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An optimal control problem governed by implicit evolution quasi-variational inequalities

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Abstract - This paper deals with an optimal control problem associated to an evolution implicit quasi-variational inequality for elastic materials. Such problems describe the quasi-static process of bilateral contact with friction between an elastic body and a rigid foundation. Existence of an optimal control is proven and necessary optimality conditions are derived.

Key words and phrases : boundary control, contact problems, variational inequalities, optimality conditions.

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1. Introduction

We consider an elastic body, which, under the influence of volume forces and surface tractions, is in bilateral contact with a rigid foundation. The friction is described by a nonlocal version of Coulomb's law. We are dealing with the quasi-static process.

Our goal is to study a related optimal control problem which allows us to obtain a desired field of displacements, by acting with a control on a part of the boundary of the body, while the norm of this control remains small enough.

The mathematical formulation of this problem is a boundary optimal control one, where the state is solution of an implicit quasi-variational inequality. The existence of an optimal control is proven.

For elastic materials, due to the lack of uniqueness of the state, the cost functional instead of depending, as usual, on the real control, depends also on the state. In order to characterize an optimal "pair", we are forced to consider a family of penalized optimal control problems governed by an implicit variational inequality. We prove the existence of an optimal control for the penalized problem and the convergence of the sequence of penalized optimal controls to an optimal control for the initial problem. To obtain the necessary optimality conditions, we use some regularization techniques leading us to a control problem of a variational equality. Existence results and necessary optimality conditions for these regularized problems are established.

Despite the fact that there are many results on the optimal control of systems governed by partial differential equations (see, for instance, [12], [18], [16], [11], [14], [4], [15], [5]) and on their applications in mechanics (see, for example, [2], [1], [9]), the optimal control of contact problems is not very often addressed in the literature. We mention here the results obtained in [6], [7], [17], [13], [3].

The main novelty brought by us consists in the presence of Coulomb friction which generates a control problem of an implicit quasi-variational inequality.

The structure of the paper is as follows. In Section 2, we present the variational formulation of our contact problem. Section 3 is devoted to the study of the boundary control problem.

2. Variational formulation of the quasi-static bilateral frictional contact problem

Let us consider a body occupying a bounded domain $\Omega \subset \mathbb{R}^p$, $p = 2, 3$, with a Lipschitz boundary $\Gamma = \overline{\Gamma}_0 \cup \overline{\Gamma}_1 \cup \overline{\Gamma}_2$, where Γ_0, Γ_1, Γ_2 are open and disjoint parts of Γ, with $meas(\Gamma_0) > 0$.

The body is subjected to the action of a volume force of density $\boldsymbol{f}$ given in $\Omega \times (0, T)$ and a surface traction of density $\boldsymbol{g}$ applied on $\Gamma_1 \times (0, T)$, where $(0, T)$ is the time interval of interest. The body is clamped on $\Gamma_0 \times (0, T)$ and, so, the displacement vector $\boldsymbol{u}$ vanishes here. On $\Gamma_2 \times (0, T)$, the body is in bilateral contact with a rigid foundation. We suppose that the contact on Γ_2 is with friction modeled by a nonlocal variant of Coulomb's law. We suppose that $\boldsymbol{f}$ and $\boldsymbol{g}$ are acting slow enough to allow us to neglect the inertial terms. We denote the velocity vector by $\dot{\boldsymbol{u}} = \partial \boldsymbol{u}/\partial t$, the strain tensor by $\boldsymbol{\epsilon} = \boldsymbol{\epsilon}(\boldsymbol{u})$, with the components $\epsilon_{ij} = \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$, and the stress tensor by $\boldsymbol{\sigma}$ with $\boldsymbol{\sigma} = \boldsymbol{\sigma}(\boldsymbol{u})$. We shall address the case of linearly elastic materials, i.e.

$$\sigma_{ij}(\boldsymbol{u}) = a_{ijkh}\epsilon_{kh}(\boldsymbol{u}), \tag{2.1}$$

where the elastic coefficients a_{ijkh}, independent of the strains and the stresses, satisfy the usual symmetry and ellipticity conditions:

$$\begin{aligned} & a_{ijkh} = a_{jihk} = a_{khij}\,, \\ & \exists\, \alpha > 0 \text{ such that } a_{ijkh}\xi_{ij}\xi_{kh} \geq \alpha\xi_{ij}\xi_{ij}\,,\ \forall \boldsymbol{\xi} = (\xi_{ij}) \in \mathbb{R}^{p^2}\,. \end{aligned} \tag{2.2}$$

Here and subsequently, the summation convention is employed.

A variational formulation of this problem is as follows (see, for instance, [10]):

Problem ($\mathbf{P}^g$): Find $\boldsymbol{u} \in W^{1,2}(0, T; \boldsymbol{V})$ such that

$$\begin{cases} a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{u}(t), \boldsymbol{v}) - j(\boldsymbol{u}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{F}^g(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\boldsymbol{V}} \\ \qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,,\ \text{a.e. } t \in (0, T), \\ \boldsymbol{u}(0) = \boldsymbol{u}_0, \end{cases}$$

where

$$\boldsymbol{V} = \{\boldsymbol{v} \in [H^1(\Omega)]^p \,;\, \boldsymbol{v} = \boldsymbol{0} \text{ a.e. on } \Gamma_0 \,;\, v_\nu = 0 \text{ a.e. on } \Gamma_2\}\,,$$
$$\boldsymbol{W} = \{\boldsymbol{u} \in \boldsymbol{V} \,;\, \operatorname{div} \boldsymbol{\sigma}(\boldsymbol{u}) \in (L^2(\Omega))^p\}\,,$$

and

$$(\boldsymbol{F}^g(t), \boldsymbol{v})_{\boldsymbol{V}} = \int_\Omega \boldsymbol{f}(t) \cdot \boldsymbol{v} \,\mathrm{dx} + \int_{\Gamma_1} \boldsymbol{g}(t) \cdot \boldsymbol{v} \,\mathrm{ds} \quad \forall \boldsymbol{v} \in \boldsymbol{V}\,. \tag{2.3}$$

$$a(\boldsymbol{u}, \boldsymbol{v}) = \int_\Omega \sigma(\boldsymbol{u})\, \epsilon(\boldsymbol{v}) \,\mathrm{dx} \quad \forall \boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}\,, \tag{2.4}$$

$$j(\boldsymbol{u}, \boldsymbol{v}) = \int_{\Gamma_2} \mu |\mathcal{R}\sigma_\nu(\boldsymbol{u})|\, |\boldsymbol{v}_\tau| \,\mathrm{ds} \quad \forall \boldsymbol{u} \in \boldsymbol{W},\ \forall \boldsymbol{v} \in \boldsymbol{V}\,. \tag{2.5}$$

We suppose that the initial displacement $\boldsymbol{u}_0 \in \boldsymbol{V}$ satisfies the following compatibility condition:

$$a(\boldsymbol{u}_0, \boldsymbol{v}) + j(\boldsymbol{u}_0, \boldsymbol{v}) \geq (\boldsymbol{F}^g(0), \boldsymbol{v})_{\boldsymbol{V}} \quad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,. \tag{2.6}$$

Here, we made the following regularity assumptions on the data:

$$\begin{cases} a_{ijkl} \in L^\infty(\Omega),\ i, j, k, l = 1, ..., p\,, \\ \boldsymbol{f} \in W^{1,2}(0, T; (L^2(\Omega))^p)\,, \\ \boldsymbol{g} \in W^{1,2}(0, T; (L^2(\Gamma_1))^p)\,, \\ \mu \in L^\infty(\Gamma_2),\ \mu \geq 0 \text{ a.e. on } \Gamma_2\,, \\ \mathcal{R} : H^{-1/2}(\Gamma_2) \to L^2(\Gamma_2) \text{ is a linear compact operator}\,, \\ \boldsymbol{u}_0 \in \boldsymbol{V}. \end{cases} \tag{2.7}$$

The following existence result holds (see [10]).

Theorem 2.1. *Under the above hypotheses, there exists $\mu_0 > 0$ such that for all $\mu > 0$ with $\|\mu\|_{L^\infty(\Gamma_2)} \leq \mu_0$, the problem $(\mathbf{P}^g)$ has at least a solution $\boldsymbol{u} \in W^{1,2}(0, T; \boldsymbol{V})$.*

3. Control problem

For $\beta > 0$ and $\boldsymbol{u}_d \in \mathbf{H_u}$ given, we shall study the following control problem:

Problem (CP) $\quad \inf\limits_{(\boldsymbol{g}, \boldsymbol{u}) \in \mathcal{V}_{ad}} J(\boldsymbol{g}, \boldsymbol{u})\,,$

where

$$J(\boldsymbol{g}, \boldsymbol{u}) = \frac{1}{2}\|\boldsymbol{u} - \boldsymbol{u}_d\|^2_{\mathbf{H_u}} + \frac{\beta}{2}\|\boldsymbol{g}\|^2_{\mathbf{H_g}}\,, \quad \forall (\boldsymbol{g}, \boldsymbol{u}) \in \mathbf{H_g} \times W^{1,2}(0, T; \mathbf{V})\,, \tag{3.1}$$

$$\mathcal{V}_{ad} = \{(\boldsymbol{g}, \boldsymbol{u}) \in \mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V})\,;\boldsymbol{u} \text{ verifies } (\mathbf{P}^g)\} \tag{3.2}$$

and

$$\mathbf{H_g} = W^{1,2}(0,T;(L^2(\Gamma_1))^p)\,, \quad \mathbf{H_u} = L^2(0,T;\boldsymbol{V})\,. \tag{3.3}$$

As already mentioned, due to the lack of uniqueness of solution for problem $(\mathbf{P}^g)$, the cost functional J depends not only on the "real" control $\boldsymbol{g}$, but also on the state $\boldsymbol{u}$.

It is easy to verify that the set $\mathcal{V}_{ad}$ is weakly closed and the functional J is weakly lower semi-continuous, but not coercive on $\mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V})$. Hence, in order to obtain an existence result for the control problem $(\mathbf{CP})$ and to derive the necessary conditions of optimality, we first introduce a family of penalized control problems governed by a variational inequality and, secondly, we consider a family of regularized problems governed by a variational equation.

We start by introducing a new control space:

$$\mathbf{H_w} = L^2(0,T;\boldsymbol{W})\,.$$

Now, for $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$, we consider the variational inequality which models our problem in the case of Tresca friction.

Problem $(\mathbf{P^{g,w}})$: Find $\boldsymbol{u} \in W^{1,2}(0,T;\mathbf{V})$ such that

$$\begin{cases} a(\boldsymbol{u}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)) + j(\boldsymbol{w}(t), \boldsymbol{v}) - j(\boldsymbol{w}(t), \dot{\boldsymbol{u}}(t)) \geq (\boldsymbol{F}^g(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t))_{\boldsymbol{V}} \\ \qquad\qquad \forall\, \boldsymbol{v} \in \boldsymbol{V}\,, \text{ a.e. } t \in (0,T)\,, \\ \boldsymbol{u}(0) = \boldsymbol{u}_0\,. \end{cases}$$

Using the same techniques as in [10] or [8] and taking into account the positivity of j, one can prove the following existence result:

Proposition 3.1. *For $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ given, there exists a unique solution $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}$ of problem $(\mathbf{P^{g,w}})$. Moreover, we have*

$$\|\dot{\boldsymbol{u}}^{\boldsymbol{g},\boldsymbol{w}}\|_{L^2(0,T;\boldsymbol{V})} \leq C(\|\dot{\boldsymbol{F}}\|_{L^2(0,T;\boldsymbol{V})} + \|\boldsymbol{w}\|_{L^2(0,T;\boldsymbol{V})})\,,$$

with C a positive constant.

In the sequel, for $(\boldsymbol{g}, \boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ given, $\boldsymbol{u}^{\boldsymbol{g},\boldsymbol{w}}$ denotes the unique solution of problem $(\mathbf{P^{g,w}})$.

Let us fix $\epsilon > 0$. We introduce the penalized functional $J_\epsilon : \mathbf{H_g} \times \mathbf{H_w} \to \mathbb{R}_+$ by

$$J_\epsilon(\boldsymbol{g}, \boldsymbol{w}) = J(\boldsymbol{u}^{g,w}, \boldsymbol{g}) + \frac{1}{2\epsilon}\|\boldsymbol{u}^{g,w} - \boldsymbol{w}\|^2_{\mathbf{H_w}} \tag{3.4}$$

and we consider the control problem:

Problem ($\mathbf{CP}_\epsilon$) $\qquad \inf\{J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,;\ (\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}\}\,.$

The following result establishes the existence of an optimal solution for this penalized control problem.

Proposition 3.2. *Let* (2.2), (2.7) *and* (2.6) *hold. Then, for all* $\epsilon > 0$, *there exists a solution* $(\boldsymbol{g}_\epsilon^*, \boldsymbol{w}_\epsilon^*)$ *of problem* $(\mathbf{CP}_\epsilon)$.

Proof. Let $\{(\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n)\}_n \subset \mathbf{H_g} \times \mathbf{H_w}$ be a minimizing sequence for the functional J_ϵ. Then, from the definition of J_ϵ, we deduce that the sequence $\{(\boldsymbol{g}_\epsilon^n, \boldsymbol{u}_\epsilon^n)\}_n$ is bounded in $\mathbf{H_g} \times L^2(0,T;\boldsymbol{V})$, where $\boldsymbol{u}_\epsilon^n = u^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n}$. Therefore, there exists $(\boldsymbol{g}_\epsilon^*, \boldsymbol{F}_\epsilon^*) \in \mathbf{H_g} \times W^{1,2}(0,T;\mathbf{V})$ such that, passing to a subsequence still denoted in the same way, we have

$$\boldsymbol{g}_\epsilon^n \rightharpoonup \boldsymbol{g}_\epsilon^* \text{ weakly in } \mathbf{H_g}\,, \tag{3.5}$$

$$\boldsymbol{F}_\epsilon^n \rightharpoonup \boldsymbol{F}_\epsilon^* \text{ weakly in } W^{1,2}(0,T;\mathbf{V})\,, \tag{3.6}$$

where

$$(\boldsymbol{F}_\epsilon^n(t), \boldsymbol{v})_{\boldsymbol{V}} = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\ \mathrm{dx} + \int_{\Gamma_1} \boldsymbol{g}_\epsilon^n(t)\cdot\boldsymbol{v}\ \mathrm{ds} \tag{3.7}$$

and

$$(\boldsymbol{F}_\epsilon^*(t), \boldsymbol{v})_{\boldsymbol{V}} = \int_\Omega \boldsymbol{f}(t)\cdot\boldsymbol{v}\ \mathrm{dx} + \int_{\Gamma_1} \boldsymbol{g}_\epsilon^*(t)\cdot\boldsymbol{v}\ \mathrm{ds}\,.$$

From $(\mathbf{P}^{\boldsymbol{g}_\epsilon^n, \boldsymbol{w}_\epsilon^n})$, we deduce that the sequence $\{\boldsymbol{u}_\epsilon^n\}_n$ is also bounded in $\mathbf{H_w}$, which, from the definition of J_ϵ, implies that the sequence $\{\boldsymbol{w}_\epsilon^n\}_n$ is bounded in $\mathbf{H_w}$.

Now, from Proposition 3.1, it results that the sequence $\{\dot{\boldsymbol{u}}_\epsilon^n\}_n$ is bounded in $L^2(0,T;V)$. Thus, there exist the elements $\boldsymbol{u}_\epsilon^* \in W^{1,2}(0,T;\boldsymbol{V})$ and $\boldsymbol{w}_\epsilon^* \in \mathbf{H_w}$ and the subsequences still denoted by $\{\boldsymbol{u}_\epsilon^n\}_n$ and $\{\boldsymbol{w}_\epsilon^n\}_n$ such that

$$\boldsymbol{w}_\epsilon^n \rightharpoonup \boldsymbol{w}_\epsilon^* \text{ weakly in } \mathbf{H_w}\,, \tag{3.8}$$

$$\boldsymbol{u}_\epsilon^n \rightharpoonup \boldsymbol{u}_\epsilon^* \text{ weakly in } W^{1,2}(0,T;\mathbf{V})\,. \tag{3.9}$$

Using the embedding $W^{1,2}(0,T;\boldsymbol{V}) \hookrightarrow C([0,T];\boldsymbol{V})$, we also have

$$\boldsymbol{u}_\epsilon^n(t) \rightharpoonup \boldsymbol{u}_\epsilon^*(t) \text{ weakly in } \boldsymbol{V} \quad \forall t \in [0,T]\,. \tag{3.10}$$

Now, by passing to the limit in $(\mathbf{P}^{g_\epsilon^n, w_\epsilon^n})$ with $n \to \infty$, one obtains that $\boldsymbol{u}_\epsilon^* = \boldsymbol{u}^{\boldsymbol{g}_\epsilon^* \boldsymbol{w}_\epsilon^*}$, i.e. $\boldsymbol{u}_\epsilon^*$ is the unique solution of problem $(\mathbf{P}^{g_\epsilon^*, w_\epsilon^*})$.

Finally, by using the above convergences, we have

$$\inf\{J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,;\,(\boldsymbol{g},\boldsymbol{w})\in\mathbf{H_g}\times\mathbf{H_w}\}=$$

$$\lim_{n\to\infty} J_\epsilon(\boldsymbol{g}^n_\epsilon,\boldsymbol{w}^n_\epsilon)\geq\liminf_{n\to\infty} J_\epsilon(\boldsymbol{g}^n_\epsilon,\boldsymbol{w}^n_\epsilon)\geq J_\epsilon(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)\,.$$

□

The next result shows that the penalized control problems $(\mathbf{CP}_\epsilon)$ approximate our initial problem $(\mathbf{CP})$ and also gives the existence of an optimal control for $(\mathbf{CP})$.

Theorem 3.1. *For $\epsilon>0$, let $(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)\in\mathbf{H_g}\times\mathbf{H_w}$ be an optimal control of $(\mathbf{CP}_\epsilon)$ and $\boldsymbol{u}^*_\epsilon=\boldsymbol{u}^{\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon}$. Then, there exist the elements $\boldsymbol{u}^*\in W^{1,2}(0,T;\mathbf{V})$ and $\boldsymbol{g}^*\in\mathbf{H_g}$ such that*

$$\begin{array}{l}\boldsymbol{g}^*_\epsilon\to\boldsymbol{g}^*\ \ \textit{weakly in}\ \ \mathbf{H_g}\,,\\ \boldsymbol{w}^*_\epsilon\rightharpoonup\boldsymbol{u}^*\ \ \textit{strongly in}\ \ \mathbf{H_w}\,,\\ \boldsymbol{u}^*_\epsilon\rightharpoonup\boldsymbol{u}^*\ \ \textit{weakly in}\ \ W^{1,2}(0,T;\mathbf{V})\,,\\ \boldsymbol{u}^*_\epsilon\to\boldsymbol{u}^*\ \ \textit{strongly in}\ \ L^2(0,T;\boldsymbol{V})\,.\end{array}\tag{3.11}$$

Moreover, $(\boldsymbol{g}^*,\boldsymbol{u}^*)\in\mathcal{V}_{ad}$ and

$$\lim_{\epsilon\to0} J_\epsilon(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)=J(\boldsymbol{g}^*,\boldsymbol{u}^*)=\min_{(\boldsymbol{g},\boldsymbol{u})\in\mathcal{V}_{ad}} J(\boldsymbol{g},\boldsymbol{u})\,.\tag{3.12}$$

Proof. Since $\lim\limits_{\epsilon\to0}\|\boldsymbol{w}^*_\epsilon-\boldsymbol{u}^*_\epsilon\|_{\mathbf{H_w}}=0$, by using standard arguments, one obtains the convergences (3.11) and that $(\boldsymbol{g}^*,\boldsymbol{u}^*)\in\mathcal{V}_{ad}$. This implies that $\lim\limits_{\epsilon\to0}\dfrac{1}{\epsilon}\|\boldsymbol{w}^*_\epsilon-\boldsymbol{u}^*_\epsilon\|^2_{\mathbf{H_w}}=0$ and, therefore, we have

$$J(\boldsymbol{g}^*,\boldsymbol{u}^*)\leq\liminf_{\epsilon\to0} J_\epsilon(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)\leq\limsup_{\epsilon\to0} J_\epsilon(\boldsymbol{g}^*,\boldsymbol{u}^*)=J(\boldsymbol{g}^*,\boldsymbol{u}^*)$$

and

$$J_\epsilon(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)\leq J_\epsilon(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}})=J(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}})\quad\forall(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}})\in\mathcal{V}_{ad}\,,$$

which complete the proof. □

Despite the fact that the problem $(\mathbf{CP}_\epsilon)$ is simpler than the initial one, it still involves a nondifferentiable functional J_ϵ. Therefore, for obtaining the optimality conditions, we need to consider a family of regularized problems associated to $(\mathbf{P}^{g,w})$, defined, for $\rho>0$, by

Problem $(\mathbf{P}^{g,w}_\rho)$: Find $\boldsymbol{u}\in W^{1,2}(0,T;\mathbf{V})$ such that

$$\left\{\begin{array}{l}\rho(\dot{\boldsymbol{u}}(t),\boldsymbol{v})_{\boldsymbol{V}}+a(\boldsymbol{u}(t),\boldsymbol{v})+\langle\nabla_2 j_\rho(\boldsymbol{w}(t),\dot{\boldsymbol{u}}(t)),\boldsymbol{v}\rangle=\\ \qquad(\boldsymbol{F}^g(t),\boldsymbol{v})_{\boldsymbol{V}}\,,\forall\,\boldsymbol{v}\in\boldsymbol{V}\,,\ \text{a.e.}\ t\in(0,T)\,,\\ \boldsymbol{u}(0)=\boldsymbol{u}_0\,,\end{array}\right.$$

where, for $\boldsymbol{w} \in \boldsymbol{W}$, $\{j_\rho(\boldsymbol{w},\cdot)\}_\rho$ is a family of convex functionals $j_\rho(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \mathbb{R}_+$, of class C^2, i.e. the gradients with respect to the second variable, $\nabla_2 j_\rho(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \boldsymbol{V'}$ and $\nabla_2^2 j_\rho(\boldsymbol{w},\cdot) : \boldsymbol{V} \to \mathcal{L}(\boldsymbol{V},\boldsymbol{V'})$, are continuous. In addition, we suppose that the following conditions hold true:

$$i)\ j_\rho(\boldsymbol{w},\mathbf{0}) = 0 \quad \forall \boldsymbol{w} \in \boldsymbol{W}\,, \tag{3.13}$$

$$ii)\ |j_\rho(\boldsymbol{w},\boldsymbol{v}) - j(\boldsymbol{w},\boldsymbol{v})| \le C\rho \|\boldsymbol{w}\|_{\boldsymbol{V}} \quad \forall \boldsymbol{w} \in \boldsymbol{W}, \forall \boldsymbol{v} \in \boldsymbol{V} \text{ with } C \text{ a constant independent of } \boldsymbol{v}\,, \tag{3.14}$$

$$iii)\ \lim_{n\to\infty} \int_0^T \langle \nabla_2 j_\rho(\boldsymbol{w}_n(t),\boldsymbol{u}_n(t)),\boldsymbol{v}\rangle\,\mathrm{dt} = \int_0^T \langle \nabla_2 j_\rho(\boldsymbol{w}(t),\boldsymbol{u}(t)),\boldsymbol{v}\rangle\,\mathrm{dt} \quad \forall(\boldsymbol{w}_n,\boldsymbol{u}_n) \rightharpoonup (\boldsymbol{w},\boldsymbol{u}) \text{ weakly in } \mathbf{H_w} \times \mathbf{L^2(0,T;}\boldsymbol{V}\mathbf{)}\,,\ \forall \boldsymbol{v} \in \boldsymbol{V}\,, \tag{3.15}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pair on $\boldsymbol{V'} \times \boldsymbol{V}$.

In what follows, for $(\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H_g} \times \mathbf{H_w}$ and $\rho > 0$ given, we shall denote by $\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}} \in W^{1,2}(0,T;\mathbf{V})$ the unique solution of problem $(\mathbf{P}_\rho^{g,w})$.

We consider the corresponding regularized optimal control problem:

Problem $(\mathbf{PC}_{\epsilon\rho})$ $\qquad \inf\{J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w})\,;\,(\boldsymbol{g},\boldsymbol{w}) \in \mathbf{H}_{\boldsymbol{g}} \times \mathbf{H}_{\boldsymbol{w}}\}$. where

$$\begin{aligned} J_{\epsilon\rho}(\boldsymbol{g},\boldsymbol{w}) &= J(\boldsymbol{g},\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}) + \frac{1}{2\epsilon}\|\boldsymbol{w} - \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}\|^2_{\mathbf{H}_{\boldsymbol{w}}} = \\ &\frac{1}{2}\|\boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}} - \boldsymbol{u}_d\|^2_{\mathbf{H}_u} + \frac{\beta}{2}\|\boldsymbol{g}\|^2_{\mathbf{H}_{\boldsymbol{g}}} + \frac{1}{2\epsilon}\|\boldsymbol{w} - \boldsymbol{u}_\rho^{\boldsymbol{g},\boldsymbol{w}}\|^2_{\mathbf{H}_{\boldsymbol{w}}}\,, \end{aligned} \tag{3.16}$$

Using similar techniques as in the proof of Proposition 3.2, we obtain, for $\rho > 0$, the existence of a solution $(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})$ of problem $(\mathbf{PC}_{\epsilon\rho})$.

We have the following convergence result:

Theorem 3.2. *Let $(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})$ be a solution of problem $(\mathbf{CP}_{\epsilon\rho})$ and $\boldsymbol{u}^*_{\epsilon\rho} = \boldsymbol{u}_\rho^{\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}}$. Then,*

$$\begin{cases} \boldsymbol{g}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{g}^*_\epsilon \text{ weakly in } \mathbf{H_g}\,, \\ \boldsymbol{w}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{w}^*_\epsilon \text{ weakly in } \mathbf{H_w}\,, \\ \boldsymbol{u}^*_{\epsilon\rho} \rightharpoonup \boldsymbol{u}^*_\epsilon \text{ weakly in } W^{1,2}(0,T;\boldsymbol{V})\,, \end{cases} \tag{3.17}$$

where $\boldsymbol{u}^*_\epsilon = \boldsymbol{u}^{\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon}$. Moreover, $(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)$ is an optimal control for J_ϵ and

$$\lim_{\rho\to 0} J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}) = J_\epsilon(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon) = \min_{(\boldsymbol{g},\boldsymbol{w})\in\mathbf{H_g}\times\mathbf{H_w}} J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,.$$

Proof. Since

$$J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho}) \le J_{\epsilon\rho}(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}) \quad \forall(\tilde{\boldsymbol{g}},\tilde{\boldsymbol{u}}) \in \mathcal{V}_{ad}$$

and the sequence $\{\boldsymbol{u}_\rho^{\tilde{g},\tilde{u}}\}_\rho$ is bounded in $W^{1,2}(0,T;\boldsymbol{V})\cap\mathbf{H_w}$, it follows that the sequence $\{J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\}_\rho$ is bounded. Proceeding like in the proof of Proposition 3.2, we deduce that the sequence $\{(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{u}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\}_\rho$ is bounded in $\mathbf{H_g}\times W^{1,2}(0,T;\mathbf{V})\cap\mathbf{H_w}\times\mathbf{H_w}$. Thus, there exist the elements $(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon,\boldsymbol{u}^*_\epsilon)\in\mathbf{H_g}\times\mathbf{H_w}\times\mathbf{H_u}$ such that the convergences (3.17) are true and $\boldsymbol{u}^*_\epsilon=\boldsymbol{u}^{\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon}$.

Let $(\bar{\boldsymbol{g}}_\epsilon,\bar{\boldsymbol{w}}_\epsilon)$ be a solution of problem $(\mathbf{CP}_\epsilon)$, $\bar{\boldsymbol{u}}_\epsilon=\boldsymbol{u}^{\bar{\boldsymbol{g}}_\epsilon,\bar{\boldsymbol{w}}_\epsilon}$ and $\bar{\boldsymbol{u}}_{\epsilon\rho}=\boldsymbol{u}_\rho^{\bar{\boldsymbol{g}}_\epsilon,\bar{\boldsymbol{w}}_\epsilon}$. Since

$$\bar{\boldsymbol{u}}_{\epsilon\rho}\to\bar{\boldsymbol{u}}_\epsilon \text{ strongly in } L^\infty(0,T;\boldsymbol{V})\cap\mathbf{H_w}\,, \tag{3.18}$$

we have

$$\begin{aligned} &J_\epsilon(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)\le\liminf_{\rho\to0}J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})\le\limsup_{\rho\to0}J_{\epsilon\rho}(\bar{\boldsymbol{g}}_\epsilon,\bar{\boldsymbol{w}}_\epsilon)=\\ &\lim_{\rho\to0}J_{\epsilon\rho}(\bar{\boldsymbol{g}}_\epsilon,\bar{\boldsymbol{w}}_\epsilon)=J_\epsilon(\bar{\boldsymbol{g}}_\epsilon,\bar{\boldsymbol{w}}_\epsilon)\le J_\epsilon(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)\,, \end{aligned} \tag{3.19}$$

i.e.

$$\lim_{\rho\to0}J_{\epsilon\rho}(\boldsymbol{g}^*_{\epsilon\rho},\boldsymbol{w}^*_{\epsilon\rho})=J_\epsilon(\boldsymbol{g}^*_\epsilon,\boldsymbol{w}^*_\epsilon)=\min\{J_\epsilon(\boldsymbol{g},\boldsymbol{w})\,;\;(\boldsymbol{g},\boldsymbol{w})\in\mathbf{H_g}\times\mathbf{H_w}\}\,.$$

□

Using a well-known theorem due to Lions [12], we can state now the main result of this section, namely the necessary conditions of optimality for the problem $(\mathbf{CP}_{\epsilon\rho})$ that give a convergent algorithm for the original problem and can be numerically exploited.

Theorem 3.3. *Let $(\boldsymbol{g}^*,\boldsymbol{w}^*)\in\mathbf{H_g}\times\mathbf{H_w}$ be a solution of the optimal control problem $(\mathbf{PC}_{\epsilon\rho})$. Then, there exist the unique elements $\boldsymbol{u}^*\in\mathbf{X}$ and $\boldsymbol{q}^*\in L^2(0,T;\boldsymbol{V}')$ such that*

$$\left\{\begin{aligned} &\rho\int_0^T(\dot{\boldsymbol{u}}^*(t),\boldsymbol{v}(t))_{\boldsymbol{V}}\,\mathrm{dt}+\int_0^T a(\boldsymbol{u}^*(t)+\boldsymbol{u}_0,\boldsymbol{v}(t))\,\mathrm{dt}+\\ &\int_0^T\langle\nabla_2 j_\rho(\boldsymbol{w}^*(t),\dot{\boldsymbol{u}}^*(t)),\boldsymbol{v}(t)\rangle\,\mathrm{dt}=\int_0^T(\boldsymbol{f}(t),\boldsymbol{v}(t))_{(L^2(\Omega))^p}\,\mathrm{dt}+\\ &\int_0^T(\boldsymbol{g}^*(t),\boldsymbol{v}(t))_{(L^2(\Gamma_1))^p}\,\mathrm{dt}\quad\forall\boldsymbol{v}\in L^2(0,T;\boldsymbol{V})\,, \end{aligned}\right. \tag{3.20}$$

$$\begin{cases} \int_0^T \rho(\dot{\boldsymbol{v}}(t), \boldsymbol{q}^*(t))_V \, \mathrm{dt} + \int_0^T a(\boldsymbol{v}(t), \boldsymbol{q}^*(t)) \, \mathrm{dt} + \\ \int_0^T \langle \nabla_2^2 j(\boldsymbol{w}^*(t), \dot{\boldsymbol{u}}^*(t)) \dot{\boldsymbol{v}}(t) - \nabla_2 j(\boldsymbol{v}(t), \dot{\boldsymbol{u}}^*(t)), \boldsymbol{q}^*(t) \rangle \, \mathrm{dt} = \\ \int_0^T (\boldsymbol{u}^*(t) + \boldsymbol{u}_0 - \boldsymbol{u}_d, \boldsymbol{v}(t))_{\boldsymbol{V}} \, \mathrm{dt} \quad \forall \boldsymbol{v} \in \mathbf{X} \end{cases} \tag{3.21}$$

and

$$\beta(\boldsymbol{g}^*, \boldsymbol{g})_{\mathbf{H_g}} = (\boldsymbol{q}^*, \boldsymbol{g})_{L^2(0,T;(L^2(\Gamma_1))^p)} \quad \forall \boldsymbol{g} \in \mathbf{H_g}\,, \tag{3.22}$$

where

$$\mathbf{X} = \{\boldsymbol{v} \in W^{1,2}(0,T;\boldsymbol{V}) \cap L^2(0,T;\boldsymbol{W})\,;\ \boldsymbol{v}(0) = \mathbf{0}\}.$$

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