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On Dümbgen's exponentially modified Laplace continued fraction for Mill's ratio

Florin Avram

Abstract - The approximation of the Gaussian cumulative distribution $\Phi(x)$ or of the related Mills ratio

$$R(x) := \frac{1-\Phi(x)}{\phi(x)} := h(x)^{-1} \tag{0.1}$$

where $\phi(x)$ is the standard Gaussian density, and $h(x)$ is its hazard rate, have a long history starting with Gauss and Laplace and continuing nowadays [6, 12, 1, 2, 16, 10]. Below, we improve an important family of bounds provided recently by Dümbgen [5].

Key words and phrases : continued fraction, Mill's ratio, hazard rate.

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1. Introduction

A convenient starting point for the study of the Gaussian Mill's ratio (0.1) is the first order ODE

$$R'(x) = x\,R(x) - 1, \quad R(0) = \sqrt{\pi/2}. \tag{1.1}$$

The equation (1.1) allows building a Taylor expansion around 0, and a formal Laurent expansion in negative powers at ∞, due to Laplace:

$$R(x) = \frac{1}{x}\left(1 - \frac{1}{x^2} + \frac{3!!}{x^4} - \frac{5!!}{x^6} + ...\right).$$

The latter is divergent (though asymptotic in the sense of Poincaré); however, this problem may be remedied by considering continued fractions, whose domain of convergence typically is larger than that of the series. The passage from series to a continued fraction with denominators 1 [14, pg. 21] may be achieved by using recursively the formula

$$1+\sum_{i=1}^{\infty}(-1)^i a_i x^i \approx$$
$$\left(1+a_1x+(a_1^2-a_2)x^2+(a_1^3-2a_1a_2+a_3)x^3+(a_1^4-3a_1^2a_2+a_2^2+2a_1a_3-a_4)x^4...\right)^{-1}$$

yielding

$$\begin{aligned}
R(x) &= \frac{1}{x}\left(1-\frac{1}{x^2}+\frac{3!!}{x^4}-\frac{5!!}{x^6}+...\right)=\frac{1}{x}\frac{1}{1+\frac{1}{x^2}-\frac{2}{x^4}+\frac{10}{x^6}-\frac{74}{x^8}...}\\
&= \frac{1}{x}\frac{1}{1+\frac{1}{x^2}(1-\frac{2}{x^2}+\frac{10}{x^4}-\frac{74}{x^6}...)}=\frac{1}{x}\frac{1}{1+\frac{x^{-2}}{1+2x^{-2}(1-\frac{3}{x^2}+\frac{21}{x^4}...)}}\\
&= \frac{1}{x}\frac{1}{1+\frac{x^{-2}}{1+\frac{2x^{-2}}{1+\frac{3}{x^2}(1-\frac{4}{x^2}...)}}}=\frac{1}{x}\frac{1}{1+\frac{x^{-2}}{1+\frac{2x^{-2}}{1+\frac{3x^{-2}}{1+\frac{4x^{-2}}{1+...}}}}}\\
&= \frac{1}{1+\frac{v}{1+\frac{2v}{1+\frac{3v}{1+...}}}}, v=\frac{1}{x^2}. \qquad (1.2)
\end{aligned}$$

This equation is related to the famous Laplace's continued fraction (2.2), which yields alternating upper and lower bounds for Mill's ratio. Tighter alternating bounds were derived recently by [5], by judicious modifications of the last denominators. We propose further modifications which improve numerically on Dümbgen's, and seem (but are not yet proved) to provide alternating bounds as well.

Contents. A brief review of continued fractions is given in Section 2. Lee's and Dümbgen's approaches to the Gaussian Mill's ratio are reviewed in Section 3. The new family of bounds is introduced and illustrated numerically in Section 4. In Section 5 we discuss briefly the possibility of extending this approach for providing continued fraction bounds for other Pearson densities, like the Gamma density, which is of interest in queueing, for example in asymptotic studies of retrial queues in the Halfin-Whitt regime.

2. A brief review of continued fractions

Definitions. Recall that a continued fraction

$$b_0+\mathbb{K}_1\left(\frac{a_k}{b_k}\right)=b_0+\frac{a_1}{b_1+\frac{a_2}{b_2+...}}=b_0+\frac{a_1}{b_1+\frac{a_2}{b_2+\mathbb{K}_3\left(\frac{a_k}{b_k}\right)}}$$

where $\mathbb{K}_n(\frac{a_k}{b_k}) := \frac{a_n}{b_n+\mathbb{K}_{n+1}(\frac{a_k}{b_k})}$ is defined, when convergent, as the limit of the **convergents** $R_n(x)=\frac{A_n(x)}{B_n(x)}$ obtained by replacing $\mathbb{K}_n\left(\frac{a_k}{b_k}\right)$ with 0.

A_n and B_n satisfy both the forward Wallis-Euler recursion $x_n = b_n x_{n-1}+a_n x_{n-2}, n \geq 2$, with respective initial conditions $A_0 = b_0, A_1 = a_1+b_0b_1$, and

$B_0 = 1, B_1 = b_1$, and may also be written as "continuant" determinants:

$$A_n = \det \begin{pmatrix} b_0 & -1 & & & & & \\ a_1 & b_1 & -1 & & & & \\ & a_2 & b_2 & -1 & & & \\ & & & \vdots & & & \\ & & & & a_{n-1} & b_{n-1} & -1 \\ & & & & & a_n & b_n \end{pmatrix}$$

$$B_n = \det \begin{pmatrix} b_1 & -1 & & & & & \\ a_2 & b_2 & -1 & & & & \\ & a_3 & b_3 & -1 & & & \\ & & & \vdots & & & \\ & & & & a_{n-1} & b_{n-1} & -1 \\ & & & & & a_n & b_n \end{pmatrix}$$

Transformations. For any sequence $p_k \neq 0, p_0 = 1$, the two fractions

$$b_0 + \mathbb{K}_1 \left(\frac{a_k}{b_k} \right), \quad b_0 + \mathbb{K}_1 \left(\frac{p_{k-1} p_k a_k}{p_k b_k} \right) \tag{2.1}$$

are equivalent (have the same convergents). Thus, appropriate choices of p_k will simplify either the numerators or denominators, as desired.

Laplace's continued fraction. Applying the transformation (2.1) to (1.2) with $p_k = x$ and putting $a_k = k + \delta_0(k)$, one arrives to Laplace's continued fraction

$$\mathbb{K}_1 \left(\frac{a_k}{x} \right) = \frac{1}{x + \frac{1}{x + \frac{2}{x + \frac{3}{x + \dots}}}}, \quad x > 0, \tag{2.2}$$

which converges to $R(x)$ on $(0, \infty)$. Another continued fraction associated to the Taylor expansion around 0 was provided by Shenton.

Remark 2.1. Note that due to the repetition of the numerator 1, it is more natural here to start indexing R_n by $n = 0$, so that the terminating fraction with numerator n is denoted by R_n. Thus,

$$R_0 = \frac{1}{x}, R_1 = \frac{1}{x + \frac{1}{x}}, R_2 = \frac{1}{x + \frac{1}{x + \frac{2}{x}}}, \dots$$

Remark 2.2. Another derivation of Laplace's continued fraction may be obtained, following Euler, by differentiating (1.1), which yields

$$\begin{aligned} R^{(n)}(x) &= B_n(x)\, R(x) - A_n(x) \Leftrightarrow R(x) = \frac{A_n(x)}{B_n(x)} + \frac{R^{(n)}(x)}{B_n(x)}, \quad (2.3) \\ A_{n+1}(x) &= x A_n(x) + n A_{n-1}(x), \quad (A_0(x), A_1(x)) = (0, 1), \\ B_{n+1}(x) &= x B_n(x) + n B_{n-1}(x), \quad (B_0(x), B_1(x)) = (1, x), \end{aligned}$$

see [10].

Modified continued fractions. The computation of continued fractions is often achieved by the backward recurrence

$$R_{m,n} = b_m + \frac{a_m}{R_{m+1,n}}, m = n-1, n-2, ..., 0$$

where $R_{m,n} = \mathbb{K}_m^n \left(\frac{a_k}{b_k}\right) = b_m + \frac{a_{m+1}}{b_{m+1}+\frac{a_{m+2}}{b_{m+2}+\cdots\frac{a_n}{b_n}}}$.

The classic starting point is $R_{n,n} = b_n$, but the result may often be improved by starting with modified last denominators $R_{n,n} = \beta_n = b_n + \gamma$, i.e. by using

$$\begin{aligned} R_n(x) &= b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}... + \frac{a_{n-1}}{b_{n-1}+} + \frac{a_n}{b_n+\gamma} \\ &= \frac{(b_n+\gamma)A_{n-1} + a_n A_{n-2}}{(b_n+\gamma)B_{n-1} + a_n B_{n-2}} = \frac{A_n + \gamma A_{n-1}}{B_n + \gamma B_{n-1}}, \end{aligned}$$

[14, Ch. 5.5]. Note that we have switched here to the one line convention of writing continued fractions (in which the subcontinued fractions following a + or − are realigned on the first line), and that parametrizing the last modified denominator by $\beta_n = b_n + \gamma$ (developping around the "usual" continued fraction coefficient b_n) simplifies some expressions.

The idea is to replace b_n by an "ansatz" β_n approximating more closely the exact value $R_{n,n}$ [16]. We will call this unknown value the "correct ansatz".

The limit ansatz. Assuming n is big enough so that $R_{n,k}$ varies slowly in n, one such approximation is the limit ansatz obtained by solving

$$R_{n,\infty} = b_n + \frac{a_n}{R_{n,\infty}}. \tag{2.4}$$

Alternating bounds. As noticed already by Brouncker and Euler [4, 9], the positivity of the continued fraction numerators and denominators implies that the convergents yield upper and lower bounds

$$R_2 \le R_4 \le ...R_{2n}... \le R... \le R_{2n+1} \le ... \le R_3 \le R_1, \tag{2.5}$$

valid on the domain of convergence of the continued fraction.

In particular, the convergents of the Laplace continued fraction yield bounds valid on $(0,\infty)$ (see also [10, Prop. 7]).

General error estimates

$$|R - \mathbb{R}_n| < \frac{n!}{B_n B_{n+1}} \tag{2.6}$$

are also available [§] .

Uniform bounds on $[0, \infty)$. The Laplace and Shenton continued fractions are quite efficient in their "natural domains", and this allows constructing efficient approximations based on both. However, if one entertains the somewhat academic wish to use a single approximation valid on $[0, \infty)$, one must improve the quality of approximation at 0 if the continued fraction is based on the series at ∞, and viceversa.

Following [12, 5] in their tribute to Laplace, we will consider here the continued fraction based on the series at ∞. Two strategies suggest themselves:

1. use **rational two-point Padé approximants** $[k_0, k_\infty]$ fitting k_0 derivatives at 0 and k_∞ derivatives at ∞ (these seem to have been introduced by Murphy and McCabe [15]).

   Reasonable uniform approximations are already obtained with $k_\infty = 2, k_0 = 1, 2, ...$ [1], the simplest one with $k_0 = 1$ being $R_2(x) = \frac{(\pi-2)\sqrt{2\pi}+x(4-\pi)}{2(\pi-2)+x\sqrt{2\pi}+x^2(4-\pi)} = \frac{1}{x+\frac{1}{\beta_1(x)}}$, where $\beta_1(x) = \frac{(\pi-2)\sqrt{2\pi}+x(4-\pi)}{2(\pi-2)+x(3-\pi)\sqrt{2\pi}}$. The fit at 0 is due here to $\beta_1(0) = \frac{\sqrt{2\pi}}{2}$.

2. use cleverly chosen modified continued fractions, which, besides fitting at 0, achieve possibly also a good approximation of the "correct ansatz".

   Applying the limit ansatz (2.4) to Laplace's continued fraction amounts to replacing the denominator x below the numerator n by the "terminating denominator"

   $$\beta_n(x) = \frac{x}{2} + \sqrt{\left(\frac{x}{2}\right)^2 + n}. \tag{2.8}$$

   An even better starting point $\beta_n(x) = \frac{x}{2} + \sqrt{(\frac{x}{2})^2 + \gamma_n}$, with $\gamma_n = \beta_n^2(0)$ defined in (3.6) has been proposed by [5], by exploiting both the functional form of the limit ansatz, and the correct behavior at 0.

   The simplest choice is chosing linear modifications

   $$\beta_n(x) = \lambda_n x + \beta_n(0).$$

[§] The relations (2.5), (2.6) are consequences of the Euler identities

$$R_n - \mathbb{R}_{n+1} = \frac{\prod_{i=1}^{n+1}(-a_i)}{B_n B_{n+1}} \tag{2.7}$$

$$R_{2n+1} - \mathbb{R}_{2n-1} = -\frac{b_{2n+1}\prod_{i=1}^{2n} a_i}{B_{2n-1}B_{2n+1}}$$

$$R_{2n} - \mathbb{R}_{2n-2} = \frac{b_{2n}\prod_{i=1}^{2n-1} a_i}{B_{2n-2}B_{2n}}$$

Dümbgen's best results, confirmed here, are finally obtained with **exponential type modifications**. Since there is no clear reason for that, we might call this an "inspiration ansatz".

3. Lee's and Dümbgen's modified Laplace continued fractions

One possible approach, taken by [12], is to consider **doubly modified convergents**

$$\begin{aligned} R_n(x) &= b_0 + \frac{a_1}{b_1+}\frac{a_2}{b_2+}...+\frac{a_{n-1}}{b_{n-1}+}+\frac{\alpha}{b_n+\gamma} \\ &= \frac{(b_n+\gamma)A_{n-1}+\alpha A_{n-2}}{(b_n+\gamma)B_{n-1}+\alpha B_{n-2}} = \frac{A_n+\gamma A_{n-1}+(\alpha-a_n)A_{n-2}}{B_n+\gamma B_{n-1}+(\alpha-a_n)B_{n-2}}, \end{aligned}$$

with both the last numerator and denominator modified, and where $b_n = b_n(x), \gamma = \gamma(x)$ may depend on x.

Consider the sign of the approximation error, supposing, more generally, that $R(x)$ is the Mill's ratio of a density $f(x)$ satisfying

$$f'(x) = -q(x)\ f(x), \tag{3.1}$$

where $q(x)$ is rational. Then, $R(x)$ satisfies the first order differential equation

$$R'(x) = q(x)\ R(x) - 1, \tag{3.2}$$

generalizing (1.1). Then, if $\lim_{u\to\infty}\phi(u)R_n(u) = 0$, the approximation error

$$\Delta_n(x) = \int_x^\infty \phi(u)du - \phi(x)R_n(x)$$

may be expressed as

$$\begin{aligned} \Delta_n(x) &= \int_x^\infty \phi(u)du + \int_x^\infty (\phi(u)R_n(u))'du \\ &= \int_x^\infty \phi(u)(1+R_n(u)' - q(u)R_n(u))du. \end{aligned}$$

While the sign of the last integral is hard to analyze, it is easier to control the sign of the integrand

$$\delta_n(u) = -\frac{\Delta_n'(u)}{\phi(u)} = 1 + R_n'(u) - q(u)R_n(u) := (GR_n)(u) \tag{3.3}$$

where we note that G is precisely the operator defining our function of interest (3.2). Providing upper/lower bounds may thus be achieved by ensuring that $\delta_n(u)$ is negative/positive for all u in the domain of convergence.

Remark 3.1. Let us note also an expression for the second derivative:

$$\begin{aligned}\delta_n^{(2)}(u) &= -\frac{\Delta_n''(u)}{\phi(u)} \\ &= R_n''(u) - 2q(u)R_n'(u) + (q^2(u) - q'(u))R_n(u) - q(u) \\ &:= (G^{(2)}R_n)(u).\end{aligned} \tag{3.4}$$

We turn now to Dümbgen's impressive "creative denominator modifications", whose numerical results suggest that Lee's double modifications are not necessary. The basis is again an analysis of the sign of the derivative of the approximation error $\delta_n(u)$ defined in (3.3), this times in terms of the modification $\beta_n(x)$ [5, Lem. 1].

Lemma 3.1. *Let $\beta(u) = \beta_n(u)$ denote differentiable terminating modified denominators for Laplace's continued fraction*

$$\cfrac{1}{x + \cfrac{1}{x + \cfrac{2}{x + \cfrac{3}{x + \dots \cfrac{n-1}{x + \cfrac{n}{\beta_n(x)}}}}}}$$

of the Gaussian Mills ratio. Then:

$$\delta_n(u) = \frac{(-1)^{n-1}(n)!}{B_{n-1}(u)^2}\tilde{G}_n\beta(u), \quad \tilde{G}_n\beta(u) = u\beta(u) + \beta'(u) + n - \beta^2(u). \tag{3.5}$$

Proof. The equation may be established by induction. The operator $\tilde{G}_n\beta(u)$, which provides the sign of $\delta_n(u)$, is also given on [5, pg. 7].

The next step towards producing uniform bounds valid on $[0,\infty)$ is to find conditions on the modified denominators $\beta_n(x)$ which give rise to a zero of the error at 0.

Lemma 3.2. *The equations for ensuring $\Delta_n(0) = 0, \Delta_n'(0) = 0, \Delta_n''(0) = 0$ are linear in $\beta_n(0), \beta_n'(0), r_n := \beta_n''(0)/\beta_n(0)$, with solutions:*

$$\Delta_n(0) = 0 \Leftrightarrow \beta_n(0) = \sqrt{2}\frac{\Gamma(n/2+1)}{\Gamma(n/2+1/2)} \tag{3.6}$$

$$\delta_n(0) = 0 \Leftrightarrow \beta_n'(0) = \beta_n^2(0) - n \tag{3.7}$$

$$\delta_n^{(2)}(0) = 0 \Leftrightarrow r_n = 2(\beta_n^2(0) - n - \frac{1}{2}). \tag{3.8}$$

The constants $\beta_n'(0)$ and r_n are positive.

Remark 3.2. These formulas will produce two-point Padé approximants, when applied to rational modifications $\beta(x)$.

Proof. The first formula is obtained in [5, (13),(14)], by imposing recursively the condition $\Delta_n(0) = 0 \Leftrightarrow R_n(0) = \sqrt{\frac{\pi}{2}}$ on the successive errors

$$\Delta_0 = 1 - \Phi(x) - \frac{\phi(x)}{\beta_0(x)}, \Delta_1 = 1 - \Phi(x) - \frac{\phi(x)}{x + \frac{1}{\beta_1(x)}},$$

yielding $\beta_0(0) = \sqrt{\frac{2}{\pi}}, \beta_1(0) = \sqrt{\frac{\pi}{2}}, \ldots$. In general, we may note that one has $R_n(\beta_n, x) = R_{n-1}(x + \frac{n}{\beta_n}, x)$, yielding $\beta_k(0) = \frac{k}{\beta_{k-1}(0)}, k = 1, 2, \ldots$.

The second formula follows from (3.5). In [5, Thm 2], it is presented as a favorite choice among several possible linear modifications $\beta_n(x) = \lambda_n x + \beta_n(0)$, and a proof that it yields alternating bounds is offered, but without mention of the two-point Padé connection.

For the third formula, which does not appear in [5], it is enough to consider the case $\beta_n(x) = \beta(0) + x(\beta^2(0) - n) + x^2 \frac{\beta''(0)}{2}$. A tedious computation yields that

$$\delta^{(2)}(0) = 0 \Leftrightarrow R''(0) = R(0) \Longrightarrow \frac{\beta''(0)}{\beta(0)} = 2(\beta^2(0) - n - \frac{1}{2}).$$

Intriguingly, the same expression appears in a different context on the bottom of [5, pg. 9]. This topic deserves further attention, and we are investigating currently whether the second order two-point Padé condition leads to alternating bounds, as suggested by our numerical results.

The positivity follows from [5, Lem. 3].

Question 3.1. These results suggest the interesting problem of obtaining minimal solutions to the Riccatti inequations $\tilde{G}_n \beta(u) \geq (\leq) 0, \forall u \geq 0,$, with constraints $\beta(0) = \sqrt{2} \frac{\Gamma(n/2+1)}{\Gamma(n/2+1/2)}$, which would provide an optimal modification of Laplace's continued fraction.

Next, [5, Lem. 2] offers a simplified method of establishing alternating bounds, by replacing the requirement of strictly negative/positive derivatives $\Delta_n'(x)$ by the weaker requirement of strictly negative/positive and unimodal derivatives, which is easier to impose. This idea is not exploited in our paper.

Finally, [5] raises the dilemma of choosing between several possible functional forms for $\beta_n(x)$.

1. The approximations $\beta_n(x) = x + \beta_n(0)$ are not far from Lee's bound $\beta_n(x) = x + \sqrt{n+1}$, since it may be shown that $\beta_n(0) \in (\sqrt{n + 1/2}, \sqrt{n+1})$. However, both Lee's and Dümbgen's linear approximations fare not so well numerically.

2. [5, Thm.1] considers square root modifications, in which n in the ansatz (2.8) is replaced by the constants $\beta_n^2(0)$ of (3.6).

3. [5, Thm.2] considers more general linear modifications $\beta_n(x) = \lambda_n x + \beta_n(0)$, where $\lambda_n = \beta_n'(0)$ is choosen to make also the first derivative $\Delta_n'(0)$ equal to 0. By Lemma 3.1, this requires solving $\beta_n'(0) + n - \beta_n^2(0) = 0$, yieding $\lambda_n = \beta_n^2(0) - n$ [5, Sec 5].

4. Finally, [5, Thm.3] shows that the rational bounds may be considerably improved by using **exponential-type modifications** of the last denominators.

4. Improved Dümbgen's exponentially modified continued fractions

We have implemented one step further Dümbgen's idea of considering exponentially modified continued fractions, by looking for exponential + linear modifications:

$$\beta_n(x) = c_n x + \beta_n(0)e^{-\sqrt{r_n}x} = (\lambda_n + r_n\beta_n(0))x + \beta_n(0)e^{-\sqrt{r_n}x}, \tag{4.1}$$

where the new constants r_n are chosen to make the second derivative $\Delta_n''(0)$ equal to 0, which requires, cf. Lemma 3.2,

$$r_n = \frac{\beta_n''(0)}{\beta_n(0)} = 2(\beta_n^2(0) - n - \frac{1}{2}). \tag{4.2}$$

The figures below compare the exponential, our improved exponential (practically indistinguishable from 0), and the linear and square root modifications. As expected, the square root (who does not fit any derivatives at 0) loses always near 0, but catches up with the linear later. The exponential modifications are always better, especially the new one proposed here. The maximum errors of the first four terms are .00021, .000048, .000030, .000016.

5. Bounds for the Gamma density Mills ratio/Prym's function

Besides the normal, bounds for Mill's ratio of other "Pearson distributions" (with connections to orthogonal polynomials, etc...) are also of great interest to probabilists.

The Gamma density for example $\gamma(s, x)$ is of special interest due to its appearance in many classic problems: the birthday paradox, Ramanujan's Q function, Erlang loss probability, reliability, etc. For the convenience of the reader interested in this problem, we summarize here some relevant information.

The Mills ratio $R(x) = R_s(x)$ for the Gamma density $\gamma_s(x)$ satisfies the equation

$$R'(x) = q(x)\ R(x) - 1,\ q(x) = 1 + \frac{1-s}{x},\ R(\infty) = 1\ (R(0) = 0,\ \text{for } s < 1) \tag{5.1}$$

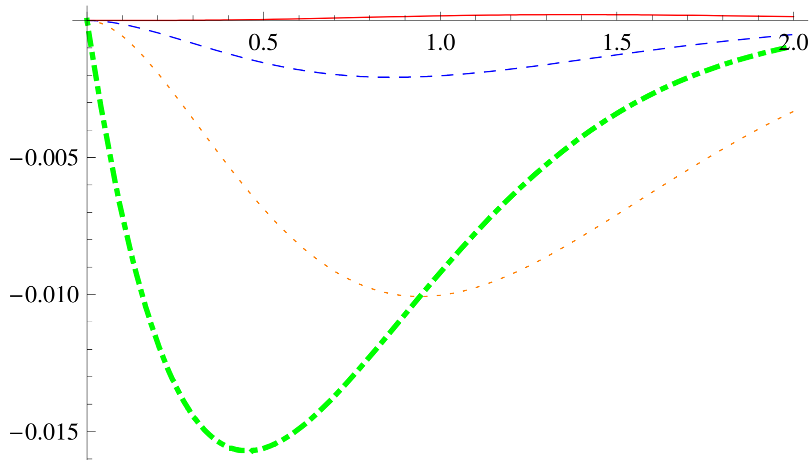

Figure 1: Errors for Dümbgen's bounds for $\Delta_0(x)$; blue, dashed: Dümbgen's expo, red:second order expo, yellow, dotted: linear, green, dotdashed: square root

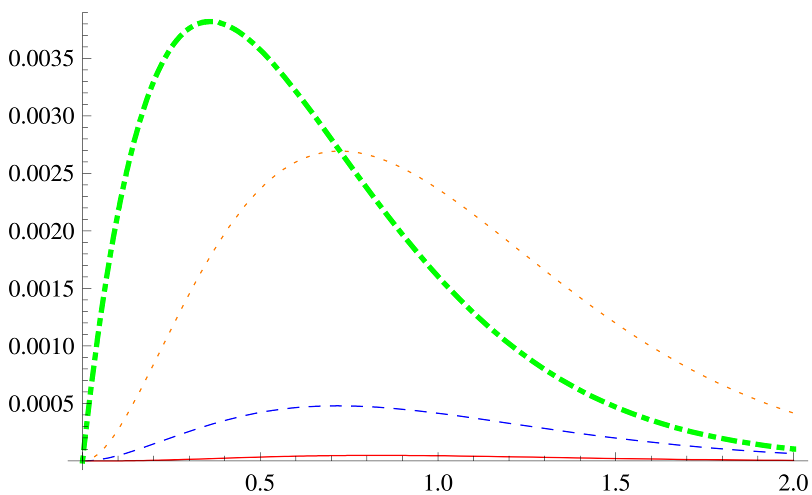

Figure 2: Errors for Dümbgen's bounds $\Delta_1(x)$; blue, dashed: Dümbgen's expo, red:second order expo, yellow, dotted: linear, green, dotdashed: square root

and a continued fraction for it was already developped in [13]. Note the integral representation:

$$R_s(\lambda) = \lambda \int_0^\infty (1+t)^s e^{-\lambda t} dt = \int_0^\infty (1+\frac{u}{\lambda})^s e^{-u} du. \tag{5.2}$$

For integer s, this may be easily derived by noting that the normaliza-

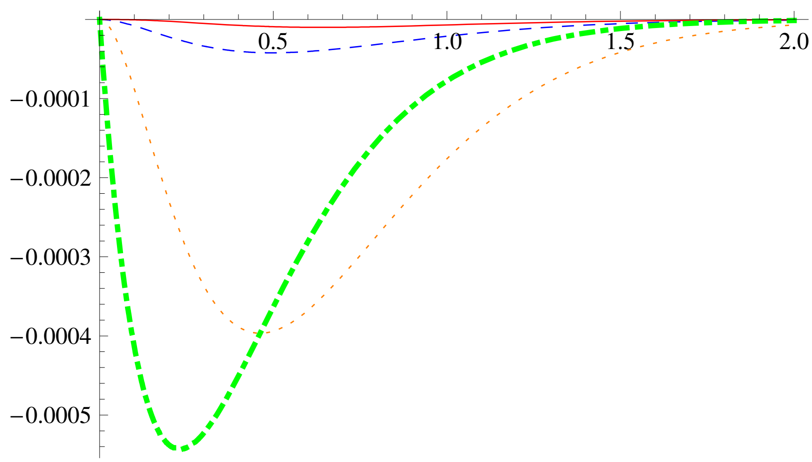

Figure 3: Errors for Dümbgen's bounds $\Delta_4(x)$; blue, dashed: Dümbgen's expo, red:second order expo, yellow, dotted: linear, green, dotdashed: square root

tion of the Gamma density $\gamma_{k+1}(x)$ may be written as $\frac{s(s-1)...(s-k+1)}{\lambda^k} = \lambda\binom{s}{k}\int_0^\infty t^k e^{-\lambda t}dt$ and summing for $k = 0, 1, ...s$. For noninteger s, see [7].

Some changes of variables [8, pg 143] put (5.1) in the form of a homogeneous Riccati equation:

$$t^2 z'(t) - (1 + (1-s)t)z(x) + z^2(t) = 0, \tag{5.3}$$

from which the continuous fraction

$$R(x) = \frac{x}{x + \frac{1-s}{1+\frac{1}{x+\frac{2-s}{1+...}}}} = \frac{x}{x+}\ \frac{1-s}{1}\ _+ \frac{1}{x+}\ + \frac{2-s}{1} + ... + \frac{n}{x+}\ \frac{n+1-s}{1}\ _{+...} \tag{5.4}$$

may be obtained via a classic method of Lagrange [3]. Cf [8, (11.6)], contracting the continuous fraction yields

$$e^x\int_x^\infty u^{s-1}e^{-u}du = x^{s-1}R(x)$$
$$= \frac{x^s}{x+1-s_+}\ \frac{s-1}{x+3-s_+}\ \frac{2(s-2)}{x+5-s_{+...+}}\ \frac{n(s-n)}{x+2n+1-s_{-...}}, \tag{5.5}$$

a result which goes back to Laguerre [11]. A similar continued fraction expansion holds for the cumulative Gamma distribution:

$$x^{-s+1}e^x\int_0^x u^{s-1}e^{-u}du = \frac{x}{s-}\ \frac{sx}{1+s+x-}\ \frac{(1+s)x}{2+s+x-}...\frac{(n-1+s)x}{n+s+x-}... \tag{5.6}$$

An equivalent continued fraction used by [16] is:

$$x^{1-s}e^x\int_x^\infty u^{s-1}e^{-u}du = \frac{1}{1+}\ \frac{(1-s)v}{1+}\ \frac{v}{1+}\ \frac{(2-s)v}{1+}\ \frac{2v}{1+}...,\qquad v = \frac{1}{x}. \tag{5.7}$$

This generalizes Laplace's continued fraction. Indeed, putting $s = \frac{1}{2}, u = \frac{v}{2}$ yields $\frac{1}{1+} \frac{u}{1+} \frac{2u}{1+} \frac{(3u}{1+} ...$, which is equivalent to Laplace's Continued Fraction after substituting $u = (\frac{1}{2x})^2$.

The problem of providing bounds for the Gamma Mills ratio based on the continued fractions (5.4), (5.6) has been considered by [6]. Several cases need to be distinguished, according to their difficulty:

1. for $s \in (0, 1]$, the continued fraction approximations continue to have positive coefficients, like in the Gaussian case (which corresponds in fact to $s = 2$, via a simple transformation). This case is thus straightforward [6, (3.5)].

2. for $s > 1$, fixed, the computation may be reduced to the case $s \in (0, 1]$ by induction on the integer part of s [6, (3.7)].

3. the case $s \approx x \to \infty$ is more subtle, and it is precisely this case that is of interest in queueing, for example in asymptotic studies of retrial queues in the Halfin-Whitt regime.

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Florin Avram
Département de Mathématiques, Université de Pau
Avenue de l'Université - BP 1155, 64013 Pau Cedex, France
E-mail: `florin.avram@univ-pau.fr`