} - \beta_{u_2}) - \gamma_0(u_1, \beta_{u_1}, \beta_{u_1} - \beta_{u_2})] \, ds \\
&\qquad = \int_0^t [\gamma_0(u_2, \beta_{u_2}, \beta_{u_1} - \beta_{u_2}) - \gamma_0(u_2, \beta_{u_1}, \beta_{u_1} - \beta_{u_2})] \, ds \\
&\qquad + \int_0^t [\gamma_0(u_2, \beta_{u_1}, \beta_{u_1} - \beta_{u_2}) - \gamma_0(u_1, \beta_{u_1}, \beta_{u_1} - \beta_{u_2})] \, ds \\
&\qquad \leq m_\gamma \int_0^t |\beta_{u_1} - \beta_{u_2}|^2_{\Pi_0} \, ds + m_\gamma \int_0^t \|u_1 - u_2\| \, |\beta_{u_1} - \beta_{u_2}|_{\Pi_0} \, ds \\
&\qquad \leq \frac{m_\gamma}{2} \int_0^t \|u_1(s) - u_2(s)\|^2 \, ds + \frac{3\, m_\gamma}{2} \int_0^t |\beta_{u_1}(s) - \beta_{u_2}(s)|^2_{\Pi_0} \, ds.
\end{aligned}$$

By Gronwall's inequality the estimate (2.16) is established. □

Now we can prove the following existence and uniqueness result for the abstract problem Q.

Theorem 2.1. *Assume that conditions* (2.1)-(2.11) *hold. Then there exists a unique solution of the problem Q.*

Proof. For every $\beta \in X_0$ let $u_\beta \in W^{2,2}(0,T;H_0) \cap W^{1,2}(0,T;V_0)$ be the solution of the inequality (2.12) corresponding to β such that $u_\beta(0) = u_0$,

$\dot{u}_\beta(0) = u_1$ and let $\beta_{u_\beta} \in X_0 \cap W^{1,\infty}(0,T;\Pi_0)$ be the solution of the inequality (2.13) corresponding to u_β. Define the mapping $\mathcal{T} : X_0 \to X_0$ as $\forall\, \beta \in X_0 \quad \mathcal{T}\beta = \beta_{u_\beta}$. We shall prove that $\mathcal{T} : X_0 \to X_0$ has a unique fixed point, which is equally the solution of the problem Q.

For all $\beta_1,\, \beta_2 \in X_0$, for all $t \in [0,T]$, (2.16) and (2.15) imply that

$$\begin{aligned}
|\mathcal{T}\beta_1(t) - \mathcal{T}\beta_2(t)|^2_{\Pi_0} &\le C_2 \int_0^t \|u_{\beta_1}(s) - u_{\beta_2}(s)\|^2\, ds \\
&\le C_1\, C_2 \int_0^t \left(\int_0^s \exp(-2kr)\cdot \exp(2kr)\, |\beta_1(r) - \beta_2(r)|^2_{\Pi_0}\, dr \right) ds \\
&\le C_1\, C_2\, \|\beta_1 - \beta_2\|_k^2 \int_0^t \frac{\exp(2ks)}{2k}\, ds \\
&\le \frac{C_1\, C_2}{4k^2} \cdot \exp(2kt)\, \|\beta_1 - \beta_2\|_k^2.
\end{aligned}$$

Then

$$\begin{aligned}
\|\mathcal{T}\beta_1 - \mathcal{T}\beta_2\|_k &= \max_{t\in[0,T]} \left[\exp(-kt)\, |\mathcal{T}\beta_1(t) - \mathcal{T}\beta_2(t)|_{\Pi_0}\right] \\
&\le \frac{\sqrt{C_1\, C_2}}{2k}\, \|\beta_1 - \beta_2\|_k.
\end{aligned}$$

Hence, for all $\beta_1,\, \beta_2 \in X_0$

$$\|\mathcal{T}\beta_1 - \mathcal{T}\beta_2\|_k \le \frac{\sqrt{C_1\, C_2}}{2k}\, \|\beta_1 - \beta_2\|_k,$$

so that if k is sufficiently large it follows that $\mathcal{T}$ is a contraction and its fixed point is the solution of the problem Q. □

3. A dynamic contact problem with adhesion and friction

Consider two viscoelastic bodies, characterized by a Kelvin-Voigt constitutive law, which occupy the reference domains Ω^α of $\mathbb{R}^d$, $d = 2$ or 3, with Lipschitz continuous boundaries $\Gamma^\alpha = \partial\Omega^\alpha$, $\alpha = 1, 2$. Assume the small deformation hypothesis. Let Γ_1^α, Γ_2^α and Γ_3^α be three open disjoint sufficiently smooth parts of Γ^α such that $\Gamma^\alpha = \overline{\Gamma}_1^\alpha \cup \overline{\Gamma}_2^\alpha \cup \overline{\Gamma}_3^\alpha$ and, to simplify the estimates, $\mathrm{meas}(\Gamma_1^\alpha) > 0$, $\alpha = 1, 2$.

Let $\boldsymbol{y}^\alpha(\boldsymbol{x}^\alpha, t)$ be the position at time $t \in [0,T]$, where $T > 0$, of the material point represented by $\boldsymbol{x}^\alpha$ in the reference configuration and $\boldsymbol{u}^\alpha(\boldsymbol{x}^\alpha, t) := \boldsymbol{y}^\alpha(\boldsymbol{x}^\alpha, t) - \boldsymbol{x}^\alpha$ be the displacement vector of $\boldsymbol{x}^\alpha$ at time t, with the Cartesian coordinates $u^\alpha = (u_1^\alpha, ..., u_d^\alpha) = (\overline{u}^{\,\alpha}, u_d^\alpha)$.

Let $\boldsymbol{\varepsilon}^\alpha$, with the Cartesian coordinates $(\varepsilon_{ij}\,(u^\alpha))$, and $\boldsymbol{\sigma}^\alpha$, with the Cartesian coordinates (σ_{ij}^α), be the infinitesimal strain tensor and the stress tensor, respectively, corresponding to Ω^α, $\alpha = 1, 2$. The usual summation convention will be used for $i, j, k, l = 1, \dots, d$.

Assume that the displacement $\boldsymbol{U}^\alpha = \boldsymbol{0}$ is prescribed on $\Gamma_1^\alpha \times]0, T[$, $\alpha = 1, 2$, and, to simplify, that the densities of both bodies are equal to 1. Let $\boldsymbol{f} = (\boldsymbol{f}^1, \boldsymbol{f}^2)$ and $\boldsymbol{F} = (\boldsymbol{F}^1, \boldsymbol{F}^2)$ denote the given body forces in $\Omega^1 \cup \Omega^2$ and tractions on $\Gamma_2^1 \cup \Gamma_2^2$, respectively. The initial displacements and velocities of the bodies are denoted by $\boldsymbol{u}_0 = (\boldsymbol{u}_0^1, \boldsymbol{u}_0^2)$, $\boldsymbol{u}_1 = (\boldsymbol{u}_1^1, \boldsymbol{u}_1^2)$.

Suppose that the solids can be in contact between the potential contact surfaces Γ_C^1 and Γ_C^2 that can be parametrized by two C^1 functions, φ^1, φ^2, defined on an open subset Ξ of $\mathbb{R}^{d-1}$ such that $\varphi^1(\xi) - \varphi^2(\xi) \geq 0 \ \forall \xi \in \Xi$ and each Γ_C^α is the graph of φ^α on Ξ that is $\Gamma_C^\alpha = \{ (\xi, \varphi^\alpha(\xi)) \in \mathbb{R}^d; \xi \in \Xi\}$, $\alpha = 1, 2$. Let $\boldsymbol{m}^\alpha : \Xi \to \mathbb{R}^d$, with $\boldsymbol{m}^1(\xi) := (\nabla\varphi^1(\xi), -1)$, $\boldsymbol{m}^2(\xi) := (-\nabla\varphi^2(\xi), 1)$, $\forall \xi \in \Xi$, be the outward normal to Γ_C^α, $\alpha = 1, 2$. Since the displacements, their derivatives and the gap are assumed small, by using a method similar to the one presented in [3] (see also [10]) the following unilateral contact condition at time t on the set Ξ is obtained:

$$
\begin{aligned}
0 \leq \varphi^1(\xi) - \varphi^2(\xi) &+ u_d^1(\xi, \varphi^\alpha(\xi), t) - u_d^2(\xi, \varphi^\alpha(\xi), t) \\
&- \nabla\varphi^1(\xi) \cdot \overline{u}^1(\xi, \varphi^1(\xi), t) + \nabla\varphi^2(\xi) \cdot \overline{u}^2(\xi, \varphi^2(\xi), t) \quad \forall \xi \in \Xi.
\end{aligned}
$$

Using the definition of $\boldsymbol{m}^1$, $\boldsymbol{m}^2$, this relation can be written under the following form: for all $\xi \in \Xi$

$$
\boldsymbol{m}^1(\xi) \cdot \boldsymbol{u}^1(\xi, \varphi^1(\xi), t) + \boldsymbol{m}^2(\xi) \cdot \boldsymbol{u}^2(\xi, \varphi^2(\xi), t) \leq \varphi^1(\xi) - \varphi^2(\xi). \tag{3.1}
$$

Let $\boldsymbol{n}^\alpha := \boldsymbol{m}^\alpha / |\boldsymbol{m}^\alpha|$ denote the unit outward normal vector to Γ_C^α, $\alpha = 1, 2$, and define the initial normalized gap between the two contact surfaces by

$$
g_0(\xi) := \frac{\varphi^1(\xi) - \varphi^2(\xi)}{\sqrt{1 + |\nabla\varphi^1(\xi)|^2}} \quad \forall \xi \in \Xi.
$$

Let the normal and tangential components of a displacement field $\boldsymbol{v}^\alpha$, $\alpha = 1, 2$, of the relative displacement corresponding to $\boldsymbol{v} := (\boldsymbol{v}^1, \boldsymbol{v}^2)$, including the initial gap g_0 in the normal direction, and of the stress vector $\boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha$ on Γ_C^α be given by

$$
\begin{aligned}
&\boldsymbol{v}^\alpha := \boldsymbol{v}^\alpha(\xi, t) = \boldsymbol{v}^\alpha(\xi, \varphi^\alpha(\xi), t), \\
&v_N^\alpha := v_N^\alpha(\xi, t) = \boldsymbol{v}^\alpha(\xi, \varphi^\alpha(\xi), t) \cdot \boldsymbol{n}^\alpha(\xi), \quad \boldsymbol{v}_T^\alpha := \boldsymbol{v}_T^\alpha(\xi, t) = \boldsymbol{v}^\alpha - v_N^\alpha \boldsymbol{n}^\alpha, \\
&[v_N] := [v_N](\xi, t) = v_N^1 + v_N^2 - g_0, \quad [\boldsymbol{v}_T] := [\boldsymbol{v}_T](\xi, t) = \boldsymbol{v}_T^1 - \boldsymbol{v}_T^2, \\
&\sigma_N^\alpha := \sigma_N^\alpha(\xi, t) = (\boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha) \cdot \boldsymbol{n}^\alpha, \quad \boldsymbol{\sigma}_T^\alpha := \boldsymbol{\sigma}_T^\alpha(\xi, t) = \boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha - \sigma_N^\alpha \boldsymbol{n}^\alpha,
\end{aligned} \tag{3.2}
$$

for all $\xi \in \Xi$ and for all $t \in [0, T]$. Let $g := -[u_N] = g_0 - u_N^1 - u_N^2$ be the gap corresponding to the solution $\boldsymbol{u} := (\boldsymbol{u}^1, \boldsymbol{u}^2)$. Assuming that

$\nabla\varphi^1(\xi) \simeq \nabla\varphi^2(\xi)$, it follows that the unilateral contact condition (3.1) at time t can be written as

$$[u_N]\,(\xi,t) = -g(\xi,t) \leq 0 \quad \forall\, \xi \in \Xi. \tag{3.3}$$

Let β denote an internal state variable, see, e.g. [13]–[15], representing the intensity of adhesion or the adhesion field ($\beta = 1$ means that the adhesion is total, $\beta = 0$ means that there is no adhesion and $0 < \beta < 1$ is the case of partial adhesion).

3.1. Classical formulation

Let $\mathcal{A}^\alpha$, $\mathcal{B}^\alpha$ denote two fourth-order tensors, the elasticity tensor and the viscosity tensor corresponding to Ω^α, with the components $(\mathcal{A}^\alpha_{ijkl})$ and $(\mathcal{B}^\alpha_{ijkl})$, respectively. Assume that these components satisfy the following classical symmetry and ellipticity conditions: $\mathcal{C}_{ijkl} = \mathcal{C}_{jikl} = \mathcal{C}_{klij} \in L^\infty(\Omega^\alpha), \forall\, i,\, j,\, k,\, l = 1, \ldots, d,\ \exists\, \alpha_{\mathcal{C}} > 0$ such that $\mathcal{C}_{ijkl}\tau_{ij}\tau_{kl} \geq \alpha_{\mathcal{C}}\, \tau_{ij}\tau_{ij}$, $\forall\, \tau = (\tau_{ij})$ verifying $\tau_{ij} = \tau_{ji}, \forall\, i,\, j = 1, \ldots, d$, where $\mathcal{C}_{ijkl} = \mathcal{A}^\alpha_{ijkl}$, $\mathcal{C} = \mathcal{A}^\alpha$ or $\mathcal{C}_{ijkl} = \mathcal{B}^\alpha_{ijkl}, \mathcal{C} = \mathcal{B}^\alpha\ \ \forall\, i,\, j,\, k,\, l = 1, \ldots, d,\ \alpha = 1, 2.$

We choose the following state variables: the infinitesimal strain tensor $(\boldsymbol{\varepsilon}^1, \boldsymbol{\varepsilon}^2) = (\boldsymbol{\varepsilon}(\boldsymbol{u}^1), \boldsymbol{\varepsilon}(\boldsymbol{u}^2))$ in $\Omega^1 \cup \Omega^2$, the relative normal displacement $[u_N] = u^1_N + u^2_N - g_0$, the relative tangential displacement $[\boldsymbol{u}_T] = \boldsymbol{u}^1_T - \boldsymbol{u}^2_T$, and the intensity of adhesion β in Ξ.

Let $\mu = \mu(\xi, [\dot{\boldsymbol{u}}_T]) \geq 0$ be the slip rate dependent coefficient of friction and assume that $\mu : \Xi \times \mathbb{R}^d \to \mathbb{R}_+$ is a bounded function such that for a.e. $\xi \in \Xi$, $\mu(\xi, \cdot)$ is Lipschitz continuous with the Lipschitz constant independent of ξ and for any $v \in \mathbb{R}^d$, $\mu(\cdot, v)$ is measurable.

Define a truncation operator $\vartheta = \vartheta_{l_0}$ by $\vartheta : \mathbb{R} \to \mathbb{R}$, $\vartheta(s) = -l_0$ if $s \leq -l_0$, $\vartheta(s) = s$ if $|s| < l_0$ and $\vartheta(s) = l_0$ if $s \geq l_0$, where $l_0 > 0$ is a given characteristic length (see, e.g. [26, 29]).

Let $\kappa : \mathbb{R} \times [0,1] \to \mathbb{R}$ be a bounded Lipschitz continuous function such that $\kappa(0,0) = 0$. Note that various normal compliance conditions, friction and adhesion with damage laws can be obtained by choosing particular functions as κ, see [17, 18, 21, 27, 29].

We consider the following classical formulation of the dynamic contact problem coupling adhesion and friction.

Problem $\boldsymbol{P_c}$: Find $(\boldsymbol{u}^1, \boldsymbol{u}^2)$, β such that $\boldsymbol{u}(0) = \boldsymbol{u}_0 = (\boldsymbol{u}^1_0, \boldsymbol{u}^2_0)$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1 = (\boldsymbol{u}^1_1, \boldsymbol{u}^2_1)$ in $\Omega^1 \times \Omega^2$, $\beta(0) = \beta_0$ in Ξ and, for all $t \in]0, T[$,

$$\ddot{\boldsymbol{u}}^\alpha - \operatorname{div} \boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha) = \boldsymbol{f}^\alpha \ \text{ in } \ \Omega^\alpha, \tag{3.4}$$

$$\boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha) = \boldsymbol{\mathcal{A}}^\alpha \varepsilon(\boldsymbol{u}^\alpha) + \boldsymbol{\mathcal{B}}^\alpha \varepsilon(\dot{\boldsymbol{u}}^\alpha) \ \text{ in } \ \Omega^\alpha, \tag{3.5}$$

$$\boldsymbol{u}^\alpha = \boldsymbol{0} \ \text{ on } \ \Gamma^\alpha_1,\ \boldsymbol{\sigma}^\alpha \boldsymbol{n}^\alpha = \boldsymbol{F}^\alpha \ \text{ on } \ \Gamma^\alpha_2,\ \alpha = 1, 2, \tag{3.6}$$

$$\boldsymbol{\sigma}^1 \boldsymbol{n}^1 + \boldsymbol{\sigma}^2 \boldsymbol{n}^2 = \boldsymbol{0} \ \text{ in } \ \Xi, \tag{3.7}$$

$$\sigma_N = \kappa([u_N], \beta) \ \text{ in } \ \Xi, \tag{3.8}$$

$$|\,\boldsymbol{\sigma}_T\,| \leq \mu([\dot{\boldsymbol{u}}_T])\,|\,\sigma_N\,| \quad \text{in } \ \Xi \ \text{ and} \tag{3.9}$$
$$\begin{aligned} &|\,\boldsymbol{\sigma}_T\,| < \mu([\dot{\boldsymbol{u}}_T])\,|\,\sigma_N\,| \Rightarrow [\dot{\boldsymbol{u}}_T] = \boldsymbol{0},\\ &|\,\boldsymbol{\sigma}_T\,| = \mu([\dot{\boldsymbol{u}}_T])\,|\,\sigma_N\,| \ \Rightarrow \exists\,\theta \geq 0,\ [\dot{\boldsymbol{u}}_T] = -\theta\boldsymbol{\sigma}_T, \end{aligned}$$

$$\beta \in [0,1] \ \text{ in } \ \Xi \ \text{ and} \tag{3.10}$$
$$\begin{aligned} &b\,\dot{\beta} \geq w \ \text{ if } \ \beta = 0,\\ &b\,\dot{\beta} = w - C_N\,\vartheta([u_N]^2)\,\beta \ \text{ if } \ \beta \in]0,1[,\\ &b\,\dot{\beta} \leq w - C_N\,\vartheta([u_N]^2) \ \text{ if } \ \beta = 1, \end{aligned}$$

where $\beta_0 \in [0,1]$ in Ξ, $C_N > 0$, $b > 0$, $w > 0$, $\boldsymbol{\sigma}^\alpha = \boldsymbol{\sigma}^\alpha(\boldsymbol{u}^\alpha, \dot{\boldsymbol{u}}^\alpha)$, $\alpha = 1,2$, $\sigma_N := \sigma_N^1$, $\boldsymbol{\sigma}_T := \boldsymbol{\sigma}_T^1$, $\boldsymbol{\sigma} := \boldsymbol{\sigma}^1$.

3.2. Variational formulation

We adopt the following notations:

$$\begin{aligned} &\boldsymbol{H}^s := [H^s(\Omega^1)]^d \times [H^s(\Omega^2)]^d \quad \forall\, s \in \mathbb{R},\\ &\langle \boldsymbol{v}, \boldsymbol{w}\rangle_{-s,s} = \langle \boldsymbol{v}^1, \boldsymbol{w}^1\rangle_{\boldsymbol{H}^{-s}(\Omega^1),\boldsymbol{H}^s(\Omega^1)} + \langle \boldsymbol{v}^2, \boldsymbol{w}^2\rangle_{\boldsymbol{H}^{-s}(\Omega^2),\boldsymbol{H}^s(\Omega^2)}\\ &\forall\,\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{H}^{-s},\ \forall\,\boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2) \in \boldsymbol{H}^s. \end{aligned}$$

Define the Hilbert spaces $(\boldsymbol{H},\ |.|)$ with the associated inner product denoted by $(.\,,.)$, $(\boldsymbol{V},\ \|.\|)$ with the associated inner product (of $\boldsymbol{H}^1$) denoted by $\langle .\,,.\rangle$ and the set Λ as follows:

$$\begin{aligned} &\boldsymbol{H} := \boldsymbol{H}^0 = \left[L^2(\Omega^1)\right]^d \times \left[L^2(\Omega^2)\right]^d,\ \boldsymbol{V} = \boldsymbol{V}^1 \times \boldsymbol{V}^2,\ \text{ where}\\ &\boldsymbol{V}^\alpha = \{\boldsymbol{v}^\alpha \in \left[H^1(\Omega^\alpha)\right]^d;\ \boldsymbol{v}^\alpha = \boldsymbol{0} \ \text{ a.e. on } \Gamma_1^\alpha\},\ \alpha = 1,2,\\ &\Lambda = \{\eta \in L^2(\Xi)\,;\ \eta \in [0,1] \ \text{ a.e. in } \ \Xi\}. \end{aligned}$$

Assume that $\boldsymbol{F}^\alpha \in W^{1,\infty}(0,T;[L^2(\Gamma_2^\alpha)]^d)$, $\boldsymbol{f}^\alpha \in W^{1,\infty}(0,T;[L^2(\Omega^\alpha)]^d)$, $\alpha = 1,2$, $\boldsymbol{u}_0$, $\boldsymbol{u}_1 \in \boldsymbol{V}$ and $\beta_0 \in \Lambda$.

Let a, b be two bilinear, continuous and symmetric mappings defined on $\boldsymbol{H}^1 \times \boldsymbol{H}^1 \to \mathbb{R}$ as

$$a(\boldsymbol{v}, \boldsymbol{w}) = a^1(\boldsymbol{v}^1, \boldsymbol{w}^1) + a^2(\boldsymbol{v}^2, \boldsymbol{w}^2),\ b(\boldsymbol{v}, \boldsymbol{w}) = b^1(\boldsymbol{v}^1, \boldsymbol{w}^1) + b^2(\boldsymbol{v}^2, \boldsymbol{w}^2)$$

$\forall\,\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2)$, $\boldsymbol{w} = (\boldsymbol{w}^1, \boldsymbol{w}^2) \in \boldsymbol{H}^1$, where, for $\alpha = 1,2$,

$$a^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{w}^\alpha) = \int_{\Omega^\alpha} \boldsymbol{\mathcal{A}}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha)\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}^\alpha)\,dx,\ b^\alpha(\boldsymbol{v}^\alpha, \boldsymbol{w}^\alpha) = \int_{\Omega^\alpha} \boldsymbol{\mathcal{B}}^\alpha \boldsymbol{\varepsilon}(\boldsymbol{v}^\alpha)\cdot\boldsymbol{\varepsilon}(\boldsymbol{w}^\alpha)\,dx.$$

Consider $\boldsymbol{L}$ as an element of $W^{1,\infty}(0,T;\boldsymbol{H}^1)$ such that $\forall\,t \in [0,T]$

$$\langle \boldsymbol{L}, \boldsymbol{v}\rangle = \sum_{\alpha=1,2} \int_{\Omega^\alpha} \boldsymbol{f}^\alpha \cdot \boldsymbol{v}^\alpha\,dx + \sum_{\alpha=1,2} \int_{\Gamma_2^\alpha} \boldsymbol{F}^\alpha \cdot \boldsymbol{v}^\alpha\,ds \quad \forall\,\boldsymbol{v} = (\boldsymbol{v}^1, \boldsymbol{v}^2) \in \boldsymbol{H}^1.$$

We define the following mappings:

$$J : L^2(\Xi) \times (\boldsymbol{H}^1)^3 \to \mathbb{R},$$

$$J(\beta, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = \int_\Xi \mu([\boldsymbol{v}_T])\,|\,\kappa([u_N], \beta)\,|\,|\,[\boldsymbol{w}_T]\,|\,d\xi$$

$$\forall\,\beta \in L^2(\Xi),\ \forall\,(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) \in (\boldsymbol{H}^1)^3,$$

$$\gamma : \boldsymbol{H}^1 \times [L^2(\Xi)]^2 \to \mathbb{R},\ \gamma(\boldsymbol{u}, \delta, \eta) = \int_\Xi \frac{C_N}{b}\,\vartheta([u_N]^2)\,\delta\,\eta\,d\xi,$$

$$\varpi : L^2(\Xi) \to \mathbb{R},\ \varpi(\eta) = \int_\Xi \frac{w}{b}\,\eta\,d\xi \quad \forall\,\boldsymbol{u} \in \boldsymbol{H}^1,\ \ \forall\,\delta,\,\eta \in L^2(\Xi).$$

Assume the following compatibility condition: $\exists\,\boldsymbol{l} \in \boldsymbol{H}$ such that

$$\begin{aligned}(\boldsymbol{l}, \boldsymbol{v}) + a(\boldsymbol{u}_0, \boldsymbol{v}) + b(\boldsymbol{u}_1, \boldsymbol{v}) - (\kappa([u_{0N}], \beta_0), v_N)_\Xi \\ +J(\beta_0, \boldsymbol{u}_0, \boldsymbol{u}_1, \boldsymbol{v}) = \langle \boldsymbol{L}(0), \boldsymbol{v}\rangle \quad \forall\,\boldsymbol{v} \in \boldsymbol{V}.\end{aligned} \tag{3.11}$$

A variational formulation of the problem P_c is the following.

Problem $\boldsymbol{P_v}$: Find $\boldsymbol{u} \in W^{2,2}(0,T;\boldsymbol{H}) \cap W^{1,2}(0,T;\boldsymbol{V})$, $\beta \in W^{1,\infty}(0,T;L^\infty(\Xi))$ such that $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \boldsymbol{u}_1$ in $\Omega^1 \cup \Omega^2$, $\beta(0) = \beta_0$ in Ξ, $\beta(\tau) \in \Lambda$ for all $\tau \in]0,T[$ and a.e. $t \in]0,T[$

$$\begin{aligned}(\ddot{\boldsymbol{u}}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + a(\boldsymbol{u}, \boldsymbol{v} - \dot{\boldsymbol{u}}) + b(\dot{\boldsymbol{u}}, \boldsymbol{v} - \dot{\boldsymbol{u}}) - (\kappa([u_N], \beta), v_N - \dot{u}_N)_\Xi \\ +J(\beta, \boldsymbol{u}, \dot{\boldsymbol{u}}, \boldsymbol{v}) - J(\beta, \boldsymbol{u}, \dot{\boldsymbol{u}}, \dot{\boldsymbol{u}}) \geq \langle \boldsymbol{L}, \boldsymbol{v} - \dot{\boldsymbol{u}}\rangle \quad \forall\,\boldsymbol{v} \in \boldsymbol{V},\end{aligned} \tag{3.12}$$

$$(\dot{\beta}, \eta - \beta)_{L^2(\Xi)} + \gamma(\boldsymbol{u}, \beta, \eta - \beta) \geq \varpi(\eta - \beta) \quad \forall \eta \in \Lambda. \tag{3.13}$$

The formal equivalence between the variational system (3.12),(3.13) and the classical problem (3.4)–(3.10) can be easily proved by using Green's formula.

3.3. Existence and uniqueness of variational solutions

The following existence and uniqueness result holds.

Theorem 3.1. *Under the above assumptions, there exists a unique solution of the Problem P_v.*

Proof. We apply Theorem 2.1 to $H_0 = \boldsymbol{H}$, $V_0 = \boldsymbol{V}$, $U_0 = \boldsymbol{H}^{1-\iota}$, where $1/2 > \iota > 0$, $\Pi_0 = L^2(\Xi)$, $\Lambda_0 = \Lambda$, $u_0 = \boldsymbol{u}_0$, $u_1 = \boldsymbol{u}_1$, $a_0 = a$, $b_0 = b$, $L_0 = \boldsymbol{L}$, $\gamma_0 = \gamma - \varpi$ and

$$\phi_0(t, \eta, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = -(\kappa([u_N], \eta), w_N)_\Xi + J(\eta, \boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w})$$

$$\forall\,t \in [0,T],\ \forall\,\eta \in L^2(\Xi),\ \forall\,\boldsymbol{u},\,\boldsymbol{v},\,\boldsymbol{w} \in \boldsymbol{V}.$$

One can easily verify the properties (2.1)-(2.11). Thus, by Theorem 2.1 there exists a unique solution of the problem P_v. □

Note that the same method can be used to study the dynamic contact problem with irreversible adhesion (see, e.g. [29]), for which the evolution of the intensity of adhesion is governed by a differential equation.

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