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An optimality result about sample path properties of Operator Scaling Gaussian Random Fields

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Abstract - We study the sample paths properties of Operator scaling Gaussian random fields. Such fields are anisotropic generalizations of anisotropic self-similar random fields as anisotropic Fractional Brownian Motion. Some characteristic properties of the anisotropy are revealed by the regularity of the sample paths. The sharpest way of measuring smoothness is related to these anisotropies and thus to the geometry of these fields.

Key words and phrases : Operator scaling Gaussian random field, anisotropy, sample paths properties, anisotropic Besov spaces.

Mathematics Subject Classification (2010) **:** 60G15, 60G18, 60G60, 60G17, 42C40, 46E35.

1. Introduction and motivations

Random fields are now used for modeling in a wide range of scientific areas including physics, engineering, hydrology, biology, economics and finance (see [31] and its bibliography). An important requirement is that the data thus modelled present strong anisotropies which therefore have to be present in the model. Many anisotropic random fields have therefore been proposed as natural models in various areas such as image processing, hydrology, geostatistics and spatial statistics (see, for example, Davies and Hall [15], Bonami and Estrade [8], Benson et al. [4]). Let us also quote the example of Levy random fields, deeply studied by Durand and Jaffard (see [17]), which is the only known model of anisotropic multifractal random field. In many cases, Gaussian models have turned to be relevant when investigating anisotropic problems. For example the stochastic model of surface waves is usually assumed to be Gaussian and is surprisingly accurate (see [21]). More generally anisotropic Gaussian random fields are involved in many others concrete situations and then arise naturally in stochastic partial differential equations (see, e.g., Dalang [14], Mueller and Tribe [24], Ôksendal and Zhang [27], Nualart [26]).

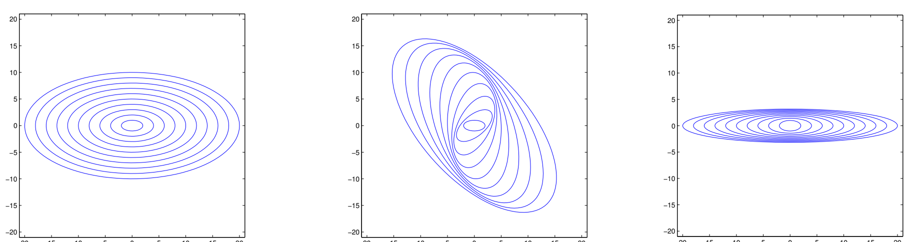
In many situations, the data present invariant features across the scales (see for example [2]). These two requirements (anisotropy and self–similarity) may seem contradictory, since the classical notion of self–similarity defined for a random field $\{X(x)\}_{x\in\mathbb{R}^d}$ on $\mathbb{R}^d$ by

$$\{X(ax)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a^{H_0}X(x)\}_{x\in\mathbb{R}^d}\ , \tag{1.1}$$

for some $H_0 \in \mathbb{R}$ (called the Hurst index) is by construction isotropic and has then to be changed in order to fit anisotropic situations. To this end, several extensions of self–similarity property in an anisotropic setting have been proposed. In [19], Hudson and Mason defined operator self-similar processes $\{X(t)\}_{t\in\mathbb{R}}$ with values in $\mathbb{R}^d$. In [20], Kamont introduced Fractional Brownian Sheets which satisfies different scaling properties according to the coordinate axes. More recently, in [7] Biermé, Meerschaert and Scheffler introduced the notion of Operator Scaling Random Fields (OSRF). These fields satisfy the following anisotropic scaling relation :

$$\{X(a^{E_0}x)\}_{x\in\mathbb{R}^d} \overset{(f.d.)}{=} \{a^{H_0}X(x)\}_{x\in\mathbb{R}^d}\ , \tag{1.2}$$

for some matrix E_0 (called an exponent or an anisotropy of the field) whose eigenvalues have a positive real part and some $H_0 > 0$ (called an Hurst index of the field). The usual notion of self-similarity is extended replacing **usual scaling**, (corresponding to the case $E_0 = Id$) by **a linear scaling** involving the matrix E_0 (see figure 1 below). It allows to define new classes of random fields with new geometry and structure.



$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\lambda \in \{1, \cdots, 10\}$

$E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$, $\lambda \in \{1, \cdots, 10\}$

$E = \begin{pmatrix} 1 & 0 \\ 0 & 1/2 \end{pmatrix}$, $\lambda \in \{1, \cdots, 10\}$

Figure 1.
Action of a linear scaling $x \mapsto \lambda^E x$ on the smallest ellipsis.

This new class of random fields have been introduced in order to model various phenomena such as fracture surfaces (see [27]) or sedimentary aquifers (see [4]). In [7], the authors construct a large class of Operator Scaling Stable Random Fields with stationary increments presenting both a moving average and an harmonizable representation of these fields.

In order to use such models in practice, the first problem is to recover the parameters H_0 and E_0 from the inspection of one sample paths. Note that this model is overparametriezd, that is for a given OSSRGF, there is an infinity of couples (E_0, H_0) satisfying (1.2). We then need an additional condition, $\mathrm{Tr}(E_0) = d$ to ensure the uniqueness of the Hurst index. Recently, in [23], an estimation method for E_0, using the different parametrization with $H_0 = 1$, has been proposed, based on non-linear regression.

Here, we want to propose another approach based on the indentification of some specific features of exponents and indices which can be recovered on sample paths. This paper is then a first step to an alternative method of estimation : we will prove that from the regularity point of view these exponents and Hurst indices satisfy what we call optimality properties. More precisely, we prove that (see Theorem 4.1), the Hurst index H_0 maximizes the local critical exponent of the field in specific functional spaces related with the anisotropy matrix E_0 among all possible critical exponents in general anisotropic functional spaces.

Therefore, the results of the present paper open the way to the following strategy to recover the Hurst index. One first have to consider a discretized version of the set of all possible anisotropies. In each case an estimator of the critical exponent related with these anisotropies has to be given. Therefore, one has to locate the maximum of all these estimators–which can be based on anisotropic quadratic variations–and to identify the corresponding values of the anisotropy. This method can be turned into an effective algorithm using hyperbolic wavelet analysis (see [1] and [29] for more details). The problem can thus be reformulated in terms of finding extreme values of some multivariate Gaussian series related to the set of discrete anisotropies (see [34] for some reference about extremes of multivariate series). The study of these estimators from a statistical point of view will be the purpose of a forthcoming paper and will be compared to the method introduced in [23].

Our optimality result comes from sample paths properties of the model under study in an anisotropic setting. This approach is natural : In [20], Kamont studied the regularity of the sample paths of the well-known anisotropic Fractional Brownian Sheet in anisotropic Hölder spaces related to Fractional Brownian Sheet. Moreover, some results of regularity in specific anisotropic Hölder spaces related to matrix E_0 have already be established for operator scaling self-similar random fields (which may be not Gaussian) in [6] or in the more general setting of strongly non deterministic anisotropic Gaussian fields in [38]. We then extend already existing results by measuring smoothness in general anisotropic spaces not necessarily related to the exponent matrix E_0 of the field.

This paper is organized as follows. In Section 2, we briefly recall some facts about Operator Scaling Random Gaussian Fields (OSRGF) and describe the construction of [7] of the model. In Section 3, we present the different concepts used for measuring smoothness in an anisotropic setting and especially anisotropic Besov spaces. Section 4 is devoted to the statement of our optimality and regularity results. Finally, Section 5 contains proofs of the results stated in Section 4.

In the sequel, we will use some notations. For any matrix M

$$\lambda_{\min}(M) = \min_{\lambda \in Sp(M)}(|\mathrm{Re}(\lambda)|),\ \lambda_{\max}(M) = \max_{\lambda \in Sp(M)}(|\mathrm{Re}(\lambda)|)\ ,$$

where $\mathrm{Sp}(M)$ denotes the spectrum of matrix M.
For any real $a > 0$, a^M denotes the matrix

$$a^M = \exp(M \log(a)) = \sum_{k \geq 0} \frac{M^k \log^k(a)}{k!}.$$

In the following pages, we denote $\mathcal{E}^+$ the collection of matrices of $M_d(\mathbb{R})$ whose eigenvalues have positive real part.

2. Presentation of the studied model

The existence of operator scaling stable random fields, that is random fields satisfying relationship (1.2), is proved in [7]. The following theorem (Theorem 4.1 and Corollary 4.2 of [7]) completes this result by yielding a practical way to construct a Operator Scaling Stable Random Field (OSRF) with stationary increments for any $E_0 \in \mathcal{E}^+$ and $H_0 \in (0, \lambda_{\min}(E))$. We state it only in the Gaussian case, having in mind the problem of the estimation of the Hurst index H_0 and the anisotropy E_0. In what follows, we are given $\mathrm{d}\widehat{W}$ a complex–valued Brownian measure such that all the processes and the fields we consider are real-valued.

Theorem 2.1. *Let E_0 be in $\mathcal{E}^+$ and ρ a continuous function with positive values such that for all $x \neq 0$, $\rho(x) \neq 0$. Assume that ρ is ${}^tE_0$–homogeneous, that is :*

$$\forall a > 0,\ \forall \xi \in \mathbb{R}^d,\ \rho(a^{{}^tE_0}\xi) = a\rho(\xi)\ .$$

Then the Gaussian field

$$X_\rho(x) = \int_{\mathbb{R}^d} (\mathrm{e}^{\mathrm{i}<x,\xi>} - 1)\rho(\xi)^{-H_0 - \frac{\mathrm{Tr}(E_0)}{2}} \mathrm{d}\widehat{W}(\xi)\ , \qquad (2.1)$$

exists and is stochastically continuous if and only if $H_0 \in (0, \rho_{\min}(E_0))$. Moreover this field has the following properties :

1. *Stationary increments :*

$$\forall h \in \mathbb{R}^d,\ \{X_\rho(x+h) - X_\rho(h)\}_{x \in \mathbb{R}^d} \overset{(f.d.)}{=} \{X_\rho(x)\}_{x \in \mathbb{R}^d}\ .$$

2. The operator–scaling relation (1.2) *is satisfied.*

Remark 2.1. The assumption of homogeneity on the function ρ is necessary to recover linear self-similarity properties of the Gaussian field $\{X_\rho(x)\}_{x\in\mathbb{R}^d}$. The assumption of continuity on ρ ensures that the constructed field is stochastically continuous.

Remark 2.2. In general, the couple (H_0, E_0) of an OSRF is not unique. Indeed, if H_0 and E_0 are respectively an Hurst index and an exponent of the OSRF $\{X(x)\}_{x\in\mathbb{R}^d}$, then for any $\lambda > 0$ so do λH_0 and λE_0.
Uniqueness of the Hurst index H_0 can be recovered by choosing a normalization for E_0, for example $\mathrm{Tr}(E_0) = d$. However, even under this assumption, E_0 is not necessarily unique. Nevertheless remark that, under the assumption $\mathrm{Tr}(E_0) = d$, two anisotropies of an OSRF have necessarily the same real diagonalizable part (see Section 5.2 for a definition). We refer to Remark 2.10 of [7] for more details on the structure of the set of exponents of an OSRF.

Remark that Theorem 2.1 relies on the existence of ${}^tE_0$–homogeneous functions. Constructions of such functions have been proposed in [7] via an integral formula (Theorem 2.11). An alternative construction, more fitted for numerical simulations, can be found in [13].

3. Anisotropic concepts of smoothness

Our main goal here is to study the sample paths properties of this class of Gaussian fields in suitable anisotropic functional spaces. This approach is quite natural (see [20, 6]) since the studied model is anisotropic. To this end, suitable concepts of anisotropic smoothness are needed. The aim of this section is to give some background about the appropriate anisotropic functional spaces : Anisotropic Besov spaces. These spaces generalize classical (isotropic) Besov spaces and have been studied in parallel with them (see [9, 10] for a complete account on the results presented in this section). The definition of anisotropic Besov spaces is based on the concept of pseudo-norm. We first recall some well–known facts about pseudo-norms which can be found with more details in [22].

3.1. Preliminary results about pseudo-norms

In order to introduce anisotropic functional spaces, an anisotropic topology on $\mathbb{R}^d$ is needed. We need to introduce a slight variant of the notion of pseudo–norm introduced in [22], fitted to the case of discrete dilatations.

Definition 3.1. *Let $E \in \mathcal{E}^+$. A function ρ defined on $\mathbb{R}^d$ is a $(\mathbb{R}^d, E)$ pseudo-norm if it satisfies the three following properties :*

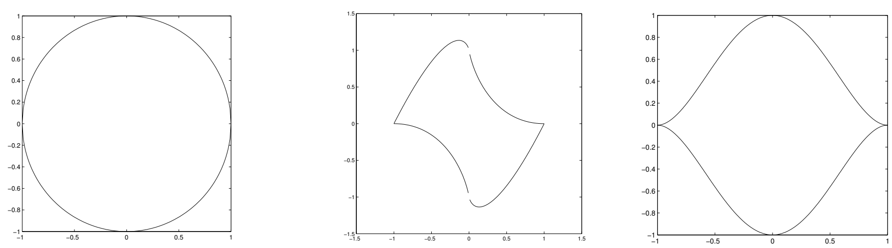
1. ρ is continuous on $\mathbb{R}^d$,

2. ρ is E-homogeneous, i.e. $\rho(a^E x) = a\rho(x) \quad \forall x \in \mathbb{R}^d, \forall a > 0$,

3. ρ is strictly positive on $\mathbb{R}^d \setminus \{0\}$.

For any $(\mathbb{R}^d, E)$ pseudo–norm, define the anisotropic sphere $S_0^E(\rho)$ as

$$S_0^E(\rho) = \{x \in \mathbb{R}^d;\ \rho(x) = 1\}\ . \tag{3.1}$$



$$E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

Figure 2.
Examples of anisotropic spheres for different anisotropies.

Proposition 3.1. *For all $x \in \mathbb{R}^d \setminus \{0\}$, there exists an unique couple $(r, \theta) \in \mathbb{R}_+^* \times S_0^E(\rho)$ such that $x = r^E \theta$.*

Moreover $S_0^E(\rho)$ is a compact of $\mathbb{R}^d$ and the map

$$(r, \theta) \to x = r^E \theta\ ,$$

is an homeomorphism from $\mathbb{R}_+^* \times S_0^E(\rho)$ to $\mathbb{R}^d \setminus \{0\}$.

The term “pseudo-norm” is justified by the following proposition :

Proposition 3.2. *Let ρ a $(\mathbb{R}^d, E)$ pseudo-norm. There exists a constant $C > 0$ such that*

$$\rho(x + y) \leq C(\rho(x) + \rho(y)), \quad \forall x, y \in \mathbb{R}^d\ . \tag{3.2}$$

The following key property allows to define an anisotropic topology on $\mathbb{R}^d$ based on pseudo-norms and then anisotropic functional spaces :

Proposition 3.3. *Let ρ_1 and ρ_2 be two $(\mathbb{R}^d, E)$ pseudo-norms. They are equivalent in the following sense : There exists a constant $C > 0$ such that*

$$\frac{1}{C}\rho_1(x) \leq \rho_2(x) \leq C\rho_1(x), \quad \forall x \in \mathbb{R}^d .$$

In particular, two different $(\mathbb{R}^d, E)$ pseudo-norms define the same topology on $\mathbb{R}^d$.

3.2. Anisotropic Besov spaces

Let $E \in \mathcal{E}^+$ and $\rho_{^tE}$ a fixed $(\mathbb{R}^d,{}^tE)$–pseudo-norm. For $x_0 \in \mathbb{R}^d$ and $r > 0$, $B_{\rho_{t_E}}(x_0, r)$ denotes the anisotropic ball of center x_0 and radius r, namely

$$B_{\rho_{t_E}}(x_0, r) = \{x \in \mathbb{R}^d, \rho_{^tE}(x - x_0) \ \leq r\} .$$

The definition of anisotropic Besov spaces is based on the following result :

Proposition 3.4. *Let $\psi_0^E \in \mathcal{S}(\mathbb{R}^d)$ be such that*

$$\begin{cases} \widehat{\psi_0^E}(\xi) = 1 \ \text{if} \ \rho_{^tE}(\xi) \ \leq 1, \\ \widehat{\psi_0^E}(\xi) = 0 \ \text{if} \ \rho_{^tE}(\xi) \ \geq 2 . \end{cases}$$

For any positive integer j, set

$$\widehat{\psi_j^E}(\xi) = \widehat{\psi_0^E}(2^{-j^tE}\xi) - \widehat{\psi_0^E}(2^{-(j-1)^tE}\xi) .$$

Then

$$\sum_{j=0}^{+\infty} \widehat{\psi_j^E} \equiv 1 ,$$

is an anisotropic partition of the unity satisfying,

$$\operatorname{supp}(\widehat{\psi_j^E}) \subset B_{\rho_{t_E}}(0, 2^{j+1}) \setminus B_{\rho_{t_E}}(0, 2^{j-1}) .$$

One can then deduce the definition of anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d, E)$ as follows :

Definition 3.2. *Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Define*

$$\|f\|_{B^s_{p,q}(\mathbb{R}^d,E)} = \left(\sum_{j=0}^{\infty} 2^{jsq} \|f * \psi_j^E\|^q_{L^p(\mathbb{R}^d)} \right)^{1/q} . \tag{3.3}$$

Then

$$B^s_{p,q}(\mathbb{R}^d, E) = \{f \in \mathcal{S}'(\mathbb{R}^d), \|f\|_{B^s_{p,q}(\mathbb{R}^d,E)} < +\infty\} .$$

The matrix E is called the anisotropy of the Besov space $B^s_{p,q}(\mathbb{R}^d, E)$.
In a more general way, if $\beta \in \mathbb{R}$, define

$$\|f\|_{B^s_{p,q,|\log|^\beta}(\mathbb{R}^d,E)} = \left(\sum_{j=0}^{\infty} j^{-\beta q} 2^{jsq} \|f * \psi_j^E\|^q_{L^p(\mathbb{R}^d)} \right)^{1/q} .$$

Then

$$B^s_{p,q,|\log|^\beta}(\mathbb{R}^d, E) = \{ f \in \mathcal{S}'(\mathbb{R}^d), \|f\|_{B^s_{p,q,|\log|^\beta}(\mathbb{R}^d,E)} < +\infty \} .$$

Remark 3.1. One can prove that this definition is independent of the choice of the function ψ_0^E involved in the definition of the Besov space $B^s_{p,q}(\mathbb{R}^d, E)$.

Remark 3.2. Let $E \in \mathcal{E}^+$ and $\rho_{^tE}$ a $(\mathbb{R}^d, {}^tE)$ pseudo-norm. For any $\lambda > 0$, $\rho_{^tE}^{1/\lambda}$ is a $(\mathbb{R}^d, \lambda\, {}^tE)$ pseudo-norm. Hence for any $s > 0$, $B^{\lambda s}_{p,q}(\mathbb{R}^d, \lambda E) = B^s_{p,q}(\mathbb{R}^d, E)$.

So, as stated in Section 1 without loss of generality, we assume in the sequel that $\mathrm{Tr}(E) = d$. We then define

$$\mathcal{E}^+_d = \{ E \in \mathcal{E}^+,\, \mathrm{Tr}(E) = d \} ,$$

where $\mathcal{E}^+$ is the collection of $d \times d$ matrices whose eigenvalues have positive real part. As it is the case for isotropic spaces, anisotropic Hölder spaces $\mathcal{C}^s(\mathbb{R}^d, E)$ can be defined as particular anisotropic Besov spaces.

Definition 3.3. *Let s be in $\mathbb{R}$ and $\beta \in \mathbb{R}$. The anisotropic Hölder spaces $\mathcal{C}^s(\mathbb{R}^d, E)$ and $\mathcal{C}^s_{|\log|^N}(\mathbb{R}^d, E)$ are defined by*

$$\mathcal{C}^s(\mathbb{R}^d, E) = B^s_{\infty,\infty}(\mathbb{R}^d, E) \quad \textit{and} \quad \mathcal{C}^s_{|\log|^\beta}(\mathbb{R}^d, E) = B^s_{\infty,\infty,|\log|^\beta}(\mathbb{R}^d, E) .$$

Proposition 3.5. *Let $0 < s < \lambda_{\min}(E)$ and $\beta \in \mathbb{R}$. Then, for any $(\mathbb{R}^d, E)$ pseudo-norm ρ_E, the two norms*

- $$\|f\|_{L^\infty(\mathbb{R}^d)} + \sup_{\rho_E(h) \le 1} \sup_{x \in \mathbb{R}^d} \left(\frac{|f(x+h) - f(x)|}{\rho_E(h)^s |\log(\rho_E(h))|^\beta} \right) ,$$
- $\|f\|_{B^s_{\infty,\infty,|\log|^\beta}}$ *defined by* (3.3),

are equivalent in $\mathcal{C}^s_{|\log|^\beta}(\mathbb{R}^d, E)$.

Remark 3.3. Anisotropic Hölder spaces admit a characterization by finite differences of order $M \geq 1$ under the general assumption $s > 0$. Here, we only need to deal with the case $0 < s < \lambda_{\min}(E)$ and have thus stated Proposition 3.5 in this special setting.

Let us comment Proposition 3.5. Let $0 < s < \lambda_{\min}(E)$ and $N \in \mathbb{R}$. A bounded function f belongs to $\mathcal{C}^s_{|\log|^\beta}(\mathbb{R}^d, E)$ if and only if for any $r \in (0,1)$, $\Theta \in S_0^E(\rho_E)$ and $x \in \mathbb{R}^d$, one has

$$|f(x + r^E\Theta) - f(x)| \leq C_0 r^s |\log(r)|^\beta \ ,$$

for some $C_0 > 0$.
Hence, a function f belongs to the Hölder space $\mathcal{C}^s_{|\log|^\beta}(\mathbb{R}^d, E)$ if and only if its restriction f_Θ along any parametric curve of the form

$$r > 0 \mapsto r^E\Theta \ ,$$

with $\Theta \in S_0^E(|\cdot|_E)$ is in the usual Hölder space $\mathcal{C}^s_{|\log|^\beta}(\mathbb{R})$ and $\|f_\Theta\|_{\mathcal{C}^s_{|\log|^\beta}(\mathbb{R})}$ does not depend on Θ. Roughly speaking, the anisotropic "directional" regularity in any anisotropic "direction" has to be larger than s. In other words, we replace straight lines of isotropic setting by curves with parametric equation $r > 0 \mapsto r^E\Theta$ adapted to anisotropic setting.

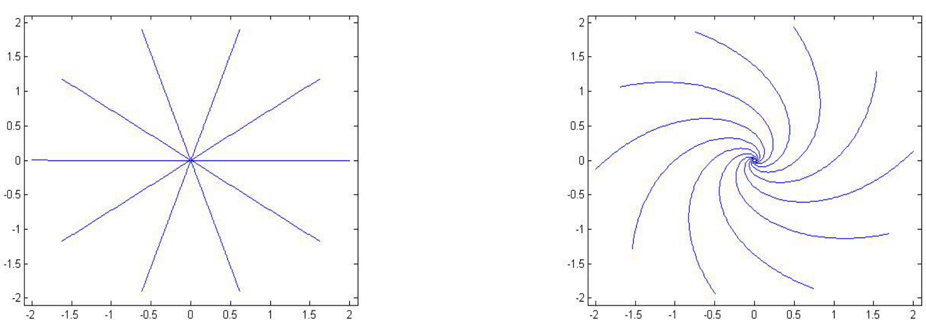


Figure 3.
"Isotropic lines" and "anisotropic lines" in the case $E = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$.

To state our optimality results we need a local version of anisotropic Besov spaces :

Definition 3.4. *Let $E \in \mathcal{E}^+$ be a fixed anisotropy, $0 < p, q \leq \infty$, $0 < s < \infty$ and $f \in L^p_{loc}(\mathbb{R}^d)$.*
The function f belongs to $B^s_{p,q,loc}(\mathbb{R}^d, E)$ if for any $\varphi \in \mathcal{D}(\mathbb{R}^d)$, the function

φf belongs to $B^s_{p,q}(\mathbb{R}^d, E)$.
The spaces $B^s_{p,q,|\log|^\beta,loc}(\mathbb{R}^d, E)$ can be defined in an analogous way for any $0 < p, q \leq \infty$, $0 < s < \infty$, $\beta \in \mathbb{R}$.

The anisotropic local critical exponent in anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d, E)$ of $f \in L^p_{loc}(\mathbb{R}^d)$ is then defined by

$$\alpha_{f,loc}(E, p, q) = \sup\{s,\ f \in B^s_{p,q,loc}(\mathbb{R}^d, E)\}\ .$$

In the special case $p = q = \infty$, this exponent is also called the anisotropic local critical exponent in anisotropic Hölder spaces of $f \in L^\infty_{loc}(\mathbb{R}^d)$ and is denoted by $\alpha_{f,loc}(E)$.

4. Statement of our results

In what follows, we are given $E_0 \in \mathcal{E}^+$ and ρ_{E_0} a $(\mathbb{R}^d,\ {}^tE_0\ )$ pseudo-norm. We denote $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ the OSRGF with exponent E_0 and Hurst index H_0 defined by (2.1) with $\rho = \rho_{E_0}$.

We first state our optimality result and characterize in some sense an anisotropy E_0 and an Hurst index of the field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$. These results come from an accurate study of sample paths properties of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Besov spaces (see Theorem 4.2 just below).

We assume–without loss of generality–that $E_0 \in \mathcal{E}^+_d$, namely that all the eigenvalues of E_0 have a positive real part and that $\mathrm{Tr}(E_0) = d$. Our results will be based on a comparison between the topology related to the pseudo–norm ρ_{E_0} involved in the construction of the Gaussian field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ and this of the analyzing spaces $B^s_{p,q}(\mathbb{R}^d, E)$. To be able to compare these two topologies, we also assume that $E \in \mathcal{E}^+_d$.

The main result of this paper is the following one :

Theorem 4.1. *Let $(p, q) \in [1, +\infty]^2$ and $E_0 \in \mathcal{E}^+_d$. Then almost surely*

$$\begin{aligned}\alpha_{X_{\rho_{E_0},H_0},loc}(E_0,p,q) &= \sup\{\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q), E \in \mathcal{E}^+_d, E \textit{ commuting with } E_0\}\\ &= H_0\ ,\end{aligned}$$

that is the value $E = E_0$ maximizes the anisotropic local critical exponent of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ among all possible anisotropic local critical exponent in anisotropic Besov spaces with an anisotropy E commuting with E_0.

Remark 4.1. Since E and E_0 are commuting, these matrices admit the same spectral decomposition. Hence, in fact we proved that any anisotropy E_0 maximize the critical exponent among matrices having the same spectral decomposition. Thus, in the general case, we implicitly assumed that the spectral decomposition of anisotropy matrix is known. In dimension two, we have a stronger optimality result about anisotropy E_0 and Hurst index H_0, involving matrices of $\mathcal{E}_d^+$ which do not commute necessilary.

To prove Theorem 4.1, we investigate the local regularity of the sample paths of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in general anisotropic Besov spaces. But before any statement, we first need some background about the concept of real diagonalizable part of a square matrix. This notion is based on real additive Jordan decomposition of a square matrix (see for e.g. to Lemma 7.1 chap 9 of [18] where a multiplicative version of Proposition 4.1 is given) :

Proposition 4.1. *Any matrix M of $M_d(\mathbb{R})$ can be decomposed into a sum of three commuting real matrices*

$$M = D + S + N\ ,$$

where D is a diagonalizable matrix in $M_d(\mathbb{R})$, S is a diagonalizable matrix in $M_d(\mathbb{C})$ with zero or imaginary complex eigenvalues, and N is a nilpotent matrix. Matrix D is called the real diagonalizable part of M, S its imaginary semi-simple part, and N its nilpotent part.

Now we are given two **commuting** matrices E_0, E of $\mathcal{E}_d^+$. Let D_0 (resp D) be the real diagonalizable part of matrix E_0 (resp E). Since matrices E_0 and E are commuting, so do matrices D_0 and D. Furthermore, matrices D_0 and D are diagonalizable in $M_d(\mathbb{R})$ then they are simultaneously diagonalizable. Up to a change of basis, we may assume that D_0 and D are two diagonal matrices. More precisely, suppose that

$$D_0 = \begin{pmatrix} \lambda_1^0 Id_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_m^0 Id_{d_m} \end{pmatrix},\ D = \begin{pmatrix} \lambda_1 Id_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_m Id_{d_m} \end{pmatrix}, \tag{4.1}$$

with

$$\frac{\lambda_m}{\lambda_m^0} \le \cdots \le \frac{\lambda_1}{\lambda_1^0}. \tag{4.2}$$

Since $\mathrm{Tr}(E_0) = \mathrm{Tr}(E) = d$, one has $\lambda_m/\lambda_m^0 \le 1$.

The regularity results about sample path of the field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ are summed up in the following theorem.

Theorem 4.2. *Let $1 \le p \le +\infty$, $1 \le q \le +\infty$. Almost surely the anisotropic local critical exponent $\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q)$ in anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d,E)$ of the OSRGF $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ satisfies*

$$\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q) = \frac{\lambda_m H_0}{\lambda_m^0} \le H_0\ .$$

In particular, in the special case $E = E_0$, one has $\alpha_{X_{\rho_{E_0},H_0},loc}(E,p,q) = H_0$.

In other words Theorem 4.2 asserts that when one measures local regularity of the sample paths along anisotropic directions different from those associated to an anisotropy of the field E_0, one loses smoothness. The further the anisotropic direction of measure from the genuine anisotropic direction associated to the field are, the smaller the anisotropic local critical exponent is. This anisotropic local critical exponent can take any value in the range $(0, H_0]$.

The special case $p = q = +\infty$ yields us the following result about anisotropic Hölderian regularity of the sample paths.

Corollary 4.1. *Almost surely the anisotropic local critical exponent of the sample paths of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Hölder spaces equals $(\lambda_m H_0)/\lambda_m^0$ and is always lower than H_0. In particular, if $E = E_0$ this critical exponent equals the Hurst index H_0.*

Remark 4.2. This estimate on anisotropic local critical exponent was already known in the case $E = E_0$ (see [6]).

Theorem 4.2 allows us to obtain regularity results which extend those proved in the case $p = q = \infty$ in the usual isotropic setting. Since matrices E_0 and Id are commuting, we can apply the above result to the case $E = Id$. Note that in this case $\lambda_m^0 = \lambda_{\max}(E_0)$. We obtain the following proposition:

Proposition 4.2. *Almost surely the local critical exponent of the sample paths of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in classical Besov spaces equals $H_0/\lambda_{\max}(E_0)$.*

In particular, for $p = q = \infty$, almost surely the local critical exponent of the sample paths of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ in classical Hölder spaces equals $H_0/\lambda_{\max}(E_0)$.

Remark 4.3. In the special case $p = q = \infty$, we recover already known results about classic Hölderian regularity (see Theorem 5.4 of [7]). Recall that this theorem is based on directional regularity results about the Gaussian field $\{X_{\rho_{E_0},H_0}\}$ and comes from an estimate of the variogram $v_{X_{\rho_{E_0},H_0}}(h) = \mathbb{E}(|X_{\rho_{E_0},H_0}(h)|^2)$ along special directions related to the spectral decomposition of matrix E_0. Here our approach is different and based on wavelet technics.

5. Complements and proofs

5.1. Role of the real diagonalizable part of the anisotropy E of the analysing spaces $B^s_{p,q}(\mathbb{R}^d, E)$

We will first prove that measuring smoothness in the general Besov spaces $B^s_{p,q}(\mathbb{R}^d, E)$ may be deduced from the special case where the matrix E is diagonalizable. To this end, we show the following embedding property:

Proposition 5.1. *Assume that $E_1 \in \mathcal{E}_d^+$ and $E_2 \in \mathcal{E}_d^+$ have the same real diagonalizable part D. Let $\rho_{{}^tE_1}$ (resp $\rho_{{}^tE_2}$)) a $(\mathbb{R}^d, {}^tE_1)$ (resp $(\mathbb{R}^d, {}^tE_2)$) pseudo–norm. Then for any $\alpha > 0$ and any $(p,q) \in [1,+\infty]^2$ one has,*

$$B^\alpha_{p,q,|\log|^{-\frac{d}{\rho_{\min}(D)}-1}}(\mathbb{R}^d, E_1) \hookrightarrow B^\alpha_{p,q}(\mathbb{R}^d, E_2) \hookrightarrow B^\alpha_{p,q,|\log|^{\frac{d}{\rho_{\min}(D)}+1}}(\mathbb{R}^d, E_1)\ . \tag{5.1}$$

As a direct consequence, we obtain Corollary 5.1.

Corollary 5.1. *The anisotropic local critical exponent*

$$\alpha_{X,loc}(E,p,q) = \sup\{s > 0,\ X(\cdot) \in B^s_{p,q,loc}(\mathbb{R}^d, E)\}\ ,$$

of any Gaussian field $\{X(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d, E)$ depends only on the real diagonalizable part of E.

Note that this result does not depend on the studied Gaussian field but of the analyzing functional spaces. Hence, it does not give any information about the anisotropic properties of the field.

We now show Proposition 5.1. The proof of this result relies on the following lemma :

Lemma 5.1. *Assume that E_1 and E_2 are two matrices of $\mathcal{E}_d^+$ having the same real diagonalizable part D. Then there exists two positive constants c_1 and c_2 such that, for all $x \in \mathbb{R}^d$,*

$$c_1\rho_{{}^tE_2}(x)(1+|\log(\rho_{{}^tE_2}(x)|)^{-\frac{d}{\lambda_{\min}(D)}} \le \rho_{{}^tE_1}(x) \le c_2\rho_{{}^tE_2}(x)(1+|\log(\rho_{{}^tE_2}(x)|)^{\frac{d}{\lambda_{\min}(D)}} \tag{5.2}$$

Proof of Lemma 5.1. Using polar coordinates associated to ${}^tE_1$, one has, for $x \in \mathbb{R}^d$,

$$x = r^{{}^tE_1}\Theta, (r,\Theta) \in \mathbb{R}_+^* \times S_0^{{}^tE_1}(\rho_{{}^tE_1})\ .$$

Denote $F_1 = E_1 - D$, $F_2 = E_2 - D$. Then

$$\begin{aligned} \rho_{{}^tE_2}(x) &= \rho_{{}^tE_2}\left(r^{{}^tE_2} \cdot (r^{-D} r^{-{}^tF_2}) \cdot (r^D r^{{}^tF_1}\Theta)\right) \\ &\le r\rho_{{}^tE_2}(r^{-{}^tF_2} r^{{}^tF_1}\Theta)\ , \end{aligned}$$

because F_1, F_2, D are pairwise commuting matrices. Observe now that F_1, F_2 have only pure imaginary eigenvalues. Hence, by Lemma 2.1 of [7], one deduces that for any $\varepsilon > 0$

$$\rho_{{}^tE_2}(x) \quad \leq \quad Cr\max(|r^{-{}^tF_2}r^{{}^tF_1}\Theta|^{\frac{1}{\rho_{\min}(D)-\varepsilon}}, |r^{-{}^tF_2}r^{{}^tF_1}\Theta|^{\frac{1}{\rho_{\max}(D)+\varepsilon}})\ ,$$

where $|\cdot|$ denotes the usual Euclidean norm. Denote $\|\cdot\|$ an operator norm on $M_d(\mathbb{R})$. Since Θ belongs to the anisotropic sphere $S_0(\rho_{{}^tE_1})$ which is compact, one has

$$\begin{aligned}\rho_{{}^tE_2}(x) &\leq Cr\max(\|r^{-{}^tF_2}r^{{}^tF_1}\|^{\frac{1}{\rho_{\min}(D)-\varepsilon}}, \|r^{-{}^tF_2}r^{{}^tF_1}\|^{\frac{1}{\rho_{\max}(D)+\varepsilon}}) \\ &\leq Cr\ (1+|\log(r)|)^{\frac{d-1}{\rho_{\min}(D)-\varepsilon}} \\ &\leq Cr\ (1+|\log(r)|)^{\frac{d}{\rho_{\min}(D)}}\ ,\end{aligned}$$

for $\varepsilon > 0$ sufficiently small. We then proved Lemma 5.1. We now show Proposition 5.1.

Proof of Proposition 5.1. Using two anisotropic Littlewood-Paley analysis associated respectively to matrices E_1, E_2 and D and the lemma above, we deduce (5.1). Indeed, for any $i \in \{1,2\}$, let $(\psi_j^{E_i})_{j\in\mathbb{N}}$ an anisotropic Littlewood–Paley analysis of Besov spaces $B^{\alpha}_{p,q}(\mathbb{R}^d, E_i)$. By definition,

$$\mathrm{supp}(\widehat{\psi_1^{E_i}}) \subset \{\xi \in \mathbb{R}^d,\ 1 \leq \rho_{{}^tE_i}(x)\ \leq 4\}\ ,$$

for $i \in \{1,2\}$. Then there exists some $j_0 \in \mathbb{N}$ such that for any $j \in \mathbb{N}$, one has

$$\begin{aligned}\mathrm{supp}(\widehat{\psi_j^{E_2}}) &\subset \{\xi, 2^{j-1} \leq \rho_{{}^tE_2}(\xi)\ \leq 2^{j+1}\} \\ &\subset \bigcup_{\ell=j-j_0-\frac{d\log_2(j)}{\lambda_{\min}(D)}}^{j+j_0+\frac{d\log_2(j)}{\lambda_{\min}(D)}} \{\xi, 2^{\ell-1} \leq \rho_{{}^tE_1}(\xi) \leq 2^{\ell+1}\}\ .\end{aligned}$$

Hence

$$\widehat{\psi_j^{E_2}}(\xi)\,\widehat{f}(\xi) = \widehat{\psi}_j^{E_2}(\xi)\left(\sum_{\ell=j-j_0-\frac{d\log_2(j)}{\lambda_{\min}(D)}}^{j+j_0+\frac{d\log_2(j)}{\lambda_{\min}(D)}} \widehat{\psi}_\ell^{E_1}(\xi)\,\widehat{f}(\xi)\right)\ .$$

Define q' the conjugate of q, that is the positive real satisfying $1/q+1/q' = 1$. The last inequality and Cauchy–Schwartz inequality imply that for some

$C > 0$,

$$\begin{aligned}\|f * \psi_j^{E_2}\|_{L^p}^q &\leq C(\log_2 j)^{q/q'} \left(\sum_{\ell=j-j_0-\frac{d\log_2(j)}{\lambda_{\min}(D)}}^{j+j_0+\frac{d\log_2(j)}{\lambda_{\min}(D)}} \|\psi_j^{E_2} * (\psi_\ell^{E_1} * f)\|_{L^p}^q \right) \\ &\leq C\,(\log_2 j)^{q/q'} \|\psi_0^{E_2}\|_{L^1}^q \left(\sum_{\ell=j-j_0-\frac{d\log_2(j)}{\lambda_{\min}(D)}}^{j+j_0+\frac{d\log_2(j)}{\lambda_{\min}(D)}} \|\psi_\ell^{E_1} * f\|_{L^p}^q \right) .\end{aligned}$$

Then we can give the following upper bound of $\sum_{j=1}^{J} 2^{jsq}\|f * \psi_j^{E_2}\|_{L^p}^q$:

$$\begin{aligned}\sum_{j=1}^{J} 2^{jsq}\|f * \psi_j^{E_2}\|_{L^p}^q &\leq C\sum_{j=1}^{J}(\log_2 j)^{q/q'}2^{jsq} \left(\sum_{\ell=j-j_0-\frac{d\log_2(j)}{\lambda_{\min}(D)}}^{j+j_0+\frac{d\log_2(j)}{\lambda_{\min}(D)}} \|(f * \psi_\ell^{E_1})\|_{L^p}^q \right) \\ &\leq C \sum_{\ell=1}^{J+j_0+\frac{d\log_2(J)}{\lambda_{\min}(D)}} \|f * \psi_\ell^{E_1}\|_{L^p}^q \left(\sum_{j=\ell-j_0-\frac{d\log_2(\ell)}{\lambda_{\min}(D)}}^{\ell+j_0+\frac{d\log_2(\ell)}{\lambda_{\min}(D)}} (\log_2 j)^{q/q'}2^{jsq} \right) \\ &\leq C \sum_{\ell=1}^{J+j_0+\frac{d\log_2(J)}{\lambda_{\min}(D)}} \|f * \psi_\ell^{E_1}\|_{L^p}^q 2^{\ell sq}\ell^{d/\lambda_{\min}(D)+1} .\end{aligned}$$

Let now J tends to ∞. It yields the embedding

$$B^{\alpha}_{p,q,|\log|^{-\frac{d}{\lambda_{\min}(D)}-1}}(\mathbb{R}^d, E_1) \hookrightarrow B^{\alpha}_{p,q}(\mathbb{R}^d, E_2) .$$

Permuting E_1 and E_2 yields the other inclusion.

5.2. Local regularity in anisotropic Besov spaces of the studied field

In the previous section, we proved that we can restrict our study to diagonal Besov spaces. This point is crucial for the proof of the regularity results stated in Section 4. Indeed it allows us to use tools that are only defined in the diagonal case, as anisotropic multi-resolution analysis and anisotropic wavelet bases. The aim of the following subsection is to recall the constructions of these wavelet bases.

5.2.1. Orthonormal Wavelet bases of (diagonal) anisotropic spaces

In this section, we assume that the anisotropy D of the analyzing space is diagonal (with positive eigenvalues), namely that

$$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_d \end{pmatrix} .$$

In addition we also assume that $\mathrm{Tr}(D) = d$. Our main tool will be anisotropic multi–resolution analyses defined by Triebel in [37].

Let $\{V_j, j \geq 0\}$ be a one–dimensional multi–resolution analysis of $L^2(\mathbb{R})$. Denote by ψ^F (resp. ψ^M) the corresponding scaling function (resp. wavelet function).

Notation 5.1. We denote by $\{F, M\}^{d^*}$ the set

$$\{F, M\}^{d^*} = \{F, M\}^d \setminus \{(F, \cdots, F)\} .$$

For $j \in \mathbb{N}$, we define the set $I^j(D)$ of $\{F, M\}^d \times \mathbb{N}^d$ in the following way.

- If $j = 0$, $I^0(D) = \{((F, \cdots, F), (0, \cdots, 0))\}$.

- If $j \geq 1$, $I^j(D)$ is the set of all the elements (G, γ) with $G \in \{F, M\}^{d^*}$ and $\gamma \in \mathbb{N}^d$ such that for any $r \in \{1, \cdots, d\}$:

$$\begin{array}{l} \text{If } G_r = F, \gamma_r = [(j-1)\lambda_r], \\ \text{If } G_r = M, [(j-1)\lambda_r] \leq \gamma_r < [j\lambda_r] . \end{array}$$

Finally, for $j \in \mathbb{N}$ and $(G, \gamma) \in I^j(D)$, we will denote by $D_{j,G,\gamma}$ the matrix defined by

$$D_{j,G,\gamma} = \begin{pmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_d \end{pmatrix} .$$

Finally, let us define the family of wavelets as follows. For $j \in \mathbb{N}$, $(G, \gamma) \in I^j(D)$ and $k \in \mathbb{Z}^d$, we set

$$\Psi^k_{j,G,\gamma}(x) = (\psi^{(G)})(2^{D_{j,G,\gamma}} x - k) ,$$

with

$$\psi^{(G)} = \psi_{G_1} \otimes \cdots \otimes \psi_{G_d} .$$

The anisotropic wavelet bases yield a wavelet characterisation of anisotropic Besov spaces (see [36] and [37], Theorem 5.23).

Theorem 5.1.

1. *The family* $\left\{2^{\frac{\mathrm{Tr}(D_{j,G,\gamma})}{2}}\Psi^k_{j,G,\gamma},\ j\in\mathbb{N},\ (G,\gamma)\in I^j(D),\ k\in\mathbb{Z}^d\right\}$ *is an orthonormal basis of* $L^2(\mathbb{R}^d)$.

2. *Let* $(\Psi_k^{j,G,\gamma})_{j\in\mathbb{N},(G,\gamma)\in I^j(D),k\in\mathbb{Z}^d}$ *be the family constructed from* ψ_F *and* ψ_M *Daubechies wavelets with, for some* $u\in\mathbb{N}$,

$$\psi_F\in\mathcal{C}^u(\mathbb{R}),\psi_M\in\mathcal{C}^u(\mathbb{R})\ .$$

Let $0<p,q\leq\infty$ *and* $s,N\in\mathbb{R}$. *There exists an integer* $u(s,p,D)$ *such that if* $u>u(s,p,D)$, *for any tempered distribution* f *the two following assertions are equivalent*

   (a) $f\in B^s_{p,q,|\log|^\beta}(\mathbb{R}^d,D)$.

   (b) $f=\sum c^k_{j,G,\gamma}\Psi^k_{j,G,\gamma}$ *with*

$$\sum_{j,G,\gamma} j^{-\beta q}2^{j(s-\frac{d}{p})q}\left(\sum_k |c^k_{j,G,\gamma}|^p\right)^{\frac{q}{p}}<+\infty\ ,$$

the convergence being in $\mathcal{S}'(\mathbb{R}^d)$.

The above expansion is then unique and

$$c^k_{j,G,\gamma}=<f,2^{\mathrm{Tr}(D_{j,G,\gamma})}\Psi^k_{j,G,\gamma}>\ . \tag{5.3}$$

Remark 5.1. An analogous result is stated (see [37], Theorem 5.24) replacing Daubechies wavelets by Meyer wavelets. In that case, $u=+\infty$.

We now prove our regularity results about the sample path of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ based on wavelet characterization of Besov spaces.

5.2.2. Local regularity of the field $\{X_{E_0,H_0}(x)\}_{x\in\mathbb{R}^d}$ in anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d,D_0)$

Assume that we are given a Gaussian field $\{X_{E_0,H_0}(x)\}_{x\in\mathbb{R}^d}$ of the form (2.1) where $E_0\in\mathcal{E}_d^+$ and $H_0\in(0,\lambda_{\min}(E_0))$. The aim of this section is to prove :

Proposition 5.2. *Let* $1\leq p,q\leq+\infty$. *Define* δ *on* $(0,+\infty]$ *as follows :*

$$\delta(p)=\begin{cases}3/2 \ \textit{if}\ p=+\infty,\\ 1\ \textit{otherwise.}\end{cases}$$

Then one has

1. *For any $\beta > 1/q + d/\lambda_{\min}(E_0) + \delta(p)$, almost surely, the sample path of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ belongs to $B^{H_0}_{p,q,|\log|^{\beta},loc}(\mathbb{R}^d, D_0)$,*
2. *For $\beta = 1/q + d/\lambda_{\min}(E_0) + \delta(p)$, almost surely, the sample path of $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ does not belong to $B^{H_0}_{p,q,|\log|^{-\beta},loc}(\mathbb{R}^d, D_0)$.*

Adapting to our setting a result of [22], we first remark that there exists $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$ $(\mathbb{R}^d, E_0)$ pseudo–norms

Lemma 5.2. *Let $E_0 \in \mathcal{E}^+$ and φ be a $\mathcal{C}^\infty$ non–negative function compactly supported in $\mathbb{R}^d \setminus \{0\}$. The function ρ defined on $\mathbb{R}^d$, by*

$$\rho(x) = \int_0^\infty \varphi(a^{-E_0}x)\mathrm{d}a \ ,$$

is a $(\mathbb{R}^d, E_0)$ pseudo-norm belonging to $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$.

We now prove that the belongness of the sample paths to anisotropic Besov spaces of any OSRGF of the form $\{X_{\rho,H_0}\}_{x\in\mathbb{R}^d}$ do not depend on the $(\mathbb{R}^d, {}^tE_0)$ pseudo-norm ρ involved in the construction of the field.

Lemma 5.3. *Let $E_0 \in \mathcal{E}^+_d$ and ρ_1, ρ_2 two $({}^tE_0, \mathbb{R}^d)$ pseudo–norms. Denote respectively $\{X_1(x)\}_{x\in\mathbb{R}^d}$ and $\{X_2(x)\}_{x\in\mathbb{R}^d}$ the two OSSRGF defined from ρ_1 and ρ_2. Then, for any $s > 0$, $\beta \in \mathbb{R}$, $(p, q) \in (0,\infty]^2$, a.s. X_1 belongs to $B^s_{p,q,|\log|^\beta}(\mathbb{R}^d, D_0)$ iff a.s. X_2 belongs to $B^s_{p,q,|\log|^\beta}(\mathbb{R}^d, D_0)$.*

Proof. Remark first that using the same approach than in Lemma 2 of [8] and an anisotropic version of Kolmogorov Centsov Theorem we can prove that a.s.

$$x \mapsto \int_{|\xi|\le R} (\mathrm{e}^{\mathrm{i}<x,\xi>} - 1)\rho_1^{-H_0-d/2}(\xi)\mathrm{d}\widehat{W}(\xi), x \mapsto \int_{|\xi|\le R} (\mathrm{e}^{\mathrm{i}<x,\xi>} - 1)\rho_2^{-H_0-d/2}(\xi)\mathrm{d}\widehat{W}(\xi)$$

both belong to $\mathcal{C}^r(K, D_0) \hookrightarrow B^s_{p,q}(\mathbb{R}^d, D_0)$ for any compact subset K of $\mathbb{R}^d$ and any $r > \min_{\lambda\in Sp(\Delta)} \lambda$. Since any $({}^tE_0, \mathbb{R}^d)$ pseudo–norms ρ_1, ρ_2 are equivalent, Lemma 5.3 is then a straightforward consequence of Theorem 1.1 of [12] applied with $B = B^s_{p,q}(\mathbb{R}^d, D_0)$ which is either a separable Banach space either the dual of the separable space $B = B^{-s}_{p',q'}(\mathbb{R}^d, D_0)$ with p', q' the respective conjugates of p, q and to

$$f_X = \rho_1^{-2H_0-d} 1_{|\xi|\ge R} \text{ and } f_Y = \rho_2^{-H_0-d} 1_{|\xi|\ge R} \ ,$$

and

$$f_X = \rho_2^{-2H_0-d} 1_{|\xi|\ge R} \text{ and } f_Y = \rho_1^{-H_0-d} 1_{|\xi|\ge R} \ ,$$

successively.

Thus, using Lemmas 5.2 and 5.3, we assume without loss of generality from now that the $(\mathbb{R}^d, E_0)$ pseudo–norm ρ_{E_0}, used to define the field

$\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ belongs to $\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$. We shall use this assumption when proving that the wavelet coefficients of the field are weakly dependent (see Section 6).

From now, we are given a $(\mathbb{R}^d, D_0)$ pseudo–norm, ρ_{D_0}. Observe that, to prove our local regularity results, we have to investigate the sample paths properties of φX for any function $\varphi\in\mathcal{D}(\mathbb{R}^d)$, that is for any $\varphi\in\mathcal{D}(\mathbb{R}^d)$ satisfying $\mathrm{supp}(\varphi)\subset B_{\rho_{D_0}}(k_0,r_0)=\{x\in\mathbb{R}^d,\ \rho_{D_0}(x-k_0)\le r_0\}$ where $k_0\in\mathbb{Z}^d$, $r_0>0$. Since the Besov spaces are invariant by translations and dilatations, we may assume that $k_0=0$ and $r_0=1$. We have then to study the sample paths properties of the field φX for any function $\varphi\in\mathcal{D}(\mathbb{R}^d)$ such that $\mathrm{supp}(\varphi)\subset B_{\rho_{D_0}}(0,1)$.

Our results come from the series expansion of $X_{\rho_{E_0},H_0}$ in a Daubechies anisotropic wavelet basis (see Section 5.2.1 just above). Recall that for any $j\in\mathbb{N},(G,\gamma)\in I_j(D)$, the wavelet coefficients of $X_{\rho_{E_0},H_0}$ are defined as

$$c^k_{j,G,\gamma}=< X_{\rho_{E_0},H_0}, 2^{\mathrm{Tr}(D_{j,G,\gamma})}\Psi^k_{j,G,\gamma}> \ .$$

Define $\Gamma_0(D_0)=\emptyset$ for $j=0$ and for any $j\ge 1$

$$\Gamma_j(D_0)=\{k\in\mathbb{Z}^d,\ \rho_{D_0}(k)<j2^j\}\ . \tag{5.4}$$

Thereafter set

$$X^{(1)}_{\rho_{E_0},H_0}(x)=\sum_{j,G,\gamma}\sum_{k\in\Gamma_j(D_0)}c^k_{j,G,\gamma}(\omega)\Psi^k_{j,G,\gamma}(x), \tag{5.5}$$

and

$$X^{(2)}_{\rho_{E_0},H_0}(x)=\sum_{j,G,\gamma}\sum_{k\notin\Gamma_j(D_0)}c^k_{j,G,\gamma}(\omega)\Psi^k_{j,G,\gamma}(x)\ . \tag{5.6}$$

We will investigate separately the local sample path properties in anisotropic Besov spaces of the two Gaussian fields $X^{(1)}_{\rho_{E_0},H_0}$ and $X^{(2)}_{\rho_{E_0},H_0}$. We first prove that

Proposition 5.3. *Let $1\le p,q\le\infty$.*

1. *Almost surely, for any $\beta>1/q+d/\lambda_{\min}(E_0)+\delta(p)$, the sample path of the field $\{X^{(1)}_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ belongs to $B^{H_0}_{p,q,|\log|^\beta}(\mathbb{R}^d,D_0)$.*

2. *Let φ such that $\mathrm{supp}(\varphi)\subset B_{\rho_{D_0}}(0,1)$ and satisfying*

$$\varphi\equiv 1 \ \text{on}\ B_{\rho_{D_0}}(0,1/2)\ .$$

Then almost surely, for $\beta=1/q+d/\lambda_{\min}(E_0)+\delta(p)$ the sample path of the field $\{\varphi X^{(1)}_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ does not belong to $B^{H_0}_{p,q,|\log|^{-\beta}}(\mathbb{R}^d,D_0)$.

Proof. The proof uses several technics introduced in [11]. The result comes from a comparison between $\left[\sum_{k\in\Gamma_j(D_0)}|c^k_{j,G,\gamma}|^p\right]^{1/p}$ and $\left[\mathbb{E}(|c^k_{j,G,\gamma}|^2)\right]^{1/2}$ and from Lemma 6.1 which gives an estimate of $\left[\mathbb{E}(|c^k_{j,G,\gamma}|^2)\right]^{1/2}$. Set

$$g^k_{j,G,\gamma} = \frac{c^k_{j,G,\gamma}}{\left[\mathbb{E}(|c^k_{j,G,\gamma}|^2)\right]^{1/2}} \ . \tag{5.7}$$

for any $j \in \mathbb{N}$, $(G,\gamma) \in I_j$ and $k \in \Gamma_j(D_0)$. We need to distinguish two cases : $p \neq \infty$ and $p = \infty$. In each case, we prove successively points (i) and (ii).

Assume first that $p \neq \infty$ and let us prove point (i) in this case. The definition of the sequence $(g^k_{j,G,\gamma})$ and the stationarity for any (j,G,γ) of the sequence $(c^k_{j,G,\gamma}, k \in \mathbb{Z}^d)$ implies that for any j,G,γ

$$\left(\sum_{k\in\Gamma_j(D_0)}|c^k_{j,G,\gamma}|^p\right)^{1/p} = \left[\mathbb{E}(|c^0_{j,G,\gamma}|^2)\right]^{1/2}\cdot\left(\sum_{k\in\Gamma_j(D_0)}|g^k_{j,G,\gamma}|^p\right)^{1/p} \ .$$

Use now the weak correlation of the wavelet coefficients and the two estimates of $\left[\mathbb{E}(|c^0_{j,G,\gamma}|^2)\right]^{1/2}$ and of $n_j = \mathrm{card}(\Gamma_j(D_0))$ respectively proved in Lemmas 6.1 and 6.2. One deduces that the following inequality holds for any $j \geq 0$:

$$\left(\sum_{k\in\Gamma_j(D_0)}|c^k_{j,G,\gamma}|^p\right)^{\frac{1}{p}} \leq C2^{j(\frac{d}{p}-H_0)}j^{d^*}\left(\frac{1}{n_j}\sum_{k\in\Gamma_j(D_0)}|g^k_{j,G,\gamma}|^p\right)^{\frac{1}{p}} \ , \tag{5.8}$$

where

$$d^* = \frac{d}{2\lambda_{\min}(E_0)} + \frac{d}{p} \ .$$

Lemma 6.3 stating a central limit theorem for the sequence $(g^k_{j,G,\gamma})$ and inequality (5.8) then prove point (i) of the proposition for the case $p < \infty$.

We now prove point (ii) for $p \neq \infty$. Set now

$$\Gamma'_j(D_0) = \{k \in \mathbb{Z}^d,\ \rho_{D_0}(k) \leq 2^j/j\} \ .$$

Using the assumptions on the support of φ, remark that for j sufficiently large and for any $k \in \Gamma'_j(D_0)$, one has

$$c^k_{j,G,\gamma}(\varphi X) = c^k_{j,G,\gamma}(X) \ .$$

Use the same arguments as in the proof of Lemma 6.2 and deduce that $n'_j = \mathrm{Card}(\Gamma'_j(D_0)) \sim j^{-d}2^{jd}$. Since $\Gamma'_j(D_0) \subset \Gamma_j(D_0)$, a similar approach to above then yields that for some $C > 0$ and for any $j \geq 1$

$$\begin{aligned}\left(\sum_{k\in\Gamma_j(D_0)} |c^k_{j,G,\gamma}|^p\right)^{\frac{1}{p}} &\geq \left(\sum_{k\in\Gamma'_j(D_0)} |c^k_{j,G,\gamma}|^p\right)^{\frac{1}{p}} \\ &\geq C2^{j(\frac{d}{p}-H_0)}j^{-d^*}\left(\frac{1}{n'_j}\sum_{k\in\Gamma'_j(D_0)} |g^k_{j,G,\gamma}|^p\right)^{\frac{1}{p}} .\end{aligned}$$

which directly implies point (ii) of the proposition.

If $p = \infty$, a similar approach implies that almost surely there exists some $C_1, C_2 > 0$ such that for any j, G, γ

$$C_1 2^{-jH_0} j^{-d^*}\left(\frac{1}{\sqrt{\log(n_j)}}\sup_{k\in\Gamma_j}|g^k_{j,G,\gamma}|\right) \leq \left(\sum_{k\in\Gamma_j}|c^k_{j,G,\gamma}|^p\right)^{\frac{1}{p}} ,$$

and

$$\left(\sum_{k\in\Gamma_j}|c^k_{j,G,\gamma}|^p\right)^{\frac{1}{p}} \leq C_2 2^{-jH_0} j^{d^*}\left(\frac{1}{\sqrt{\log(n_j)}}\sup_{k\in\Gamma_j}|g^k_{j,G,\gamma}|\right) ,$$

with

$$d^* = \frac{d}{2\lambda_{\min}(E_0)} .$$

Lemma 6.4 and the inequality just above then implies the result stated in point (i) for the case $p = \infty$. The proof of point (ii) for $p = \infty$ also follows from the above inequality replacing $\Gamma_j(D_0)$ with $\Gamma'_j(D_0)$ as in the case $p \neq \infty$.

We now investigate the sample paths properties of $\{X^{(2)}_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$.

Proposition 5.4. *Almost surely, the sample path of the field* $\{X^{(2)}_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ *belong to* $B^{H'}_{p,q,loc}(\mathbb{R}^d, E_0)$ *for any*

$$0 < H_0 < H' < \lambda_{\min}(D_0) = \lambda_{\min}(E_0) ,$$

and any $1 \leq p, q \leq \infty$.

Proof. Using the transference results of [37] (see Theorem 5.28) and the usual embedding of isotropic Besov spaces defined on bounded domains one remarks that

$$\mathcal{C}^{s+\varepsilon}_{loc}(\mathbb{R}^d, D_0) \subset B^s_{p,q,loc}(\mathbb{R}^d, D_0) ,$$

for any $1 \leq p, q \leq \infty$ and any $s, \varepsilon > 0$. It then suffices to prove the result for $p = q = \infty$.

Let now consider $H' \in (H_0, \lambda_{\min}(E_0))$, $\varepsilon > 0$ and $\varphi \in \mathcal{D}(\mathbb{R}^d)$. Recall that we assumed that

$$\mathrm{supp}(\varphi) \subset B_{\rho_{D_0}}(0,1) = \{x \in \mathbb{R}^d,\ \rho_{D_0}(x) \leq 1\}\ ,$$

and $0 \leq \varphi \leq 1$ on $\mathbb{R}^d$. We denote by Y the random field $\varphi X^{(2)}_{\rho_{E_0},H_0}$.

We will give an upper bound of $|Y(x+h) - Y(x)|$ for any given x in $B_{D_0}(0,1)$ and h sufficiently small. Observe that

$$Y(x+h) - Y(x) = Y_1(x,h) + Y_2(x,h)\ ,$$

with

$$Y_1(x,h) = \sum_{j,G,\gamma} \sum_{k \notin \Gamma_j(D_0)} c^k_{j,G,\gamma}(\varphi(x+h) - \varphi(x))\Psi^k_{j,G,\gamma}(x)\ ,$$
$$Y_2(x,h) = \sum_{j,G,\gamma} \sum_{k \notin \Gamma_j(D_0)} c^k_{j,G,\gamma}\varphi(x+h)(\Psi^k_{j,G,\gamma}(x+h) - \Psi^k_{j,G,\gamma}(x))\ .$$

We first bound $Y_1(x,h)$. Let $\varepsilon = 1 - H'/\lambda_{\min}(E_0)$. We now use that $\varphi \in B^{1-\varepsilon}_{\infty,\infty,loc}(\mathbb{R}^d)$, h sufficiently small and x belongs to the compact set $B_{\rho_{D_0}}(0,1)$. Hence , by Lemma 6.4 and the fast decay of the wavelets, almost surely for any $M > 0$ and for some $C > 0$ one has

$$|Y_1(x,h)| \leq C|h|^{1-\varepsilon} \sum_{j,G,\gamma} j^{d^*} 2^{-jH_0} \left(\sum_{k \notin \Gamma_j(D_0)} \frac{1}{(1 + |k - 2^{D_{j,G,\gamma}}x|)^M} \right)\ .$$

Here we denoted $d^* = 1/2 + d/\lambda_{\min}(E_0)$. Further, by assumption on k and x

$$\rho_{D_0}(k) \geq j2^j \geq j\rho_{D_0}(2^{D_{j,G,\gamma}}x)\ .$$

Since ρ_{D_0} is a $(\mathbb{R}^d, D_0)$ pseudo–norm, by the triangular inequality (3.2), one deduces that for j sufficiently large

$$\rho_{D_0}(k - 2^{D_{j,G,\gamma}}x) \geq C\rho_{D_0}(k)\ .$$

for some $C \in (0,1)$. Then by comparison between ρ_{D_0} and the usual Euclidean norm, one deduces that there exists some $\alpha > 0$ such that for j sufficiently large and any x in $B_{\rho_{D_0}}(0,1)$

$$|k - 2^{D_{j,G,\gamma}}x| \geq (|k|/2)^{\alpha}\ .$$

Then

$$|Y_1(x,h)| \le C|h|^{1-\varepsilon} \left(\sum_{j,G,\gamma} j^{d^*} 2^{-jH_0} \sum_{k \notin \Gamma_j(D_0)} \frac{1}{(1+|k|^{\alpha})^M} \right) .$$

Since, for M sufficiently large

$$\sum_{j,G,\gamma} j^{d^*} 2^{-jH_0} \sum_{k \notin \Gamma_j(D_0)} \frac{1}{(1+|k|^{\alpha})^M} < \infty ,$$

one has almost surely $|Y_1(x,h)| \le C' \rho_{D_0}(h)^{H'}$.

By the same approach, we can bound $Y_2(x,h)$. Indeed, using the fact that φ is bounded and the mean value theorem for $\Psi^k_{j,G,\gamma}$, we then prove that almost surely for some $C > 0$

$$|Y_2(x,h)| \le C \sum_{j,G,\gamma} j^{d^*} 2^{-jH_0} |2^{D_{j,G,\gamma}} h| \left(\sup_{y \in [x,x+h]} \sum_{k \notin \Gamma_j(D_0)} \frac{1}{(1+|k-2^{D_{j,G,\gamma}} y|)^M} \right) .$$

The end of the proof is exactly the same as above remarking that

$$|2^{D_{j,G,\gamma}} h| \le j^{\delta} 2^j \rho_{D_0}(h)^{\lambda_{\min}(E_0)} ,$$

for some $\delta > 0$. Proposition 5.2 then follows directly from Propositions 5.3 and 5.4.

5.3. Proof of regularity results in anisotropic Besov spaces with an anisotropy commuting with this of the field

The following proposition extends the results of Proposition 5.2 in anisotropic Besov spaces $B^s_{p,q}(\mathbb{R}^d, E)$ with $E \in \mathcal{E}_d^+$ commuting with E_0.

Proposition 5.5. *Let $1 \le p, q \le +\infty$, $\varepsilon > 0$ and $E \in \mathcal{E}_d^+$ commuting with E_0. Then*

1. *Almost surely the sample path of $\{X_{\rho_{E_0},H_0}(x)\}_{x \in \mathbb{R}^d}$ belongs to $B^{H_0 \frac{\lambda_m}{\lambda_m^0} - \varepsilon}_{p,q,\mathrm{loc}}(\mathbb{R}^d, E)$.*
2. *Almost surely the sample path of $\{X_{\rho_{E_0},H_0}(x)\}_{x \in \mathbb{R}^d}$ does not belong to $B^{H_0 \frac{\lambda_m}{\lambda_m^0} + \varepsilon}_{p,q,\mathrm{loc}}(\mathbb{R}^d, E)$.*

The proof is made in several steps. First we need to compare Besov spaces with different commuting anisotropies.

5.3.1. A comparison result between Besov spaces with different commuting anisotropies

Since E and E_0 are commuting, we can then assume (up to a change of basis) that D_0 and D are two diagonal matrices of the form :

$$D_0 = \begin{pmatrix} \lambda_1^0 Id_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_m^0 Id_{d_m} \end{pmatrix}, \; D = \begin{pmatrix} \lambda_1 Id_{d_1} & & 0 \\ & \ddots & \\ 0 & & \lambda_m Id_{d_m} \end{pmatrix}, \tag{5.9}$$

with

$$\frac{\lambda_m}{\lambda_m^0} \leq \cdots \leq \frac{\lambda_1}{\lambda_1^0} \, . \tag{5.10}$$

Proposition 5.6. *The notations and assumptions are as above. For any* $\alpha > 0$*,* $\beta \in \mathbb{R}$ *and* $p, q \in (0, +\infty]$*, one has the following embedding*

$$B^{\alpha}_{p,q,|\log|^\beta}(\mathbb{R}^d, D_0) \hookrightarrow B^{\alpha \frac{\lambda_m}{\lambda_m^0}}_{p,q,|\log|^\beta}(\mathbb{R}^d, D) \, .$$

The proof is straightforward and based on finite differences characterization of Besov spaces given in Theorem 5.8 (ii) of [37].

5.3.2. Proof of Proposition 5.5

We only prove the second point of Proposition 5.5 since the first one is a straigthforward consequence of Propositions 5.1 and 5.6. To this end we use the following characterization of anisotropic Besov spaces $B^s(\mathbb{R}^d, \Delta)$ with diagonal anisotropy Δ (see Theorem 5.8 of [37]) :

Proposition 5.7. *Let* Δ *a matrix belonging to* $\mathcal{E}^+$ *of the form*

$$\Delta = \begin{pmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \alpha_d \end{pmatrix} .$$

and $s \in (0, \lambda_{\min}(\Delta))$*,* $M_\ell = [s/\alpha_\ell] + 1$ *for any* $\ell = 1, \cdots, d$*. Then* $f \in B^s_{p,p}(\mathbb{R}^d, \Delta)$ *if and only if*

$$\|f\|_{L^p} + \sum_{\ell=1}^d \left(\int_0^1 \|(\Delta^{M_\ell}_{te_\ell} f)(x)\|^p_{L^p} t^{-sp/\alpha_\ell - 1} \mathrm{d}t \right)^{1/p} < \infty \, ,$$

where (e_ℓ) *is the canonical basis of* $\mathbb{R}^d$ *and where as usual, for any* $x, h \in \mathbb{R}^d$

$$(\Delta^1_h f)(x) = f(x+h) - f(x), \, \cdots, \, (\Delta^{M_\ell}_h f)(x) = (\Delta^1_h \Delta^{M_\ell - 1} f)(x) \, .$$

Remark 5.2. If for $\ell = 1, \cdots, d$,$s \in (0, \alpha_\ell)$ then $M_\ell = 1$.

Proof. This statement is proved in Theorem 5.8 in [37].

We now prove Proposition 5.5. We first remark that we have only to consider the case where $E_0 = D_0$. Indeed, let ρ_{E_0} (resp ρ_{D_0}) be a $(\mathbb{R}^d, E_0)$ (resp a $(\mathbb{R}^d, D_0)$) pseudo–norm. Lemma 5.1 then implies that for any $\varepsilon > 0$ and any $|\xi|$ sufficiently large

$$\rho_{D_0}(\xi)^{1-\varepsilon} \le \rho_{E_0}(\xi) \le \rho_{D_0}(\xi)^{1+\varepsilon} .$$

Hence Theorem 1.1 of [12] applied successively with

$$f_X = 1_{|\xi|\le R}\rho_{E_0}(\cdot)^{-2H_0-d}, f_Y = 1_{|\xi|\le R}\rho_{D_0}(\cdot)^{-(1-\varepsilon)(2H_0+d)} ,$$

and

$$f_X = 1_{|\xi|\le R}\rho_{D_0}(\cdot)^{-(1+\varepsilon)(2H_0+d)}, f_Y = 1_{|\xi|\le R}\rho_{E_0}(\cdot)^{-2H_0-d} ,$$

as above and Proposition 5.5 proved in the case $E_0 = D_0$ yield the result in the general case $E_0 \in \mathcal{E}_d^+$.

From now, we then assume that $E_0 = D_0$ and that the $(\mathbb{R}^d, D_0)$ pseudo–norm involved in the construction of the studied field belongs to $\mathcal{C}^\infty(\mathbb{R}^d \backslash \{0\})$ which ensures the weak correlation of the wavelet coefficients.

As in the proof of Proposition 5.2, we use an expansion of the Gaussian field $X_{\rho_{D_0},H_0}$ in a Daubechies wavelet basis and we define,

$$\widetilde{X}^{(1)}(x) = \sum_{j,G,\gamma} \sum_{k\in\widetilde{\Gamma}_j(D_0)} c^k_{j,G,\gamma}(\omega)\Psi^k_{j,G,\gamma}(x), \tag{5.11}$$

and

$$\widetilde{X}^{(2)}(x) = \sum_{j,G,\gamma} \sum_{k\notin\widetilde{\Gamma}_j(D_0)} c^k_{j,G,\gamma}(\omega)\Psi^k_{j,G,\gamma}(x) . \tag{5.12}$$

where $\widetilde{\Gamma}_j(D_0) = \{k \in \mathbb{Z}^d,\ \rho_{D_0}(k) \le C_1 2^j\}$.

As in the proof of Proposition 5.2, we see that, for C_1 sufficiently large, almost surely $\widetilde{X}^{(2)}$ belongs to $B^{H_0\lambda_m/\lambda^0_m+\varepsilon}_{p,q,loc}(\mathbb{R}^d, D)$ for any $1 \le p, q \le \infty$ and $\varepsilon > 0$ sufficiently small.

We then have to prove our a.s. non local regularity results for the Gaussian field $\widetilde{X}^{(1)}$. Remark now that since the multiresolution analysis is compactly supported so is $\widetilde{X}^{(1)}$. To show point (ii) of Proposition 5.5, it is then sufficient to prove that a.s. the sample paths of $\widetilde{X}^{(1)}$ does not belong to $B^{\lambda_m H_0/\lambda^0_m+\varepsilon}(\mathbb{R}^d, D)$ for any $\varepsilon > 0$.

Set $M = [s/\lambda_m] + 1$ which may be greater than one. In view of Proposition 5.7, we shall then give an almost sure lower bound of

$$\|(\Delta^M_{te_d}\widetilde{X}^{(1)})(x)\|^p_{L^p} = \int_{\mathbb{R}^d} |(\Delta^M_{te_d}\widetilde{X}^{(1)})(x)|^p \mathrm{d}x ,$$

for any $p \geq 1$ and any t of the form $t = 2^{-[j_0\lambda_m^0]}$ where j_0 is a fixed non–negative integer.

Set

$$\Delta_{j_0} = \begin{pmatrix} [j_0\lambda_1^0] & & 0 \\ & \ddots & \\ 0 & & [j_0\lambda_m^0] \end{pmatrix} .$$

Observe that if $t = 2^{-[j_0\lambda_m^0]}$, one has $te_d = 2^{-\Delta_{j_0}}e_d$. Remark also that $\widetilde{X}^{(1)}$ can be written as the sum of its low frequency component and its high frequency component, namely that

$$\widetilde{X}^{(1)} = \widetilde{X}^{(1)}_{LF} + \widetilde{X}^{(1)}_{HF}$$

with

$$\widetilde{X}^{(1)}_{LF}(x) = \sum_{j\leq j_0}\sum_{G,\gamma}\sum_{k\in\widetilde{\Gamma}_j(D_0)} c^k_{j,G,\gamma}\Psi^k_{j,G,\gamma}(x) \text{ and}$$

$$\widetilde{X}^{(1)}_{HF}(x) = \sum_{j\geq j_0+1}\sum_{G,\gamma}\sum_{k\in\widetilde{\Gamma}_j(D_0)} c^k_{j,G,\gamma}\Psi^k_{j,G,\gamma}(x) .$$

Using the triangular inequality, one has

$$\|(\Delta^M_{te_d}\widetilde{X}^{(1)})(x)\|_{L^p} \geq \|(\Delta^M_{te_d}\widetilde{X}^{(1)}_{HF})(x)\|_{L^p} - \|(\Delta^M_{te_d}\widetilde{X}^{(1)}_{LF})(x)\|_{L^p} \qquad (5.13)$$

To give a lower bound of $\|(\Delta^M_{te_d}\widetilde{X}^{(1)})(x)\|^p_{L^p}$, we shall then give a lower bound of $\|(\Delta^M_{te_d}\widetilde{X}^{(1)}_{HF})(x)\|^p_{L^p}$ and an upper bound of $\|(\Delta^M_{te_d}\widetilde{X}^{(1)}_{LF})(x)\|^p_{L^p}$.

Let us first give an upper bound of $\|(\Delta^M_{te_d}\widetilde{X}^{(1)}_{LF})(x)\|^p_{L^p}$. We suppose that the multiresolution analysis is s smooth for some $s \in (H_0, \lambda_{\min}(D_0))$. By the finite differences definition of the spaces $\dot{B}^s_{p,\infty}(\mathbb{R}^d, D_0)$ and the fact that for any $M \geq 1$ $|\Delta^M_h f(x)| \leq \sum_{\ell=1}^M |f(x+\ell h) - f(x+(\ell-1)h)|$, one has

$$\begin{aligned} &\left\|\sum_{j\leq j_0}\sum_{G,\gamma}\sum_{k\in\widetilde{\Gamma}_j(D_0)} c^k_{j,G,\gamma}\left(\Delta^M_{2^{-\Delta_{j_0}}e_d}\Psi^k_{j,G,\gamma}\right)(x)\right\|_{L^p} \\ \leq\ & C\sum_{j\leq j_0}|2^{-\Delta_{j_0}}e_d|^s_{D_0}\left\|\sum_{G,\gamma}\sum_{k\in\widetilde{\Gamma}_j(D_0)} c^k_{j,G,\gamma}\Psi^k_{j,G,\gamma}\right\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d,D_0)} \end{aligned}$$

Use now the wavelet characterization of the homogeneous Besov spaces $\dot{B}^s_{p,\infty}(\mathbb{R}^d, D_0)$. Then for some $C > 0$

$$\left\|\sum_{G,\gamma}\sum_{k\in\widetilde{\Gamma}_j(D_0)} c^k_{j,G,\gamma}\Psi^k_{j,G,\gamma}\right\|_{\dot{B}^s_{p,\infty}(\mathbb{R}^d,D_0)} \leq C2^{j(s-d/p)}\left(\sum_{G,\gamma}\sum_{k\in\widetilde{\Gamma}_j(D_0)}|c^k_{j,G,\gamma}|^p\right)^{1/p} .$$

As the proof of Proposition 5.2, we can estimate a.s. $\sum_{k\in\widetilde{\Gamma}_j(D_0)}|c^k_{j,G,\gamma}|^p$. Hence, we deduce that there exists an a.s. positive constant C' such that

$$\left|\left|\left(\Delta^M_{te_d}\widetilde{X}^{(1)}_{LF}\right)(x)\right|\right|^p_{L^p(\mathbb{R}^d)} \le C2^{-j_0 s}\sum_{j\le j_0} 2^{j(s-\frac{d}{p})}\cdot\left(2^{j\frac{d}{p}}j^{\frac{d}{2\rho_{\min}(E_0)}}2^{-jH_0}\right)\le C'\,, \tag{5.14}$$

where C' is not depending on j_0 nor s.

We now give a lower bound of $\|(\Delta^M_{te_d}\widetilde{X}^{(1)}_{HF})(x)\|^p_{L^p}$. To this end perform the change of variable $x=2^{-\Delta_{j_0}}y$ and deduce that

$$\|(\Delta^M_{te_d}\widetilde{X}^{(1)}_{HF})(x)\|^p_{L^p} = 2^{-\mathrm{Tr}(\Delta_{j_0})}\|(\Delta^M_{te_d}\widetilde{X}^{(1)}_{HF})(2^{-\Delta_{j_0}}y)\|^p_{L^p} \tag{5.15}$$

By definition of $\widetilde{X}^{(1)}_{HF}$ one has

$$(\Delta^M_{te_d}\widetilde{X}^{(1)}_{HF})(2^{-\Delta_{j_0}}y) = \sum_{j\ge j_0+1}\sum_{G,\gamma}\sum_{k\in\widetilde{\Gamma}_j(D_0)} c^k_{j,G,\gamma}\left(\Delta^M_{te_d}\Psi^k_{j,G,\gamma}\right)(2^{-\Delta_{j_0}}y)\,.$$

Define now for any $j'\ge 1$, any $G\in\{F,M\}^{d^*}$ and any $\gamma'\in\mathbb{N}^d$ such that

$$[(j'-1)\lambda_r]-2\le\gamma'_r\le[j'\lambda_r]+2$$

the family of functions

$$h^k_{j',G,\gamma'}(y) = (\Delta^M_{e_d}\Psi^{(G)})(2^{D_{j',G,\gamma'}}y-k)\,,$$

where

$$D_{j',G,\gamma'} = \begin{pmatrix}\gamma'_1 & 0 & \dots & 0\\ 0 & \ddots & & \vdots\\ \vdots & \ddots & \gamma'_{m-1} & 0\\ 0 & \dots & 0 & \gamma'_m\end{pmatrix}\,,$$

and for $j'=0$ and any $k\in\mathbb{Z}^d$ $h^k_{j',(F,\cdots,F),(0,\cdots,0)}(x)=\Psi^{(F,\cdots,F)}(x-k)$. Observe that this is a family of inhomogeneous smooth analysis molecules in the sense of Definition 5.3 in [9]. Further, if $j'=j-j_0$ and $\widetilde{\gamma}=([j_0\lambda^0_r])_r$ one has

$$\widetilde{X}^{(1)}_{HF}(2^{-\Delta_{j_0}}(y+e_d))-\widetilde{X}^{(1)}_{HF}(2^{-\Delta_{j_0}}y) = \sum_{j'\ge 1}\sum_{G,\gamma'}\sum_{k\in\widetilde{\Gamma}_j(D_0)}\widetilde{c}^k_{j',G,\gamma'}h^k_{j',G,\gamma'}(y)\,,$$

with $\widetilde{c}^k_{j',G,\gamma'}=c^k_{j'+j_0,G,\gamma'+\widetilde{\gamma}}$ if $(G,\gamma'+\widetilde{\gamma})\in I^{j'+j_0}(D_0)$ and $\widetilde{c}^k_{j',G,\gamma}=0$ otherwise. Hence

$$\begin{aligned}\|(\Delta^M_{2^{-\Delta_{j_0}}e_d}\widetilde{X}^{(1)}_{HF})(2^{-\Delta_{j_0}}y)\|_{L^p} &\ge \|(\Delta^M_{2^{-\Delta_{j_0}}e_d}\widetilde{X}^{(1)}_{HF})(2^{-\Delta_{j_0}}y)\|_{B^0_{p,p,|\log|}(\mathbb{R}^d,D_0)}\\ &\ge\left(\sum_{j'\ge 1}\sum_{(G,\gamma)\in I^{j'+j_0}} j'2^{-j'd}\sum_{k\in\widetilde{\Gamma}_{j'+j_0}(D_0)}|c^k_{j'+j_0,G,\gamma}|^p\right)^{1/p}.\end{aligned}$$

We use once more an a.s. estimate of $\sum_k |c^k_{j,G,\gamma}|^p$ as in the proof of Proposition 5.2. Since as in Lemma 6.2, we can prove that $\mathrm{Card}(\widetilde{\Gamma}_{j'+j_0}(D_0) \geq 2^{(j'+j_0)d}$. Hence there exists an a.s. positive constant C such that

$$\sum_{k\in\widetilde{\Gamma}_{j'+j_0}(D_0)} |c^k_{j'+j_0,G,\gamma}|^p \geq 2^{-(j'+j_0)(H_0p-d)} \ .$$

Hence

$$\|(\Delta^M_{2^{-\Delta_{j_0}}e_d}\widetilde{X}^{(1)}_{HF})(2^{-\Delta_{j_0}}y)\|^p_{L^p} \geq C\sum_{j',G,\gamma} j'2^{-j'd}2^{-(j'+j_0)(H_0p-d)} \ .$$

Use now the last inequality and relation (5.15). Then a.s.

$$\begin{aligned}\|(\Delta^M_{2^{-\Delta_{j_0}}e_d}\widetilde{X}^{(1)}_{HF})(2^{-\Delta_{j_0}}y)\|^p_{L^p} &\geq C2^{-j_0d}j_0^{-d}\left(\sum_{j'\geq 0} j'2^{-j'd}2^{-(j'+j_0)(H_0p-d)}\right)\\ &\geq C2^{-j_0H_0p} \ .\end{aligned}$$

We deduce that a.s.

$$\begin{aligned}&\int_0^1 \|(\Delta^M_{2^{-[j_0\lambda^0_m]}e_\ell}\widetilde{X}^{(1)}_{HF})(x)\|^p_{L^p}t^{-s/\lambda_mp-1}\mathrm{d}t\\ \geq\ & \sum_{j_0=0}^{+\infty}\int_{2^{-[(j_0+1)\lambda^0_m]}}^{2^{-[j_0\lambda^0_m]}} \|(\Delta^M_{2^{-[j_0\lambda^0_m]}e_\ell}\varphi\widetilde{X}^{(1)}_{\rho_{D_0},H_0})(x)\|^p_{L^p(\mathbb{R}^m)}t^{-s/\lambda_mp-1}\mathrm{d}t\\ \geq\ & C\sum_{j_0=0}^{+\infty} j_0^{-d}2^{-j_0H_0p}\left(2^{-[j_0\lambda^0_m]}\right)^{-sp/\lambda_m-1}2^{-[j_0\lambda^0_m]}\\ =\ & C\sum_{j_0=0}^{+\infty} 2^{-j_0H_0p}j_0^{-d}\left(2^{-[j_0\lambda^0_m]}\right)^{-sp/\lambda_m} \ .\end{aligned}$$

Since

$$\sum_{j_0=0}^{+\infty} j_0^d2^{-j_0H_0p}\left(2^{-[j_0\lambda^0_m]}\right)^{-sp/\lambda_m} = +\infty \ ,$$

if $s\lambda^0_m/\lambda_m - H_0p > 0$, we deduce that a.s.

$$\int_0^1 \|\widetilde{X}^{(1)}_{HF}(x+2^{-[j_0\lambda^0_m]}e_\ell) - \widetilde{X}^{(1)}_{HF}(x)\|^p_{L^p}t^{-s/\lambda_mp-1}\mathrm{d}t = +\infty$$

for $s > H_0\lambda_m/\lambda^0_m$. Using (5.14) and the triangular inequality (5.13), it ends the proof of Proposition 5.5. It also implies directly Theorem 4.1.

6. Technical lemmas

Our results about smoothness of the sample path are based on the following lemma

Lemma 6.1. *Assume that the anisotropic multi–resolution analysis considered is $\mathcal{C}^1$ and admits at least one vanishing moment.*

Let $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ the Gaussian field defined by (2.1) with $\rho = \rho_{E_0}$. Assume also that the pseudo–norm ρ_{E_0} involved in the construction of this field is at least $\mathcal{C}^1(\mathbb{R}^d \setminus \{0\})$. Then the wavelet coefficients of the random field $\{X_{\rho_{E_0},H_0}(x)\}_{x\in\mathbb{R}^d}$ are weakly dependent in the following sense

1. *There exists some $C_0 > 0$ such for any $j \geq 1$, $(G, \gamma) \in I_j$ and $(k, k') \in (\mathbb{Z}^d)^2$*
$$|\mathbb{E}(c^k_{j,G,\gamma} c^{k'}_{j,G,\gamma})| \leq C_0 \frac{j^{2d/\lambda_{\min}} 2^{-2jH_0}}{1 + |k - k'|}. \tag{6.1}$$

2. *There exists some $C_1, C_2 > 0$ such that for any $j \geq 1$, $(G, \gamma) \in I_j$ and any $k \in (\mathbb{Z}^d)$*
$$C_1 j^{-d/\lambda_{\min}(E_0)} 2^{-2jH_0} \leq \mathbb{E}(|c^k_{j,G,\gamma}|^2) \leq C_2 j^{d/\lambda_{\min}(E_0)} 2^{-2jH_0} \ . \tag{6.2}$$

Remark 6.1. Theorem 1.1 of [12] imply that, studying the sample paths properties of OSSRGF, we can always assume that the pseudo–norm ρ_{E_0} involved in the construction of this field is $\mathcal{C}^\infty(\mathbb{R}^d\setminus\{0\})$. Then the assumptions of Lemma 6.1 are satisfied.

Proof. Since the anisotropic multiresolution analysis admits at least one vanishing moment, one has $\widehat{\psi}_M(0) = 0$. Further, for any $j \geq 1$, $(G, \gamma) \in I_j$ and for all $k \in \mathbb{Z}^d$

$$c^k_{j,G,\gamma} = \int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}2^{-{}^tD_{j,G,\gamma}} k\,\xi} \overline{\widehat{\psi}^{(G)}}(2^{-{}^tD_{j,G,\gamma}}\ \xi) \rho_{{}^tE_0}\ (\xi)^{-H_0-d/2} \mathrm{d}\widehat{W}(\xi) \ .$$

This formula implies that (set $\zeta = 2^{-{}^tD_{j,G,\gamma}}\ \xi$)

$$\mathbb{E}(|c^k_{j,G,\gamma}|^2) = 2^{j\mathrm{Tr}(D_{j,G,\gamma})} \int_{\mathbb{R}^d} |\widehat{\psi}^{(G)}(\zeta)|^2 \rho_{{}^tE_0}(2^{{}^tD_{j,G,\gamma}}\ \zeta)^{-2H_0-d} \mathrm{d}\zeta \ .$$

Since $2^{(j-2)d} \leq \mathrm{Tr}(D_{j,G,\gamma}) \leq 2^{jd}$, using Lemma 5.1 and the inequalities $C_1 2^j \leq \rho_{D_0}(2^{{}^tD_{j,G,\gamma}}\ \zeta) \leq C_2 2^j$ imply that

$$\begin{aligned}\mathbb{E}(|c^k_{j,G,\gamma}|^2) \quad &\geq \\ C_1 2^{-2j(H_0+d)} 2^{jd} &\int_{\mathbb{R}^d} |\widehat{\psi}^{(G)}(\zeta)|^2 \rho_{{}^tD_0}^{-2H_0-d}(\zeta)\ (1+\log(\rho_{{}^tD_0}^{-2H_0-d}(\zeta))+j)^{-d/\lambda_{\min}(E_0)}\ \mathrm{d}\zeta,\end{aligned}$$

and

$$\mathbb{E}(|c^k_{j,G,\gamma}|^2) \leq C_2 2^{-2j(H_0+d)} 2^{jd} \int_{\mathbb{R}^d} |\widehat{\psi}^{(G)}(\zeta)|^2 \rho_{{}^tD_0}^{-2H_0-d}(\zeta)\, (1+\log(\rho_{{}^tD_0}(\zeta)\,)+j)^{d/\lambda_{\min}(E_0)} \mathrm{d}\zeta.$$

We then proved inequalities (6.2).
To prove inequality (6.1) remark that for any $\ell \in \{1,\cdots,d\}$

$$(k_\ell - k'_\ell)\mathbb{E}(c^k_{j,G,\gamma} c^{k'}_{j,G,\gamma}) = \int_{\mathbb{R}^d} (k_\ell - k'_\ell) \mathrm{e}^{\mathrm{i}2^{-{}^tD_{j,G,\gamma}}(k-k')\xi} |\widehat{\psi}^{(G)}(2^{-{}^tD_{j,G,\gamma}}\ \xi)|^2 \rho_{E_0}(\xi)^{-2H_0-d} \mathrm{d}\xi\ .$$

Set $\zeta = 2^{-{}^tD_{j,G,\gamma}}\ \xi$ and integrate by parts with respect to ζ_ℓ. Hence

$$(k_\ell - k'_\ell)\mathbb{E}(c^k_{j,G,\gamma} c^{k'}_{j,G,\gamma}) = 2^{j\mathrm{Tr}(D_{j,G,\gamma})} \int_{\mathbb{R}^d} (k_\ell - k'_\ell) \mathrm{e}^{\mathrm{i}(k-k')\zeta} |\widehat{\psi}^{(G)}(\zeta)|^2 \rho_{E_0}(2^{{}^tD_{j,G,\gamma}}\ \zeta)^{-2H_0-d} \mathrm{d}\zeta.$$

Recall that the pseudo–norm may be assumed to be $\mathcal{C}^\infty(\mathbb{R}^d \setminus \{0\})$. Since the multi resolution analysis is $\mathcal{C}^1$

$$\begin{aligned}
(k_\ell - k'_\ell)&\mathbb{E}(c^k_{j,G,\gamma} c^{k'}_{j,G,\gamma}) \\
&= -2^{j\mathrm{Tr}(D_{j,G,\gamma})} \textstyle\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(k-k')\zeta} \frac{\partial}{\partial \zeta_\ell} \left(|\widehat{\psi}^{(G)}(\zeta)|^2 \rho_{E_0}(2^{{}^tD_{j,G,\gamma}}\ \zeta)^{-2H_0-d} \right) \mathrm{d}\zeta \\
&= -2^{j\mathrm{Tr}(D_{j,G,\gamma})} \textstyle\int_{\mathbb{R}^d} \mathrm{e}^{\mathrm{i}(k-k')\zeta} \left(\frac{\partial}{\partial \zeta_\ell} |\widehat{\psi}^{(G)}(\zeta)|^2 \right) \rho_{E_0}(2^{{}^tD_{j,G,\gamma}}\ \zeta)^{-2H_0-d} \mathrm{d}\zeta \\
&= -2^{jTr(D_{j,G,\gamma})} \textstyle\int_{\mathbb{R}^d} \frac{\mathrm{e}^{\mathrm{i}(k-k')\zeta} |\widehat{\psi}^{(G)}(\zeta)|^2}{\rho_{E_0}(2^{{}^tD_{j,G,\gamma}}\ \zeta)^{2H_0+d+1}} \left(2^{\gamma_\ell} \frac{\partial}{\partial \zeta_\ell}(\rho_{E_0})(2^{{}^tD_{j,G,\gamma}}\ \zeta) \right) \mathrm{d}\zeta.
\end{aligned}$$

An approach similar to the proof of inequalities (6.2) yields

$$2^{j\mathrm{Tr}(D_{j,G,\gamma})} \left| \int_{\mathbb{R}^d} e^{i(k-k')\zeta} \left(\frac{\partial}{\partial \zeta_\ell} |\widehat{\psi}^{(G)}(\zeta)|^2 \right) \rho_{E_0}(2^{{}^tD_{j,G,\gamma}}\ \zeta)^{-2H_0-d} d\zeta \right| \leq C j^{d/\lambda_{\min}(E_0)} 2^{-2jH_0}\ . \tag{6.3}$$

Further, differentiate the homogeneity relationship satisfied by $\rho_{{}^tE_0}$ and deduce that for any $a>0$ and $z \in \mathbb{R}^d$

$$a^{{}^tE_0}\ (\overrightarrow{\mathrm{grad}}(\rho_{{}^tE_0}\ ))(a^{{}^tE_0}\ z) = a(\overrightarrow{\mathrm{grad}}(\rho_{{}^tE_0}\ ))(z)\ . \tag{6.4}$$

For any $y \in \mathbb{R}^d \setminus \{0\}$, let $r = \rho_{{}^tE_0}\ (y)$. Then set $j = [\log_2(r)]$ and remark that $\rho_{{}^tE_0}\ (\Theta)\ = \rho_{{}^tE_0}\ (2^{-j\ {}^tE_0}\ y)\ \in [1/2,2]$ and hence that Θ belongs to the compact set $\mathcal{C}(1/2,2,\ {}^tE_0)\ = \{\theta,\ \rho_{{}^tE_0}\ (\theta)\ \in [1/2,2]\}$. Relationship (6.4) applied with $a = 2^j$ and $z = \Theta$ then implies

$$2^{\,{}^tD_{j,G,\gamma}}\ \overrightarrow{\mathrm{grad}}(\rho_{{}^tE_0})(2^{j\ {}^tE_0}\ \Theta) = 2^{-j\ {}^tE_0\ +\ {}^tD_{j,G,\gamma}} 2^j (\overrightarrow{\mathrm{grad}(\rho_{{}^tE_0})}\ (\Theta))$$

Take the norm of each member of the equality and deduce that for any $y \in \mathbb{R}^d \setminus \{0\}$ satisfying $j = [\log_2(\rho_{{}^tE_0}(y))]$

$$|2^{{}^tD_{j,G,\gamma}} \overrightarrow{\mathrm{grad}}(\rho_{{}^tE_0})(y)| \leq C 2^j |2^{-j {}^tE_0 + {}^tD_{j,G,\gamma}}| \,,$$

where $C = \sup_{\Theta \in \mathcal{C}(1/2,2,{}^tE_0)} |\overrightarrow{\mathrm{grad}}(\rho_{{}^tE_0})(\Theta)|$.
Lemma 2.1 of [6] and the definition of j imply that

$$|2^{{}^tD_{j,G,\gamma}} \overrightarrow{\mathrm{grad}}(\rho_{{}^tE_0})(y)| \leq C 2^j |j|^{d/\lambda_{\min}} \leq C \rho_{{}^tE_0}(y) \, |\log(\rho_{{}^tE_0}(y))|^{d/\lambda_{\min}(E_0)} \,.$$

Set now $y = 2^{{}^tD_{j,G,\gamma}} \zeta$. One has

$$\begin{aligned} &|2^{{}^tD_{j,G,\gamma}} (\overrightarrow{\mathrm{grad}}(\rho_{{}^tE_0}))(2^{{}^tD_{j,G,\gamma}} \zeta)| \\ &\qquad \leq C \rho_{{}^tE_0}(2^{{}^tD_{j,G,\gamma}} \zeta) \, |\log(\rho_{{}^tE_0}(2^{{}^tD_{j,G,\gamma}} \zeta))|^{d/\lambda_{\min}(E_0)} \\ &\qquad \leq C 2^j (j + |\log(\rho_{{}^tE_0}(\zeta)|)|)^{2d/\lambda_{\min}(E_0)} \rho_{{}^tE_0}(\zeta) \,. \end{aligned}$$

Since for any $\ell \in \{1, \cdots, d\}$

$$2^{\gamma_\ell} \left| \left(\frac{\partial}{\partial \zeta_\ell} (\rho_{E_0^*}) \right) (2^{{}^tD_{j,G,\gamma}} \zeta) \right| \leq \left| 2^{{}^tD_{j,G,\gamma}} (\overrightarrow{\mathrm{grad}}(\rho_{E_0}))(2^{{}^tD_{j,G,\gamma}} \zeta) \right| \,,$$

it yields the following inequality

$$\begin{aligned} &\left| 2^{j \mathrm{Tr}(D_{j,G,\gamma})} \int_{\mathbb{R}^d} \frac{\mathrm{e}^{\mathrm{i}(k-k')\zeta} |\widehat{\psi}^{(G)}(\zeta)|^2}{\rho_{E_0}(2^{{}^tD_{j,G,\gamma}} \zeta)^{2H_0+d+1}} \left(2^{\gamma_\ell} \frac{\partial}{\partial \zeta_\ell} (\rho_{E_0})(2^{{}^tD_{j,G,\gamma}} \zeta) \right) d\zeta \right| \\ &\qquad \leq 2^{-2jH_0} |j|^{2d/\lambda_{\min}(E_0)} \,. \qquad (6.5) \end{aligned}$$

Combining inequalities (6.3) and (6.5) then yields inequality (6.1).

Remark now that

Lemma 6.2. *Let D_0 an admissible diagonal anisotropy satisfying* $\mathrm{Tr}(D_0) = d$. *Recall that $\Gamma_j(D_0)$ is defined by* (5.4). *There exists some $C_1, C_2 > 0$ such that*

$$C_1 j^d 2^{jd} \leq \mathrm{card}(\Gamma_j(D_0)) \leq j^d 2^{jd} \,.$$

Proof. Indeed, since the norms $|\cdot|_{\ell_1}$ and $|\cdot|_{\ell^\infty}$ on $\mathbb{R}^d$ are equivalent, there exists some $C_1, C_2 > 0$ such that

$$C_1 \max_\ell |k_\ell|^{1/\lambda_\ell} \leq \rho_{D_0}(k) \leq C_2 \max_\ell |k_\ell|^{1/\lambda_\ell} \,.$$

The conclusion follows since it is quite clear since that

$$\mathrm{card}\{k, \max_\ell |k_\ell|^{1/\lambda_\ell} \leq j 2^j\} = \mathrm{card}\{k, \max_\ell |k_\ell| \leq j^{\lambda_\ell} 2^{j\lambda_\ell}\} = \prod_\ell \left(j^{\lambda_\ell} 2^{j\lambda_\ell} \right) = j^d 2^d,$$

using the fact that $\lambda_1 + \cdots + \lambda_\ell = d$.

The proof of Proposition 5.3 is then based on the two following results which are a slight modification of Theorem II.1 and II.7 of [11]. We recall the proofs for completeness.
We denote

$$c_p = \mathbb{E}(|g^0_{j,G,\gamma}|^p) \ .$$

where $(g^k_{j,G,\gamma})$ is the stationary Gaussian sequence of the normalized wavelet coefficients defined by (5.7). Since, by Lemma 6.1, the wavelet coefficients are weakly dependent, we can state a central limit theorem for the sequence $(g^k_{j,G,\gamma})_{j\in\mathbb{N},(G,\gamma)\in I_j,k\in\Gamma_j}$ which is a slight modified version of Lemma II.4 of [11]

Lemma 6.3. *Let $p \in (1, +\infty)$ and $(g^k_{j,G,\gamma})$ the Gaussian sequence defined by* (5.7)*. Set $n_j = \mathrm{Card}(\Gamma_j(D_0))$. Then almost surely when $j \to \infty$*

$$n_j^{-1}\left(\sum_{k\in\Gamma_j(D_0)} |g_{j,G,\gamma}|^p\right) \to c_p \ .$$

Proof. By Lemma 6.1 the sequence $(g^k_{j,G,\gamma})$ is weakly correlated in the sense of [11]–that is satisfies the assumption (H) of [11]. We follow the main line of [11] and first give an upper bound of

$$\mathbb{E}\left|\sum_{k\in\Gamma_j} (|g_{j,G,\gamma}|^p - c_p)\right|^2 \ .$$

Using the same approach that in [11] (see Lemma II.3) we get that

$$\mathbb{E}\left|\sum_{k\in\Gamma_j} (|g_{j,G,\gamma}|^p - c_p)\right|^2 \leq C_j c_{2p} \sum_{(k,k')\in\Gamma_j^2} \frac{1}{(1+|k-k'|)^2} \ ,$$

with $C_j = j^{2d/\rho_{\min}(E_0)}$ by weak correlation of the wavelet coefficients. Set $\ell = k - k'$. Hence

$$\sum_{(k,k')\in\Gamma_j^2} \frac{1}{(1+|k-k'|)^2} \leq \sum_{k\in\Gamma_j}\sum_{\ell\in 2.\Gamma_j} \frac{1}{(1+|\ell|)^2} \leq C_j j 2^j \sum_{\ell\in 2.\Gamma_j} \frac{1}{(1+|\ell|)^2} \ .$$

Remark now that

$$\sum_{\ell\in 2.\Gamma_j} \frac{1}{(1+|\ell|)^2} \leq \sum_{\ell\in 2.\Gamma_j} \frac{1}{(1+|\ell|_{D_0})^{2/\rho_{\max}}} \leq j^{d-\delta} 2^{j(d-\delta)} \ ,$$

with $\delta = 2/\lambda_{\max}(E_0) > 0$ by comparison with an integral and Proposition 2.3 in [7].
Thereafter the end of the proof is exactly the same that in Theorem II.1 in [11].

In an analogous way, one can give a result on the asymptotic behavior of

$$\frac{1}{\sqrt{|\log(n_j)|}}\left(\max_{k\in\Gamma_j}|g_{j,G,\gamma}|\right).$$

Lemma 6.4. *Almost surely*

$$0<\liminf_{j\to\infty}\frac{1}{\sqrt{|\log(n_j)|}}\left(\max_{k\in\Gamma_j}|g_{j,G,\gamma}|\right)\leq\limsup_{j\to\infty}\frac{1}{\sqrt{|\log(n_j)|}}\left(\max_{k\in\Gamma_j}|g_{j,G,\gamma}|\right)<\infty.$$

Proof. The proof is exactly the same than these of Lemmas II.8 and II.10 in [11].

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