

## A note on almost contact metric submersions whose total space is a Chinea-Gonzalez manifold

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**Abstract** - In this paper, we discuss some geometric properties of Riemannian submersions whose total space is a manifold defined by Chinea and Gonzalez.

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### 1. Introduction

In their classification scheme of almost contact metric manifolds, Chinea and Gonzalez (see [2]) have obtained several classes that have not yet been considered, namely:  $C_7$ ,  $C_8$ ,  $C_9$ ,  $C_{10}$ ,  $C_{11}$ , and  $C_{12}$ . Briefly, we call them the *Chinea-Gonzalez manifolds*.

This note is intended to describe Riemannian submersions whose total space is one of the manifolds under consideration. It is organized in the following way.

Section 2 is devoted to the preliminary background, a revision of almost Hermitian and almost contact metric manifolds.

In section 3, we determine the structure of the fibres and the base space according to that of the total space of the fibration. We show that, for an almost contact metric submersion of type I, if the total space is a  $C_9$ ,  $C_{10}$ ,  $C_{11}$  or a  $C_{12}$ -manifold, then the fibres are Kählerian while the base space inherits the structure of the total space.

Section 4 is concerned with the geometry of the fibres. We examine the minimality and superminimality of the fibres and establish their interrelations. It is shown that the fibres of an almost contact metric submersion of type I, whose total space is one of the above manifolds are minimal. After proving that the fibres of a type I almost contact metric submersion whose total space is a  $C_{11}$  or a  $C_{12}$ -manifold are superminimal, it is shown that in the case of a type II submersion, the fibres cannot be superminimal.

In section 5, we give some examples of almost contact metric submersions, using the projections of a product manifold.

## 2. Preliminaries

An almost Hermitian manifold is an even dimensional Riemannian manifold  $(M, g)$  endowed with a tensor field  $J$  of type  $(1, 1)$  satisfying the following two conditions:

- (i)  $J^2D = -D$ , and
- (ii)  $g(JD, JE) = g(D, E)$ , for all  $D, E \in \chi(M)$ .

We shall denote by  $\Omega$  the fundamental two-form defined by  $\Omega(D, E) = g(D, JE)$ .

From the classification of almost Hermitian structures, obtained by Gray and Hervella [3], we shall be interested in the following particular classes: (a) the Kähler manifolds, and (b) the  $W_3$ -manifold, defined by the conditions

$$(\nabla_D\Omega)(E, G) - (\nabla_{JD}\Omega)(JE, G) = 0 \text{ and } \delta\Omega = 0,$$

where  $\delta$  is the codifferential associated to  $g$ :  $\delta = -\sum E_i \lrcorner \nabla_{E_i}$  for an orthonormal basis  $\{E_i\}$ .

An almost contact structure on a odd-dimensional differentiable manifold,  $M$ , is a triple  $(\varphi, \xi, \eta)$  where:

- (i)  $\xi$  is a vector field (called “characteristic”, or Reeb field),
- (ii)  $\eta$  is a differential 1-form such that  $\eta(\xi) = 1$ , and
- (iii)  $\varphi$  is a tensor field of type  $(1, 1)$  satisfying

$$\varphi^2D = -D + \eta(D)\xi, \quad \text{for all } D \in \chi(M).$$

If in addition,  $M$  admits a Riemannian metric  $g$  such that

$$g(\varphi D, \varphi E) = g(D, E) - \eta(D)\eta(E),$$

then  $g$  is called a compatible metric. In this case,  $(M, g, \varphi, \xi, \eta)$  is an almost contact metric manifold and  $\eta$  is the metric dual of  $\xi$ .

As in the case of almost Hermitian manifolds, the fundamental 2-form,  $\phi$ , of an almost contact metric manifold is defined by  $\phi(D, E) = g(D, \varphi E)$ .

We now recall the defining relations of the Chinea-Gonzalez manifolds. An almost contact metric manifold is said to be:

- $C_7$  if  $(\nabla_D\phi)(E, G) = \eta(G)(\nabla_E\eta)\varphi D + \eta(E)(\nabla_{\varphi D}\eta)E$ , and  $\delta\phi = 0$ ;
- $C_8$  if  $(\nabla_D\phi)(E, G) = -\eta(G)(\nabla_E\eta)\varphi D + \eta(E)(\nabla_{\varphi D}\eta)G$ , and  $\delta\eta = 0$ ;
- $C_9$  if  $(\nabla_D\phi)(E, G) = \eta(G)(\nabla_E\eta)\varphi D - \eta(E)(\nabla_{\varphi D}\eta)G$ ;

- $C_{10}$  if  $(\nabla_D\phi)(E, G) = -\eta(G)(\nabla_E\eta)\varphi D - \eta(E)(\nabla_{\varphi D}\eta)G$ ;
- $C_{11}$  if  $(\nabla_D\phi)(E, G) = -\eta(D)(\nabla_\xi\phi)(\varphi E, \varphi G)$ ;
- $C_{12}$  if  $(\nabla_D\phi)(E, G) = \eta(D)\eta(G)(\nabla_\xi\eta)\varphi E - \eta(D)\eta(E)(\nabla_\xi\eta)\varphi G$ .

Observe that except  $C_{11}$ , the formulae defining the other classes contain  $\nabla\eta$ . This means that if  $\eta$  is parallel, or equivalently,  $\xi$ , is parallel, then the right hand side vanishes.

### 3. Almost contact metric submersions

A Riemannian submersion is, [4], a surjective mapping  $\pi : M \rightarrow B$  between two Riemannian manifolds such that

- (i)  $\pi$  is of maximal rank;
- (ii)  $\pi_*/(\ker\pi_*)^\perp$  is an isometry.

The tangent bundle  $T(M)$ , of the total space  $M$ , admits an orthogonal decomposition

$$T(M) = V(M) \oplus H(M),$$

where  $V(M)$  and  $H(M)$  denote respectively the vertical and horizontal distributions. We denote by  $\mathcal{V}$  and  $\mathcal{H}$  the vertical and horizontal projections respectively. A vector field  $X$  of the horizontal distribution is called a basic vector field if it is  $\pi$ -related to a vector field  $X_*$  of the base  $B$  ( $\pi_*X = X_*$ ).

On the base space, tensors and other objects will be denoted by a prime ' while those tangent to the fibres will be specified by a carret  $\hat{\cdot}$ . Herein, vector fields tangent to the fibres will be denoted by  $U, V$  and  $W$ .

Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  and  $(M'^{2m'+1}, g', \varphi', \xi', \eta')$  be two almost contact metric manifolds. By an almost contact metric submersion of type I, in the sense of Watson (see [8]), one understands a Riemannian submersion

$$\pi : M^{2m+1} \rightarrow M'^{2m'+1}$$

satisfying

- (i)  $\pi_*\varphi = \varphi'\pi_*$ ,
- (ii)  $\pi_*\xi = \xi'$ .

It is the natural analogue of a holomorphic submersion.

When the base space is an almost Hermitian manifold,  $(B^{2m'}, g', J')$ , the Riemannian submersion  $\pi : M^{2m+1} \rightarrow B^{2m'}$  is called an almost contact metric submersion of type II, if  $\pi_*\varphi = J'\pi_*$  (see [8]); again this is a generalization of the holomorphicity. Such maps are also called  $(\varphi, J')$ -holomorphic).

We recall some of the fundamental properties of these submersions.

**Proposition 3.1.** (see [8]) *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. Then*

- (i)  $\pi^*\phi' = \phi$ ;
- (ii)  $\pi^*\eta' = \eta$ ;
- (iii) *the horizontal and vertical distributions are  $\varphi$ -invariant;*
- (iv)  $\eta(U) = 0$  for all  $U \in V(M)$ ;
- (v)  $\mathcal{H}(\nabla_X\varphi)Y$  *is the basic vector field associated to  $(\nabla'_{X_*}\varphi')Y_*$  if  $X$  and  $Y$  are basic.*

**Proposition 3.2.** (see [8]) *Let  $\pi : M^{2m+1} \rightarrow B^{2m'}$  be an almost contact metric submersion of type II. Then*

- (i)  $\pi^*\Omega' = \phi$ ;
- (ii) *the horizontal and vertical distributions are  $\varphi$ -invariant;*
- (iii)  $\eta(X) = 0$  for all  $X \in H(M)$ ;
- (iv)  $\mathcal{H}(\nabla_X\varphi)Y$  *is the basic vector field associated to  $(\nabla'_{X_*}J')Y_*$  if  $X$  and  $Y$  are basic.*

Now, from a given structure of the total space we want to determine the corresponding structure on the base space and on the fibres.

**Proposition 3.3.** *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is a  $C_9$ ,  $C_{10}$ ,  $C_{11}$  or a  $C_{12}$ -manifold, then the fibres are Kählerian while the base space inherits the structure of the total space.*

**Proof.** Consider the case of a  $C_9$ -manifold. Let  $U$ ,  $V$  and  $W$  be tangent to the fibres. Then, we have  $(\nabla_U\phi)(V, W) = 0$  because  $\eta$  vanishes on the vertical vector fields according to Proposition 3.1 (iv).

As  $(\hat{\nabla}_U\hat{\phi})(V, W) = g(W, (\hat{\nabla}_U\hat{\phi})V)$ , the relation  $g(W, (\hat{\nabla}_U\hat{\phi})V) = 0$  implies  $(\hat{\nabla}_U\hat{\phi})V = 0$ . On the other hand, the fibres of an almost contact metric submersion of type I are almost Hermitian manifolds, and hence  $(\hat{\nabla}_U\hat{\phi})V = 0$  is equivalent to  $(\hat{\nabla}_UJ)V = 0$ , and thus the fibres are Kähler. The remaining cases are similarly proven.

Concerning the structure of the base space, let  $X$ ,  $Y$  and  $Z$  be basic vector fields. We have to show that

$$(\nabla'_{X_*}\phi')(Y_*, Z_*) = \eta'(Z_*)(\nabla'_{Y_*}\eta')\varphi'X_* - \eta'(Y_*)(\nabla'_{\varphi'X_*}\eta')Z_*.$$

Note that  $\mathcal{H}(\nabla_X\varphi)Y$  is basic associated to  $(\nabla'_{X_*}\varphi')Y_*$ . Since  $\pi_*\xi = \xi'$ , it is clear that  $\xi$  is basic so that  $\mathcal{H}(\nabla_X\xi)$  is basic associated to  $\nabla'_{X_*}\xi'$ . Therefore,  $g'(Y_*, \nabla'_{X_*}\xi')$  corresponds to  $(\nabla'_{X_*}\eta')Y_*$  because  $\pi^*\eta' = \eta$ . It can be shown that  $\mathcal{H}(\nabla_Y\eta)\varphi X$  is basic associated to  $(\nabla'_{Y_*}\eta')\varphi'X_*$  and  $\mathcal{H}(\nabla_{\varphi X}\eta)Z$  is basic associated to  $(\nabla'_{\varphi'X_*}\eta')Z_*$ . Hence, the statement is proved.  $\square$

The analogous of the preceding Proposition 3.3 is the following

**Proposition 3.4.** *Let  $\pi : M^{2m+1} \rightarrow B^{2m'}$  be an almost contact metric submersion of type II. If the total space is a  $C_9$ ,  $C_{10}$ ,  $C_{11}$  or a  $C_{12}$ -manifold, then the fibres inherit the structure of the total space while the base space is Kählerian.*

Recall that, in [4], O'Neill has defined two configuration tensors,  $T$  and  $A$ , by setting

$$T_D E = \mathcal{H}\nabla_{\mathcal{V}D}\mathcal{V}E + \mathcal{V}\nabla_{\mathcal{V}D}\mathcal{H}E, \quad A_D E = \mathcal{V}\nabla_{\mathcal{H}D}\mathcal{H}E + \mathcal{H}\nabla_{\mathcal{H}D}\mathcal{V}E.$$

These tensors play an important role in the study of the fibres and the horizontal distribution respectively. Using the tensor  $A$ , Chinea has defined (see [1]) an associated tensor  $A^*$  on horizontal vector fields in the following way

$$A^*(X, Y) = A_X \varphi Y - A_{\varphi X} Y,$$

and has established the following structure equations:

$$\delta\phi(U) = \delta\hat{\phi}(U) + \frac{1}{2}g(\text{tr}A^*, U), \quad (3.1)$$

$$\delta\phi(X) = \delta\phi'(X_*) + g(H, \varphi X), \quad (3.2)$$

$$\delta\eta = \delta\eta' \circ \pi - g(H, \xi), \quad (3.3)$$

where  $H$  denotes the mean curvature vector field of the fibres while  $\text{tr}A^*$  is the trace of  $A^*$ .

The above equations lead to the following

**Theorem 3.1.** (see [5, Theorem 5]) *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. If among the defining relations of the total space there is the codifferential  $\delta\phi$  or  $\delta\eta$ , then the base space inherits the structure of the total space if and only if the fibres are minimal.*

As a consequence of this Theorem 3.1, one has the following

**Proposition 3.5.** *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is a  $C_7$  or a  $C_8$ -manifold, then the base space inherits the structure of the total space if and only if the fibres are minimal.*

**Proposition 3.6.** *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is a  $C_7$ -manifold, then the fibres are  $W_3$ -manifolds if and only if  $\text{tr}A^* = 0$ .*

**Proof.** Let  $U$  be a vertical vector field tangent to the fibres. Using equation (3.1) of China, we have

$$0 = \delta\hat{\phi}(U) + \frac{1}{2}g(\text{tr}A^*, U).$$

Thus,  $\delta\hat{\phi}(U) = 0$  if, and only if  $\text{tr}A^* = 0$ .

On the other hand, since  $\eta$  vanishes on the vertical vector fields, it is clear that

$$(\hat{\nabla}_U\hat{\phi})(V, W) - (\hat{\nabla}_{\hat{\phi}U}\hat{\phi})(\hat{\phi}V, W) = 0.$$

We then conclude that the fibres are  $W_3$ -manifolds if and only if it holds  $\text{tr}A^* = 0$ .  $\square$

We now examine some properties of the configuration tensors.

**Lemma 3.1.** *Let  $\pi : M^{2m+1} \rightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is a  $C_7, C_8, C_9, C_{10}, C_{11}$  or a  $C_{12}$ -manifold, then  $T_U\varphi V = \varphi T_U V$ .*

**Proof.** On the vertical vector fields, the manifolds under consideration satisfy the relation

$$(\nabla_U\phi)(V, W) = 0.$$

This relation leads to  $g(W, (\nabla_U\phi)V) = 0$  for every  $W$ , from which we deduce  $(\nabla_U\phi)V = 0$ . The horizontal projection of this gives rise to  $T_U\varphi V = \varphi T_U V$ .  $\square$

**Lemma 3.2.** *Let  $\pi : M^{2m+1} \rightarrow B^{2m'}$  be an almost contact metric submersion of type II. If the total space is a  $C_7, C_8, C_9, C_{10}, C_{11}$  or a  $C_{12}$ -manifold, then  $A_X\varphi Y = \varphi A_X Y$ .*

**Proof.** The vanishing of  $\eta$  on the horizontal vector fields leads to the condition  $(\nabla_X\phi)(Y, Z) = 0$ . As in Proposition 3.1, we deduce that  $(\nabla_X\phi)Y = 0$ . The vertical projection of this relation gives the required statement.  $\square$

#### 4. The geometry of the fibres

**Proposition 4.1.** *Let  $\pi : M^{2m+1} \rightarrow M^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is a  $C_9, C_{10}, C_{11}$  or a  $C_{12}$ -manifold, then the fibres are minimal.*

**Proof.** From [5, Proposition 4], it is known that if the configuration tensor  $T$  is  $\varphi$ -linear in the second variable on the vertical distribution, then the fibres are minimal. This is just the case observed in Lemma 3.1.  $\square$

We now examine the superminimality of the fibres. Let  $(M^{2m+1}, g, \varphi, \xi, \eta)$  be an almost contact metric manifold and  $\bar{M}$  a  $\varphi$ -invariant submanifold of

$M$ . If  $(\nabla_V \varphi) = 0$  for all  $V$  tangent to  $\bar{M}$ , then  $\bar{M}$  is said to be superminimal. In order to verify superminimality of the almost Hermitian fibres,  $(\hat{M}, \hat{J}, \hat{g})$ , recall from [6] that there are four components of  $g((\nabla_V \varphi)E, F)$  to be considered on the total space,  $M$  (and which all have to vanish). We easily find:

$$\begin{aligned} \text{SM-1)} \quad & g((\nabla_V \varphi)U, W) = g(\hat{\nabla}_V(\hat{J}U) - \hat{J}\hat{\nabla}_V U, W), \\ \text{SM-2)} \quad & g((\nabla_V \varphi)U, X) = g(T_V(\varphi U) - \varphi(T_V U), X), \\ \text{SM-3)} \quad & g((\nabla_V \varphi)X, U) = -g((\nabla_V \varphi)U, X), \\ \text{SM-4)} \quad & g((\nabla_V \varphi)X, Y) = -g(A_{\varphi X} Y + A_X(\varphi Y), V). \end{aligned}$$

Let  $\pi : (M, \varphi, \xi, \eta, g) \rightarrow (M', J', g')$  be an almost contact metric submersion of type II. In order to verify superminimality of the almost contact metric fibres,  $(\hat{M}, \hat{\varphi}, \hat{\xi}, \hat{\eta}, \hat{g})$  there are four components of  $g((\nabla_V \varphi)E, F)$  to be considered on the total space,  $M$  (and which all have to vanish). We easily find:

$$\begin{aligned} \text{SM-5)} \quad & g((\nabla_V \varphi)U, W) = g(\hat{\nabla}_V(\hat{\varphi}U) - \hat{\varphi}\hat{\nabla}_V U, W), \\ \text{SM-6)} \quad & g((\nabla_V \varphi)U, X) = g(T_V(\varphi U) - \varphi(T_V U), X), \\ \text{SM-7)} \quad & g((\nabla_V \varphi)X, U) = -g((\nabla_V \varphi)U, X), \\ \text{SM-8)} \quad & g((\nabla_V \varphi)X, Y) = -g(A_{\varphi X} Y + A_X(\varphi Y), V). \end{aligned}$$

We note that SM-1) implies that if the fibres of a type I almost contact metric submersion are superminimal, then they are Kähler.

For an almost contact metric submersion of type I, equation (3.3) of China shows that the transference of the condition  $\delta\eta = 0$  from the total space to the base space is connected to the minimality or the superminimality of the fibres.

In fact, one has the following result.

**Theorem 4.1.** *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I whose total and base spaces have codifferentials,  $\delta\eta$  and  $\delta'\eta'$ , null. If the fibres are superminimal, then they are minimal.*

**Proof.** Let  $\{E_1, \dots, E_{m'_m}, \varphi E_1, \dots, \varphi E_{m-m'}, F_1, \dots, F_{m'}, \varphi F_1, \dots, \varphi F_{m'}, \xi\}$  be a basis for the local vector fields on  $M$  with the  $\{E_i, \varphi E_i\}$  vertical and the  $\{F_j, \varphi F_j\} \cup \{\xi\}$  horizontal. The mean curvature vector field of the fibres is given by

$$H = \sum_{i=1}^{m-m'} \{T_{E_i}(E_i) + T_{\varphi E_i} \varphi E_i\}.$$

The nullity of calculation SM-2) for the superminimal fibres implies that all of the components of  $H$  with respect to the horizontal part of the local basis vanish, except probably  $g(H, \xi)\xi$ . Using equation (3.3), we have

$$0 = \delta\eta = -g(H, \xi) + \pi^*(\delta'\eta').$$

Therefore,  $H = 0$ . □

The above Theorem 4.1 applies in the case of  $\delta\phi = 0 = \delta'\phi'$ .

**Proposition 4.2.** *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. If the total space is a  $C_{11}$  or a  $C_{12}$ -manifold, then the fibres are superminimal.*

**Proof.** Consider the case of a  $C_{11}$ -manifold. Let  $V$  be a vertical vector field tangent to the fibres. By the vanishing of the contact 1-form,  $\eta$ , on the vertical vector fields, in virtue of Proposition 3.1, we have

$$(\nabla_V\phi)(E, F) = 0,$$

from which  $(\nabla_V\varphi)F = 0$  follows. Thus, the fibres are superminimal.

In the same way, suppose that the total space is a  $C_{12}$ -manifold. The vanishing of  $\eta$ , on the vertical vector fields leads to the same conclusion. □

When studying the superminimality of the fibres of an almost Hermitian submersion, B. Watson (see [9]) introduced a criterion in terms of  $(\nabla_X J)U$  which plays an important role in the transference of structure from the base to the total space. We have adapted it to the contact geometry by replacing  $J$  by  $\varphi$ .

**Proposition 4.3.** *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I. Assume that the base space is a  $C_7$ ,  $C_8$ ,  $C_9$ , or a  $C_{10}$ -manifold and the fibres are superminimal. If  $\mathcal{V}(\nabla_X\varphi)U = 0$ , then the total space inherits the structure of the base space.*

**Proof.** Let us consider the case where the base space is a  $C_7$ -manifold. In order to prove that the total space is a  $C_7$ -manifold, there are six expressions that must vanish.

$$C_7-1) \quad (\nabla_U\phi)(V, W) - \eta(W)(\nabla_V\eta)\varphi U - \eta(V)(\nabla_{\varphi U}\eta)W;$$

$$C_7-2) \quad (\nabla_U\phi)(V, X) - \eta(X)(\nabla_V\eta)\varphi U - \eta(V)(\nabla_{\varphi U}\eta)X;$$

$$C_7-3) \quad (\nabla_U\phi)(Y, X) - \eta(X)(\nabla_Y\eta)\varphi U - \eta(Y)(\nabla_{\varphi U}\eta)X;$$

$$C_7-4) \quad (\nabla_X\phi)(U, V) - \eta(V)(\nabla_U\eta)\varphi X - \eta(U)(\nabla_{\varphi X}\eta)V;$$

$$C_7-5) \quad (\nabla_X\phi)(Y, V) - \eta(V)(\nabla_Y\eta)\varphi X - \eta(Y)(\nabla_{\varphi X}\eta)V;$$

$$C_7-6) \quad (\nabla_X\phi)(Y, Z) - \eta(Z)(\nabla_Y\eta)\varphi X - \eta(Y)(\nabla_{\varphi X}\eta)Z.$$

Since the fibres are superminimal, the first two expressions vanish.

Regarding  $C_7-3)$ , it is known that  $(\nabla_U\phi)(Y, X) = g(Y, (\nabla_U\varphi)X)$  from which  $(\nabla_U\varphi)X = 0$ , because of the superminimality of the fibres. Note that

$$(\nabla_Y\eta)\varphi U = (\nabla_Y\phi)(\xi, \varphi^2U) = g(\xi, (\nabla_Y\varphi)\varphi^2U).$$

Applying the condition  $\mathcal{V}(\nabla_X\varphi)U = 0$  we deduce that  $(\nabla_Y\eta)\varphi U = 0$ .

In  $C_7$ -4), we have to examine only  $(\nabla_X\phi)(U, V)$ ; the others terms vanish, because  $\eta$  vanishes on vertical vector fields. By using the condition  $\mathcal{V}(\nabla_X\phi)V = 0$ , one gets  $(\nabla_X\phi)(U, V) = g(U, (\nabla_X\phi)V) = 0$ .

Concerning  $C_7$ -5), we have to examine  $(\nabla_X\phi)(Y, V)$  and  $\eta(Y)(\nabla_{\varphi X}\eta)V$ . Recall that  $(\nabla_X\phi)(Y, V) = g(Y, (\nabla_X\phi)V)$ . Since  $\mathcal{V}(\nabla_X\phi)V = 0$ , we get  $(\nabla_X\phi)(Y, V) = 0$ . Considering  $(\nabla_{\varphi X}\eta)V$ , we have

$$(\nabla_{\varphi X}\eta)V = (\nabla_{\varphi X}\phi)(\xi, V) = g(\xi, (\nabla_{\varphi X}\phi)V).$$

Using the condition  $\mathcal{V}(\nabla_X\phi)U = 0$ , we conclude that  $C_7$ -5) vanishes.

The last expression  $C_7$ -6) vanishes on basic horizontal vector fields because the projected tensors by the submersion down to the base space vanish. Therefore  $(M, g, \varphi, \xi, \eta)$  is a  $C_7$ -manifold. Other calculations are treated as in the case of  $C_7$ -manifold.  $\square$

The significance of this criterion is that, when the fibres are superminimal, it ensures the transference of the structure from the base to the total space. For instance, by Proposition 4.2, it is proven that submersions whose total space is a  $C_{11}$  or a  $C_{12}$ -manifold have superminimal fibres; but it can be shown that, in this case, the structure of the base space does not transfers to the total space unless this criterion is fulfilled. We can state the following result.

**Proposition 4.4.** *Let  $\pi : M^{2m+1} \rightarrow M'^{2m'+1}$  be an almost contact metric submersion of type I with superminimal fibres. If the base space is a  $C_{11}$  or a  $C_{12}$ -manifold, then these structures do not transfer to the total space unless  $\mathcal{V}(\nabla_X\phi)U = 0$ .*

**Proof.** Let us consider the case where the base space is a  $C_{11}$ -manifold. As in the case of the preceding Proposition 4.2, the following six expressions must vanish

$$C_{11-1) \quad (\nabla_U\phi)(V, W) + \eta(U)(\nabla_\xi\phi)(\varphi V, \varphi W);$$

$$C_{11-2) \quad (\nabla_U\phi)(V, X) + \eta(U)(\nabla_\xi\phi)(\varphi V, \varphi X);$$

$$C_{11-3) \quad (\nabla_U\phi)(Y, X) + \eta(U)(\nabla_\xi\phi)(\varphi Y, \varphi X);$$

$$C_{11-4) \quad (\nabla_X\phi)(U, V) + \eta(X)(\nabla_\xi\phi)(\varphi U, \varphi V);$$

$$C_{11-5) \quad (\nabla_X\phi)(Y, V) + \eta(X)(\nabla_\xi\phi)(\varphi Y, \varphi V);$$

$$C_{11-6) \quad (\nabla_X\phi)(Y, Z) + \eta(X)(\nabla_\xi\phi)(\varphi Y, \varphi Z).$$

Considering expressions  $C_{11-4)$  and  $C_{11-5)$ , we encounter  $(\nabla_X\phi)V$  and  $(\nabla_\xi\phi)\varphi V$ , which must vanish in order to conclude that the total space is a  $C_{11}$ -manifold.  $\square$

## 5. Some examples

Product of manifolds provide trivial examples of submersions.

Let  $(M', g', J')$  be a  $2m'$ -dimensional almost Hermitian manifold and let further  $(M, g, \varphi, \xi, \eta)$  be an almost contact metric manifold of dimension  $2m+1$ . Let  $\tilde{M} = M' \times M$  and set  $n = m' + m$  so that the  $\dim(\tilde{M}) = 2n+1$ . On the product  $\tilde{M}$ , one defines an almost contact metric structure  $(\tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$  by setting:

$$\tilde{\varphi}(D', D) = (J'D', \varphi D), \quad (5.1)$$

$$\tilde{\eta}(D', D) = \frac{m}{n}\eta(D), \quad (5.2)$$

$$\tilde{g}((D', D), (E', E)) = g'(D', E') + \frac{n^2}{m^2}g(D, E), \quad (5.3)$$

$$\tilde{\xi} = \frac{n}{m}(0, \xi). \quad (5.4)$$

Examples are given in [2]. Similarly to [7, Proposition 4.1], we have:

**Proposition 5.1.** *Let  $(M', g', J')$  be an almost Hermitian manifold and let  $(M, g, \varphi, \xi, \eta)$  be an almost contact metric manifold. If  $(M' \times M, \tilde{g}, \tilde{\varphi}, \tilde{\xi}, \tilde{\eta})$  is an almost contact metric manifold obtained as above, then it is:*

- (i)  $C_9$  if and only if  $M'$  is Kähler and  $M$  is  $C_9$ ;
- (ii)  $C_{10}$  if and only if  $M'$  is Kähler and  $M$  is  $C_{10}$ ;
- (iii)  $C_{11}$  if and only if  $M'$  is Kähler and  $M$  is  $C_{11}$ ;
- (iv)  $C_{12}$  if and only if  $M'$  is Kähler and  $M$  is  $C_{12}$ ;
- (v)  $C_7$  if and only if  $M'$  is a  $W_3$ -manifold and  $M$  is  $C_7$ .

### Examples of almost contact metric submersions

**Theorem 5.1.** *Let  $(M'^{2m'}, g', J')$  be an almost Hermitian manifold and  $(M^{2m+1}, g, \varphi, \xi, \eta)$  an almost contact metric one. If  $\tilde{M} = M' \times M$  is the product almost contact metric manifold defined above, then:*

- (i) the projection  $p : M' \times M \rightarrow M$  is an almost contact metric submersion of type I;
- (ii) the projection  $q : M' \times M \rightarrow M'$  is an almost contact metric submersion of type II.

**Proof.** It is known that these two projections are Riemannian submersions. We have to show that they are  $(\tilde{\varphi}, \varphi)$ -holomorphic for the first type and  $(\tilde{\varphi}, J')$ -holomorphic for the second type.

Since  $\tilde{\varphi}(D', D) = (J'D', \varphi D)$ , then

$$p_*\tilde{\varphi}(D', D) = p_*(J'D', \varphi D) = \varphi D = \varphi p_*(D', D),$$

from which  $p_*\tilde{\varphi} = \varphi p_*$ . On the other hand,  $p_*\tilde{\xi} = p_*(0, \xi) = \xi$ , which achieves the proof of (i).

Likewise,

$$q_*\tilde{\varphi}(D', D) = q_*(J'D', \varphi D) = J'D' = J'q_*(D', D),$$

which shows that  $q_*\tilde{\varphi} = J'q_*$  and establishes (ii).

We note that, in the first case, the fibres are isomorphic with  $M'$  and in the second case with  $M$ . Since  $\tilde{\varphi}(D', 0) = (J'D', 0)$  and  $\tilde{\varphi}(0, D) = (0, \varphi D)$ , then  $M'$  and  $M$  are invariant submanifolds of  $\tilde{M} = M' \times M$ .  $\square$

More concretely, Chinea and Gonzalez have shown that  $M^3 = \{(x, y, z) \in \mathbb{R}^3 \mid x > 0\}$  can be endowed with a  $C_{12}$ -structure. Using the Kählerian structure of  $\mathbb{R}^2$ , it is clear that the product  $\mathbb{R}^2 \times M^3$  is a  $C_{12}$ -manifold as proved in Proposition 5.1.

According to the preceding Theorem 5.1, we have:

(i) The projection  $p : \mathbb{R}^2 \times M^3 \rightarrow M^3$  is an almost contact metric submersion of type I whose fibres are the Kählerian manifold  $\mathbb{R}^2$  where the total and the base space are  $C_{12}$ -manifolds.

(ii) The projection  $q : \mathbb{R}^2 \times M^3 \rightarrow \mathbb{R}^2$  is an almost contact metric submersion of type II whose total space is  $C_{12}$ -manifold, the base space is the Kählerian manifold  $\mathbb{R}^2$ , while the fibres are  $C_{12}$ -manifold,  $M^3$ .

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