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Linear multifractional multistable motion: LePage series representation and modulus of continuity

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Abstract - In this paper, we obtain an upper bound of the modulus of continuity of linear multifractional multistable random motions. Such processes are generalizations of linear multifractional α-stable motions for which the stability index α is also allowed to vary in time. In the case of linear multifractional α-stable motions, we improve the recent result of [2]. The main idea is to consider some conditionnally sub-Gaussian LePage series representations to fit the framework of [5].

Key words and phrases : stable and multistable random fields, modulus of continuity.

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1. Introduction

Self-similar random fields are required to model persistent phenomena in internet traffic, hydrology, geophysics or financial markets, e.g. [1, 22]. The fractional Brownian motion ([15, 9]) provides the most famous self-similar model. Nevertheless, in image modeling, in finance or in biology for example, the phenomena under study are rarely Gaussian. Then, α-stable random processes have been proposed as an alternative to Gaussian modeling, since they allow to model data with heavy tails, such as in internet traffic [16]. The linear fractional stable motion, which has been proposed in [21, 14], is one of the numerous stable extensions of the fractional Brownian motion. Let us recall how this self-similar random motion can be defined through a stochastic integral representation. To this way, let us consider $H_1 \in (0,1)$, $\alpha_1 \in (0,2)$ and M_{α_1} a real-valued symmetric α_1-stable random measure with Lebesgue control measure (see [17] p.281 for details on such measures). Then, a linear fractional stable motion is defined by

$$X_{\alpha_1,H_1}(t) = \int_{\mathbb{R}} f_+(\alpha_1, H_1, t, \xi) M_{\alpha_1}(d\xi), \quad t \in \mathbb{R} \tag{1.1}$$

where f_+ is defined by

$$f_+(\alpha_1, H_1, t, \xi) = (t-\xi)_+^{H_1 - 1/\alpha_1} - (-\xi)_+^{H_1-1/\alpha_1} \tag{1.2}$$

with for $c \in \mathbb{R}$,

$$(x)_+^c = \begin{cases} x^c & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Since the self-similarity property is a global property which can be too restrictive for applications, a multifractional generalization $X_{\alpha_1,h}$ of this process has also been introduced by [18] to model internet traffic, by replacing H_1 by a real function h with values on $(0,1)$. Some necessary and sufficient conditions for the stochastic continuity of the linear multifractional stable motion $X_{\alpha_1,h}$ have been given in [18] and its Hölder sample path regularity has been studied in [19]. The Hölder sample path properties have also been improved in [2] by establishing upper and lower bounds for the modulus of continuity. In the following, we will improve the upper bound, using the results we established in [5]. Let us mention that in the case where $h \equiv H_1$ is constant, that is when $X_{\alpha_1,h}$ is a linear fractional stable motion, sample path regularity properties have previously been studied in [17, 20, 10].

Moreover, the framework of [5] allows to study $X_{\alpha_1,h}$ as well as some multistable generalizations for which the stability index α_1 is also allowed to vary with t. Multistable processes have been defined in [7] using sums over Poisson processes or in [6] using a Klass-Ferguson LePage series.

In this paper we consider a random field S_m defined using a Lepage series representation of the linear fractional α_1-stable motion and such that

$$S_m(\alpha(t), h(t), t), \quad t \in \mathbb{R}$$

is a linear multifractional multistable motion. This auxiliary random field S_m allows to study the variations due to the functions α, h and to the position t separately. Then, to study sample path regularity of linear multistable motions, our first step is to establish an upper bound for the modulus of continuity of the field S_m considering a conditionnally sub-Gaussian representation and applying [5]. The main property of sub-Gaussian random variables, which have been introduced by [8], is that their tail distributions decrease exponentially as the Gaussian ones. This property is one of the main tool used in [5] to study the sample path regularity property of conditionnally sub-Gaussian random series.

The paper is organized as follows. Section 2 introduces LePage series random fields under study. An upper bound of their modulus of continuity and a rate of convergence are stated in Section 3. Section 4 focuses on linear multifractional multistable motions. Some technical proofs are postponed to the appendix for reader convenience.

2. LePage series models

In order to define LePage series, let us introduce some notation.

Hypothesis 2.1. Let $(g_n)_{n\geq 1}$, $(\xi_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ be three independent sequences of random variables satisfying the following conditions.

1. $(g_n)_{n\geq 1}$ is a sequence of independent identically distributed (i.i.d.) real-valued symmetric sub-Gaussian random variables, that is such that there exists $s \in [0, +\infty)$ for which
$$\forall \lambda \in \mathbb{R},\ \mathbb{E}(\mathrm{e}^{\lambda g_n}) \leq \mathrm{e}^{\frac{s^2\lambda^2}{2}}. \tag{2.1}$$

2. $(\xi_n)_{n\geq 1}$ is a sequence of i.i.d. random variables with common law
$$\mu(d\xi) = m(\xi)d\xi$$
equivalent to the Lebesgue measure (that is such that $m(\xi) > 0$ for almost every ξ).

3. T_n is the nth arrival time of a Poisson process with intensity 1.

Let us now introduce the random field $(S_m(\alpha, H, t))_{(\alpha,H,t)\in(0,2)\times(0,1)\times\mathbb{R}}$ we study in this paper.

Proposition 2.1 (LePage series representation) *Assume that Hypothesis* 2.1 *is fulfilled and let f_+ be defined by* (1.2)*. Then, for any $(\alpha, H, t) \in (0,2)\times(0,1)\times\mathbb{R}$, the sequence*
$$S_{m,N}(\alpha, H, t) = \sum_{n=1}^{N} T_n^{-1/\alpha} f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \quad N \geq 1 \tag{2.2}$$
converges almost surely and its limit is denoted by
$$S_m(\alpha, H, t) := \sum_{n=1}^{+\infty} T_n^{-1/\alpha} f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n. \tag{2.3}$$

Proof. Let $(\alpha, H, t) \in (0,2)\times(0,1)\times\mathbb{R}$. Then, since Hypothesis 2.1 holds, the variables
$$W_n := f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha} g_n, \quad n \geq 1,$$
are i.i.d., symmetric and such that
$$\mathbb{E}(|W_1|^\alpha) = \mathbb{E}(|g_1|^\alpha) \int_{\mathbb{R}} |f_+(\alpha, H, t, \xi)|^\alpha \, d\xi < +\infty,$$
since g_1 and ξ_1 are independent (see e.g. [17]). Therefore, by Theorem 5.1 of [13], the sequence
$$\left(\sum_{n=1}^{N} T_n^{-1/\alpha} W_n \right)_{N\geq 1}$$

converges almost surely as $N \to +\infty$, that is $(S_{m,N}(\alpha, H, t))_{N \geq 1}$ converges almost surely. $\square$

Let us conclude this section by some remarks.

Remark 2.1. According to Proposition 5.1 of [5], the finite dimensional distributions of S_m do not depend on m as soon as Condition 2 of Hypothesis 2.1 holds. Moreover, when studying the sample path regularity of S_m, Proposition 5.1 of [5] allows us to change m by a more convenient function $\tilde{m}$ if necessary.

Remark 2.2. When $\alpha = \alpha_1 \in (0, 2)$ is fixed, $(S_m(\alpha_1, H, t))_{(H,t)\in(0,1)\times\mathbb{R}}$ is an α_1-stable symmetric random field, which can also be represented as an integral under an α_1-stable random measure M_{α_1} with Lebesgue control measure. More precisely, for every $\alpha_1 \in (0, 2)$,

$$(S_m(\alpha_1, H, t))_{(H,t)\in(0,1)\times\mathbb{R}} \overset{fdd}{=} d_{\alpha_1}(Y_{\alpha_1}(H, t))_{(H,t)\in(0,1)\times\mathbb{R}} \tag{2.4}$$

where $\overset{fdd}{=}$ means equality of finite distributions and

$$Y_{\alpha_1}(H, t) := \int_{\mathbb{R}} f_+(\alpha_1, H, t, \xi) M_{\alpha_1}(d\xi), \quad (H, t) \in (0, 1) \times \mathbb{R}, \tag{2.5}$$

for M_{α_1} a real-valued symmetric α_1-stable random measure with Lebesgue control measure and

$$d_{\alpha_1} := \mathbb{E}(|g_1|^{\alpha_1})^{1/\alpha_1} \left(\int_0^{+\infty} \frac{\sin x}{x^{\alpha_1}} dx \right)^{1/\alpha_1}. \tag{2.6}$$

One can check Equation (2.4) following the proof of Proposition 5.1 of [5] or Proposition 4.2 of [4], which is a consequence of Lemma 4.1 of [11].

3. Sample path properties

Several papers [20, 10, 18, 19, 2] have already investigated sample path properties of the linear fractional stable motion X_{α_1,H_1} defined by Equation (1.1) or of its multifractional generalization $X_{\alpha_1,h}$ defined on $\mathbb{R}$ by

$$X_{\alpha_1,h}(t) := Y_{\alpha_1}(h(t), t), \quad t \in \mathbb{R} \tag{3.1}$$

where $\alpha_1 \in (0, 2)$, Y_{α_1} is given by (2.5) and h is a function with values in $(0, 1)$. In the following, we improve the upper bound of the global modulus of continuity of $X_{\alpha_1,h}$ stated in [2]. Our first step is to establish an upper bound for the global modulus of continuity of the field S_m defined by (2.3) on a compact set K of $(0, 2) \times (0, 1) \times \mathbb{R}$. To obtain our upper bound, we use the results we established in [5] on conditionally sub-Gaussian random series.

Let us first recall (see [17] for example) that the α_1-stable random process $X_{\alpha_1,H_1} = (Y_{\alpha_1}(H_1,t))_{t\in\mathbb{R}}$ is unbounded almost surely on each compact set with non-empty interior when $H_1 < 1/\alpha_1$. A similar result holds for S_m as stated in the following proposition.

Proposition 3.1. *Assume that* $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [a, b] \subset (0, 2) \times (0, 1) \times \mathbb{R}$ *with* $0 < \alpha_1 \le \alpha_2 < 2$, $0 < H_1 \le H_2 < 1$ *and* $a < b$.

1. *If* $H_1 < 1/\alpha_1$, *then the random field* S_m *is almost surely unbounded on* K.
2. *If* $H_1 = 1/\alpha_1$, *then* S_m *does not have almost surely continuous sample paths on the compact set* K.

Proof. By Equation (2.4)

$$(S_m(\alpha_1, H_1, t))_{t\in\mathbb{R}} \overset{fdd}{=} d_{\alpha_1}(X_{\alpha_1,H_1}(t))_{t\in\mathbb{R}}, \tag{3.2}$$

where d_{α_1} is defined by Equation (2.6) and X_{α_1,H_1} is the linear fractional stable motion given by (1.1).
Let us first assume that $H_1 < 1/\alpha_1$. Then, since $a < b$, by Corollary 10.2.4 of [17], $(S_m(\alpha_1, H_1, t))_{t\in\mathbb{R}}$ is unbounded almost surely on the compact set $[a, b]$. It follows that

$$\sup_{(\alpha,H,t)\in K} |S_m(\alpha, H, t)| = +\infty \text{ a.s.}$$

since $\sup_{(\alpha,H,t)\in K} |S_m(\alpha, H, t)| \ge \sup_{t\in[a,b]} |S_m(\alpha_1, H_1, t)|$.

Let us now assume that $H_1 = 1/\alpha_1$ (which implies that $\alpha_1 > 1$). Then,

$$X_{\alpha_1,H_1} = (M_{\alpha_1}([0,t))\mathbf{1}_{t>0} + M_{\alpha_1}((t,0])\mathbf{1}_{t<0})_{t\in\mathbb{R}}$$

is a Lévy α_1-stable motion and by Equation (3.2), so is the process $(S_m(\alpha_1, H_1, t))_{t\in\mathbb{R}}$. Since $\alpha_1 < 2$, the stable motion $(S_m(\alpha_1, 1/\alpha_1, t))_{t\in\mathbb{R}}$ is not a Brownian motion and then does not have almost surely continuous sample paths (see Exercice 2.7 p.64 of [12] for instance). This concludes the proof. □

Therefore, it remains to study the sample paths on a compact set

$$K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (0, 2) \times (0, 1) \times \mathbb{R}$$

such that $H_1 > 1/\alpha_1$, which implies that $\alpha_1 \in (1, 2)$ and $H_1 > 1/2$.

The main result of this paper is the following theorem, which states an upper bound for the modulus of continuity of S_m on K, and for some m a rate of uniform convergence on K for the series $S_{m,N}$ defined by (2.2).

Theorem 3.1. *Assume that Hypothesis* 2.1 *is fulfilled. Let* $S_{m,N}$ *and* S_m *be defined by* (2.2) *and* (2.3) *and let us consider the compact set*

$$K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (1,2) \times (1/2,1) \times \mathbb{R}$$

with $A > 0$ *and* $H_1 > 1/\alpha_1$.

1. *As* $N \to +\infty$, *the series* $(S_{m,N})_{N\geq 1}$ *converges uniformly on* K *to* S_m *and almost surely*

$$\sup_{\substack{x,x'\in K\\ x\neq x'}} \frac{|S_m(x) - S_m(x')|}{\tau(x-x')\sqrt{|\log(\tau(x-x'))|+1}} < +\infty$$

with $\tau(z) = |\alpha| + |H| + |t|^{H_1 - 1/\alpha_1}$ *for* $z = (\alpha, H, t) \in \mathbb{R}^3$.

2. *For* $\eta > 0$, *let us consider* $m = m_\eta$ *defined by*

$$m_\eta(\xi) = c_\eta |\xi|^{-1} \left(1 + |\log(|\xi|)|\right)^{-1-\eta}, \tag{3.3}$$

with $c_\eta > 0$ *such that* $\int_{\mathbb{R}} m_\eta(\xi) d\xi = 1$. *Then, almost surely*

$$\sup_{N\geq 1} N^{\varepsilon} \sup_{x\in K} \left| S_{m_\eta,N}(x) - S_{m_\eta}(x)\right| < +\infty$$

for any $\varepsilon \in (0, 1/\alpha_2 - 1/2)$.

Proof. For all $x = (\alpha, H, t) \in (0,2) \times (0,1) \times \mathbb{R}$ and all integer $n \geq 1$, we consider

$$V_{m,n}(x) := f_+(\alpha, H, t, \xi_n) m(\xi_n)^{-1/\alpha}, \tag{3.4}$$

so that

$$S_{m,N}(x) = \sum_{n=1}^{N} T_n^{-1/\alpha} V_{m,n}(x) g_n \quad \text{and} \quad S_m(x) = \sum_{n=1}^{+\infty} T_n^{-1/\alpha} V_{m,n}(x) g_n.$$

Let us also remark that for all $x = (\alpha, H, t) \in (0,2) \times (0,1) \times \mathbb{R}$,

$$\mathbb{E}(|V_{m,n}(x)|^\alpha) = \int_{\mathbb{R}} |f_+(\alpha, H, t, \xi)|^\alpha d\xi < +\infty.$$

Note that if in Equation (2.1) the sub-Gaussian parameter s of g_n is less than 1, Equation (2.1) also holds for $s = 1$. Moreover, if s is greater than 1 we may write $V_{m,n}(x) g_n = (s V_{m,n}(x)) g_n / s$ so that g_n/s is sub-Gaussian with parameter 1. Hence without loss of generality we may and will assume that $s = 1$. It follows that $(g_n)_{n\geq 1}$, $(T_n)_{n\geq 1}$ and $(V_{m,n})_{n\geq 1}$ are three independent sequences that satisfy Assumption 4 in [5] on $(0,2) \times (0,1) \times \mathbb{R}$. Then, by Theorem 4.2 of [5], the result follows once we prove $\mathbb{E}\left(|V_{m,1}(x_0)|^2\right) < +\infty$

for some $x_0 \in K$ and Equation (15) of [5] for $p = 1$, namely (in our setting) if there exists $r > 0$ such that

$$\mathbb{E}\left(\left[\sup_{\substack{x,x' \in K \\ 0<\|x-x'\|\le r}} \frac{|V_{m,1}(x) - V_{m,1}(x')|}{\tau(x-x')}\right]^2\right) < +\infty. \tag{3.5}$$

The following proposition, whose proof is postponed to the appendix, allows to find some m satisfying such conditions.

Proposition 3.2. *There exists a finite deterministic constant $c_{3,1}(K) > 0$ such that a.s. for all $x, x' \in K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A]$,*

$$|V_{m,1}(x) - V_{m,1}(x')| \le c_{3,1}(K)\tau(x-x')h_{m,K}(\xi_1),$$

with, for almost every $\xi \in \mathbb{R}$,

$$\begin{aligned} h_{m,K}(\xi) &= \max\left(m(\xi)^{-1/\alpha_1}, m(\xi)^{-1/\alpha_2}\right)(1 + |\log m(\xi)|) \\ &\quad \times \left(\mathbf{1}_{|\xi|\le \mathrm{e}} + |\xi|^{-1+H_2-1/\alpha_2}\log|\xi|\mathbf{1}_{|\xi|>\mathrm{e}}\right). \end{aligned} \tag{3.6}$$

Let us first consider $m = m_\eta$ given by (3.3) for some $\eta > 0$. In view of Proposition 3.2, since $V_{m_\eta,1}(\alpha, H, 0) = 0$ for all $(\alpha, H, 0) \in K$, up to use a finite covering of K, it is enough to prove that there exists $r > 0$ with

$$\mathbb{E}\left(h_{m_\eta,K}(\xi_1)^2\right) < +\infty, \tag{3.7}$$

for $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A]$ with $\alpha_2 - \alpha_1 \le r$. One has

$$\begin{aligned} \mathbb{E}(h_{m_\eta,K}(\xi_1)^2) &= \int_{\mathbb{R}} h_{m_\eta,K}(\xi)^2 m_\eta(\xi)d\xi \\ &= \int_{|\xi|\le \mathrm{e}} + \int_{|\xi|>\mathrm{e}} := I_1 + I_2. \end{aligned}$$

On the one hand,

$$\begin{aligned} I_1 &= \int_{|\xi|\le \mathrm{e}} m_\eta(\xi)\max(m_\eta(\xi)^{-2/\alpha_1}, m_\eta(\xi)^{-2/\alpha_2})(1 + |\log(m_\eta(\xi))|)^2 d\xi \\ &\le c_{3,2}(\eta, K)\int_{|\xi|\le \mathrm{e}} |\xi|^{-1+2/\alpha_2}(1 + |\log(|\xi|)|)^{(1+\eta)(2/\alpha_1-1)}(1 + |\log(m_\eta(\xi))|)^2 d\xi, \end{aligned}$$

with $c_{3,2}(\eta, K)$ a positive finite constant. It follows that $I_1 < +\infty$ since $\alpha_2 > 0$. On the other hand,

$$\begin{aligned} I_2 &= \int_{|\xi|>\mathrm{e}} m_\eta(\xi)\max(m_\eta(\xi)^{-2/\alpha_1}, m_\eta(\xi)^{-2/\alpha_2})(1+|\log(m_\eta(\xi))|)^2|\xi|^{2(H_2-1/\alpha_2)-2}\log(|\xi|)^2 d\xi \\ &\le c_{3,3}(\eta, K)\int_{|\xi|>\mathrm{e}} |\xi|^{2(H_2+1/\alpha_1-1/\alpha_2)-3}\log(|\xi|)^{(1+\eta)(2/\alpha_1-1)+2}(1+|\log(m_\eta(\xi))|)^2 d\xi, \end{aligned}$$

with $c_{3,3}(\eta, K)$ a positive finite constant. Since $\alpha_1 > 1$, note that $\alpha_2 - \alpha_1 < 1 - H_2$ implies that $H_2 + 1/\alpha_1 - 1/\alpha_2 < H_2 + \alpha_2 - \alpha_1 < 1$ and thus $I_2 < +\infty$. Therefore choosing $r \in (0, 1 - H_2)$, Equation (3.7) and then (3.5) hold for $m = m_\eta$. By Theorem 4.2 of [5], $(S_{m_\eta,N})_{N\geq 1}$ and S_{m_η} satisfy *1.* and *2.* of the theorem.
Since for almost every $\xi \in \mathbb{R}$ the map $(\alpha, H, t) \mapsto f_+(\alpha, H, t, \xi)$ is continuous on K, by Assertion *2.* of Proposition 5.1 of [5], S_m satisfies Assertion *1.* whatever m is.

□

Remark 3.1. Assertion *2.* in Theorem 3.1 holds for any m satisfying Equation (3.7) instead of m_η.

4. Linear multifractional multistable and stable motions

From now on let us consider $\alpha : \mathbb{R} \mapsto (0, 2)$ and $h : \mathbb{R} \mapsto (0, 1)$ two continuous functions. Under Hypothesis 2.1, by Proposition 2.1, we may consider the linear multifractional multistable motion defined on $\mathbb{R}$ by

$$\tilde{S}_m(t) := S_m(\alpha(t), h(t), t), \tag{4.1}$$

with S_m given by (2.3).

4.1. Regularity and rate of convergence

We may also define $\tilde{S}_{m,N}(t) := S_{m,N}(\alpha(t), h(t), t)$, for all $N \geq 1$. The following theorem is a direct consequence of Theorem 3.1.

Theorem 4.1. *Let us consider $\alpha : \mathbb{R} \mapsto (0, 2)$ and $h : \mathbb{R} \mapsto (0, 1)$ two continuous functions and two real numbers $a < b$. Then let us set*

$$\alpha_1 = \min_{t\in[a,b]} \alpha(t),\ \alpha_2 = \max_{t\in[a,b]} \alpha(t)\ \text{ and }\ H_1 = \min_{t\in[a,b]} h(t).$$

Assume that $H_1 > 1/\alpha_1$ and that α and h are $(H_1 - 1/\alpha_1)$-Hölder continuous functions on $[a, b]$.

1. *Then, as $N \to +\infty$, the series $\left(\tilde{S}_{m,N}\right)_{N\geq 1}$ converges uniformly on $[a, b]$ to $\tilde{S}_m$ and almost surely*

$$\sup_{\substack{t,t'\in[a,b]\\ t\neq t'}} \frac{\left|\tilde{S}_m(t) - \tilde{S}_m(t')\right|}{|t - t'|^{H_1 - 1/\alpha_1}\sqrt{|\log|t - t'|| + 1}} < +\infty.$$

2. Moreover if $m = m_\eta$ is defined by (3.3) with $\eta > 0$, then, almost surely

$$\sup_{N\geq 1} N^\varepsilon \sup_{t\in[a,b]} \left|\tilde{S}_{m_\eta,N}(t) - \tilde{S}_{m_\eta}(t')\right| < +\infty$$

for any $\varepsilon \in (0, 1/\alpha_2 - 1/2)$.

Note that one can use $\tilde{S}_{m_\eta,N}$ to simulate $\tilde{S}_{m_\eta}$. The error of approximation is then given by N^ε.

4.2. Stochastic integral and series representation

Assuming that α is a constant function equal to α_1, we have already seen that $\tilde{S}_m \overset{fdd}{=} d_{\alpha_1} X_{\alpha_1,h}$ where $X_{\alpha_1,h}$ is the linear multifractional α_1-stable motion defined by (3.1) and d_{α_1} is given by (2.6). Using the previous theorem we will prove the following one.

Theorem 4.2. *Let $\alpha_1 \in (0,2)$ and $h : \mathbb{R} \mapsto (0,1)$ be a continuous function. Let us also consider $X_{\alpha_1,h}$ the linear multifractional α_1-stable motion defined by (3.1) and two real numbers $a < b$. If $H_1 := \min_{t\in[a,b]} h(t) > 1/\alpha_1$ and if h is $(H_1 - 1/\alpha_1)$-Hölder continuous on $[a,b]$, then there exists a continuous modification $X^*_{\alpha_1,h}$ of $X_{\alpha_1,h}$ such that almost surely*

$$\sup_{\substack{t,t'\in[a,b]\\ t\neq t'}} \frac{\left|X^*_{\alpha_1,h}(t) - X^*_{\alpha_1,h}(t')\right|}{|t-t'|^{H_1-1/\alpha_1}\sqrt{|\log|t-t'|| + 1}} < +\infty.$$

Proof. Let $\alpha : \mathbb{R} \to (0,2)$ be the constant function equal to α_1 and let $\tilde{S}_m$ be defined by (4.1). Since $\tilde{S}_m \overset{fdd}{=} d_{\alpha_1} X_{\alpha_1,h}$ with $d_{\alpha_1} \neq 0$ defined by (2.6), by Theorem 4.1, we already know that a.s.

$$\sup_{\substack{t,t'\in[a,b]\cap\mathcal{D}\\ t\neq t'}} \frac{|X_{\alpha_1,h}(t) - X_{\alpha_1,h}(t')|}{|t-t'|^{H_1-1/\alpha_1}\sqrt{|\log|t-t'|| + 1}} < +\infty,$$

where $\mathcal{D}$ is the dense set of dyadic real numbers. Moreover, since h is continuous with values in $(0,1)$, the stochastic continuity of the linear multifractional α_1-stable motion $X_{\alpha_1,h}$ has been established in [19]. This implies that there exists a modification $X^*_{\alpha_1,h}$ of $X_{\alpha_1,h}$ such that

$$\sup_{\substack{t,t'\in[a,b]\\ t\neq t'}} \frac{\left|X^*_{\alpha_1,h}(t) - X^*_{\alpha_1,h}(t')\right|}{|t-t'|^{H_1-1/\alpha_1}\sqrt{|\log|t-t'|| + 1}} < +\infty,$$

see e.g. Section D.2 of [5] for the construction of $X^*_{\alpha_1,h}$. Then, the proof is complete. □

In [2], using a wavelet series expansion, under our assumptions of Proposition 3.1, the authors obtained a continuous modification $X^*_{\alpha_1,h}$ satisfying a.s. for all $\eta > 0$,

$$\sup_{\substack{t,t'\in[a,b]\\ t\neq t'}} \frac{\left|X^*_{\alpha_1,h}(t) - X^*_{\alpha_1,h}(t')\right|}{|t-t'|^{H_1-1/\alpha_1}\left(|\log|t-t'||+1\right)^{2/\alpha_1+\eta}} < +\infty.$$

Since $1/2 < 2/\alpha_1$, our result is sharper. Moreover it is quasi-optimal since, for $\eta > 0$, one can find h such that a.s.

$$\sup_{\substack{t,t'\in[a,b]\\ t\neq t'}} \frac{\left|X^*_{\alpha_1,h}(t) - X^*_{\alpha_1,h}(t')\right|}{|t-t'|^{H_1-1/\alpha_1}\left(|\log|t-t'||+1\right)^{-\eta}} = +\infty,$$

by Theorem 6.1 of [2]. Let us also quote that following our method based on [5], one may obtain an upper bound for the global modulus of continuity of linear fractional stable sheets, which is sharper than the one given in [3].

A. Proof of Proposition 3.2

Let us consider $K = [\alpha_1, \alpha_2] \times [H_1, H_2] \times [-A, A] \subset (1,2) \times (1/2, 1) \times \mathbb{R}$ such that $1/\alpha_1 < H_1 \le H_2 < 1$. Let us note that it is enough to prove Proposition 3.2 for A large enough. Then, in this proof, we assume, without loss of generality that $A > \mathrm{e}$ (so that $\log \xi > 1$ for $\xi > A$).

For all $x = (\alpha, H, t) \in K$, we set

$$\beta(x) = H - 1/\alpha \in (0,1)$$

and remark that $\beta(x) \in [\beta_1, \beta_2] \subset (0,1)$ with

$$\beta_1 = H_1 - 1/\alpha_1 \ \text{ and } \ \beta_2 = H_2 - 1/\alpha_2.$$

Moreover, for all $x = (\alpha, H, t) \in K$ and all $\xi \in \mathbb{R}$, let us note that

$$f_+(\alpha, H, t, \xi) = g(\beta(x), t, \xi)$$

with g defined on $(0,1) \times \mathbb{R} \times \mathbb{R}$ by

$$g(\beta, t, \xi) := (t-\xi)_+^\beta - (-\xi)_+^\beta.$$

Let us now consider $x = (\alpha, H, t) \in K$ and $x' = (\alpha', H', t') \in K$. Then, by (3.4),

$$V_{m,n}(x) - V_{m,n}(x') = \Big(g(\beta(x), t, \xi_n) m(\xi_n)^{-1/\alpha} - g\big(\beta(x'), t', \xi_n\big) m(\xi_n)^{-1/\alpha'}\Big).$$

Proposition 3.2 follows from the following lemma, which proof is given at the end of this section.

Lemma A.1. *Let* $0 < \beta_1 \le \beta_2 < 1$ *and* $A > \mathrm{e}$.

1. *There exists a finite positive constant* $c_1(A, \beta_1, \beta_2)$ *such that for all* $\beta, \beta' \in [\beta_1, \beta_2]$*, all* $t, t' \in [-A, A]$ *and all* $\xi \in \mathbb{R}$,
$$\left|g(\beta, t, \xi) - g\big(\beta', t', \xi\big)\right| \le c_1(A, \beta_1, \beta_2)\Big(\left|t - t'\right|^{\beta_1} + \left|\beta - \beta'\right|\Big)h_{A,1}(\xi, \beta_2)$$
with
$$h_{A,1}(\xi, c) = \mathbf{1}_{|\xi| \le 2A} + |\xi|^{c-1} \log|\xi|\, \mathbf{1}_{|\xi| > 2A}.$$

2. *Moreover, there exists a finite positive constant* $c_2(A, \beta_1)$ *such that for all* $\beta \in [\beta_1, \beta_2]$ *and* $t \in [-A, A]$,
$$|g(\beta, t, \xi)| \le c_2(A, \beta_1) h_{A,2}(\xi, \beta_2)$$
with
$$h_{A,2}(\xi, c) = \mathbf{1}_{|\xi| \le 2A} + |\xi|^{c-1}\, \mathbf{1}_{|\xi| > 2A}.$$

Setting for almost every $\xi \in \mathbb{R}$

$$\begin{cases} F_1(x, x', \xi) & := & |g(\beta(x), t, \xi) - g(\beta(x'), t', \xi)| m(\xi)^{-1/\alpha}, \\ F_2(x, x', \xi) & := & |g(\beta(x'), t', \xi)| \left| m(\xi)^{-1/\alpha} - m(\xi)^{-1/\alpha'} \right|, \end{cases}$$

we then have

$$\left|V_{m,1}(x) - V_{m,1}\big(x'\big)\right| \le F_1\big(x, x', \xi_1\big) + F_2\big(x, x', \xi_1\big).$$

Before we apply Lemma A.1 to bound F_1 and F_2, let us remark that for all $\xi \in \mathbb{R}$,

$$h_{A,2}(\xi, \beta_2) \le h_{A,1}(\xi, \beta_2) \le c_3(A, \beta_2)\Big(\mathbf{1}_{|\xi| \le \mathrm{e}} + |\xi|^{\beta_2 - 1} \log|\xi| \mathbf{1}_{|\xi| > \mathrm{e}}\Big) \tag{A.1}$$

with $c_3(A, \beta_2)$ a finite positive constant, which does not depend on ξ. Then, combining this remark with Lemma A.1, for almost every $\xi \in \mathbb{R}$,

$$F_1\big(x, x', \xi\big) \le c_1(A, \beta_1, \beta_2) c_3(A, \beta_2)\Big(\left|t - t'\right|^{\beta_1} + \left|\beta(x) - \beta(x')\right|\Big) h_{m,K}(\xi)$$

with $h_{m,K}$ defined by Equation (3.6). Since $\alpha_1 > 1$, by definition of the function β, it follows that for almost every $\xi \in \mathbb{R}$,

$$F_1\big(x, x', \xi\big) \le c_1(A, \beta_1, \beta_2) c_3(A, \beta_2) \tau\big(x - x'\big) h_{m,K}(\xi),$$

with $\tau(x - x') = |t - t'|^{\beta_1} + |H - H'| + |\alpha - \alpha'|$.
Moreover, applying Assertion 2 of Lemma A.1, Equation (A.1) and the mean value theorem, for almost every $\xi \in \mathbb{R}$,

$$F_2\big(x, x', \xi\big) \le c_2(A, \beta_1) c_3(A, \beta_2) \left|\alpha - \alpha'\right| h_{m,K}(\xi).$$

In view of the previous computations, we have: almost surely,

$$\left|V_{m,1}(x)-V_{m,1}\big(x'\big)\right| \leq c_{3,1}(K)\tau\big(x-x'\big)h_{m,K}(\xi_1)$$

with $c_{3,1}(K) := c_3(A,\beta_2)(c_1(A,\beta_1,\beta_2)+c_2(A,\beta_1))$. This concludes the proof of Proposition 3.2. □

We conclude this section by the proof of Lemma A.1.

Proof. [Proof of Lemma A.1] Let $0<\beta_1<\beta_2<1$ and $A>\mathrm{e}$. Let $\beta,\beta'\in[\beta_1,\beta_2]\subset(0,1)$ and $t,t'\in[-A,A]$. Let us write for all $\xi\in\mathbb{R}$,

$$\left|g(\beta,t,\xi)-g\big(\beta',t',\xi\big)\right| \leq g_1\big(\beta',t,t',\xi\big)+g_2\big(\beta,\beta',t,\xi\big)$$

with

$$\begin{cases} g_1(\beta',t,t',\xi) & := & |g(\beta',t',\xi)-g(\beta',t,\xi)| \\ g_2(\beta,\beta',t,\xi) & := & |g(\beta',t,\xi)-g(\beta,t,\xi)|. \end{cases}$$

Step 1: Control of g_1. Let us note that if $t=t'$, $g_1(\beta',t,t',\xi)=0$ for all $\xi\in\mathbb{R}$. Then, in this step, we assume now, without loss of generality that $t<t'$. This implies that

$$g_1\big(\beta',t,t',\xi\big)=\begin{cases} 0 & \text{if } \xi\geq t' \\ (t'-\xi)^{\beta'} & \text{if } t\leq\xi<t' \\ \left|(t-\xi)^{\beta'}-(t'-\xi)^{\beta'}\right| & \text{if } \xi<t. \end{cases}$$

Let $\xi\in\mathbb{R}$ with $|\xi|>2A$. If $\xi<0$ it follows that $\xi<t<t'$. Since $\beta'>0$, applying the mean value theorem,

$$g_1(\beta',t,t',\xi) \quad \leq \quad \beta'\,|t-t'|\left|c_{_{\xi,t,t'}}-\xi\right|^{\beta'-1}$$

with $c_{_{\xi,t,t'}}\in(t,t')\subset[-A,A]$. Moreover, since $|\xi|>2A$

$$\left|c_{_{\xi,t,t'}}-\xi\right| \geq |\xi|-\left|c_{_{\xi,t,t'}}\right| \geq |\xi|-A\geq|\xi|/2$$

and then

$$g_1\big(\beta',t,t',\xi\big) \leq 2^{1-\beta'}\left|t-t'\right||\xi|^{\beta'-1}$$

since $\beta'\in(0,1)$. Therefore, for $|\xi|>2A$,

$$g_1\big(\beta',t,t',\xi\big) \leq 4A\left|t-t'\right|^{\beta_1}|\xi|^{\beta_2-1} \tag{A.2}$$

since $|t-t'|\leq 2A$, $\beta'\in[\beta_1,\beta_2]\subset(0,1)$ and $2A>1$.

Now let $\xi\in\mathbb{R}$ with $|\xi|\leq 2A$. Since $0<\beta'<1$, we have

$$\left|a^{\beta'}-b^{\beta'}\right| \leq |a-b|^{\beta'}$$

for all $a, b \geq 0$. By definition of g, it follows that

$$g_1\big(\beta', t, t', \xi\big) \leq \Big|\big(t'-\xi\big)_+ - (t-\xi)_+\Big|^{\beta'} \leq \big|t'-t\big|^{\beta'} \leq 2A\big|t'-t\big|^{\beta_1}$$

since $-A \leq t < t' \leq A$, $0 < \beta_1 \leq \beta' < 1$ and $A > 1$. From this last inequality and Equation (A.2), we deduce that for all $\xi \in \mathbb{R}$,

$$g_1\big(\beta', t, t', \xi\big) \leq 4A\big|t-t'\big|^{\beta_1} h_{A,2}(\xi, \beta_2) \tag{A.3}$$

with $h_{A,2}(\xi,\beta_2) = \mathbf{1}_{|\xi|\leq 2A} + |\xi|^{\beta_2 - 1}\mathbf{1}_{|\xi|>2A}$.

Step 2: Control of g_2. Let us recall that for all $\xi \in \mathbb{R}$,

$$g_2\big(\beta, \beta', t, \xi\big) = \Big|(t-\xi)_+^{\beta'} - (t-\xi)_+^{\beta} + (-\xi)_+^{\beta} - (-\xi)_+^{\beta'}\Big|.$$

Then, applying the mean value theorem, for all $\xi \in \mathbb{R}$,

$$g_2\big(\beta, \beta', t, \xi\big) \leq \big|\beta - \beta'\big| \sup_{\beta_1 \leq c \leq \beta_2} \big|(t-\xi)_+^c \log(t-\xi)_+ - (-\xi)_+^c \log(-\xi)_+\big|$$

where for $c > 0$,

$$(x)_+^c \log(x)_+ = \begin{cases} x^c \log x & \text{if } x > 0 \\ 0 & \text{if } x \leq 0. \end{cases}$$

Let us first consider $\xi \in [-2A, 2A]$. Then, $(-\xi)_+ \in [0, 2A]$ and $(t-\xi)_+ \in [0, 3A]$ since $t \in [-A, A]$. Therefore,

$$g_2\big(\beta, \beta', t, \xi\big) \leq \tilde{c}_1(A, \beta_1, \beta_2)\big|\beta - \beta'\big| \tag{A.4}$$

with

$$\tilde{c}_1(A,\beta_1,\beta_2) = 2 \max_{\beta_1 \leq c \leq \beta_2} \max_{0<u\leq 3A} u^c |\log u| = 2\max\left(\frac{1}{\mathrm{e}\beta_1}, (3A)^{\beta_2}\log(3A)\right) < +\infty.$$

Let us now assume that $\xi < -2A$. Then, $\xi < t$ and

$$g_2\big(\beta, \beta', t, \xi\big) \leq \big|\beta - \beta'\big| \sup_{\beta_1 \leq c \leq \beta_2} |(t-\xi)^c \log(t-\xi) - (-\xi)^c \log(-\xi)|$$

with $t - \xi > 0$ and $-\xi > 0$. Let us remark that $-\xi \in (-\xi/2, -3\xi/2)$ since $-\xi > 0$ and that

$$-\xi/2 < -A - \xi \leq t - \xi \leq A - \xi < -3\xi/2$$

since $t \in [-A, A]$ and $\xi < -2A$. Then, for each $c \in [\beta_1, \beta_2] \subset (0,1)$, by the mean value theorem,

$$|(t-\xi)^c \log(t-\xi) - (-\xi)^c \log(-\xi)| \leq |u_{t,\xi,c}|^{c-1}(c|\log u_{t,\xi,c}| + 1)$$

with $u_{t,\xi,c} \in (-\xi/2, -3\xi/2)$. Since $u_{t,\xi,c} \in (-\xi/2, -3\xi/2)$ and $-\xi/2 > A > \mathrm{e}$, we get

$$|(t-\xi)^c \log(t-\xi) - (-\xi)^c \log(-\xi)| \leq 4|\xi|^{\beta_2 - 1} \log|\xi|$$

for all $c \in [\beta_1, \beta_2] \subset (0,1)$. Hence, for $\xi < -2A$,

$$g_2\big(\beta, \beta', t, \xi\big) \leq 4\big|\beta - \beta'\big| |\xi|^{\beta_2 - 1} \log|\xi|.$$

Note that this last inequality still holds for $\xi > 2A$ since in this case, $g_2(\beta, \beta', t, \xi) = 0$.

Then, we have proved that for all $\xi \in \mathbb{R}$,

$$g_2\big(\beta, \beta', t, \xi\big) \leq \tilde{c}_2(A, \beta_1, \beta_2)\big|\beta - \beta'\big| h_{A,1}(\xi, \beta_2) \tag{A.5}$$

with $\tilde{c}_2(A, \beta_1, \beta_2) = \max\left(\tilde{c}_1(A, \beta_1, \beta_2), 4\right)$ and

$$h_{A,1}(\xi, \beta_2) = \mathbf{1}_{|\xi| \leq 2A} + |\xi|^{\beta_2 - 1} \log|\xi| \mathbf{1}_{|\xi| > 2A}.$$

Step 3: Proof of Assertion 1. It follows from Equations (A.3) and (A.5) choosing $c_1(A, \beta_1, \beta_2) = \tilde{c}_2(A, \beta_1, \beta_2) + 4A \in (0, +\infty)$ and using the fact that $h_{A,2}(\xi, \beta_2) \leq h_{A,1}(\xi, \beta_2)$ since $A > \mathrm{e}$.

Step 4: Proof of Assertion 2. Let us remark that

$$g\big(\beta', t', \xi\big) = g\big(\beta', t', \xi\big) - g\big(\beta', 0, \xi\big)$$

since $g(\beta', 0, \xi) = (-\xi)_+^{\beta'} - (-\xi)_+^{\beta'} = 0$. Hence, applying Equation (A.3) with $t = 0$ and $\beta' = \beta$,

$$\big|g\big(\beta', t', \xi\big)\big| \leq 4A\big|t'\big|^{\beta_1} h_{A,2}(\xi, \beta_2) \leq 4A^{\beta_1 + 1} h_{A,2}(\xi, \beta_2),$$

which concludes the proof. □

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