

## On the stability of an equilibrium problem

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**Abstract** - This paper investigates the upper and lower semicontinuity in the sense of Berge of the solutions set and  $\varepsilon$ -solutions set of an equilibrium problem. Also we study the continuity of the solutions set of a nested optimization problem having equilibrium problem constraints.

**Key words and phrases** : quasivariational inequality, equilibrium problem, semicontinuity, closed map.

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### 1. Introduction and preliminaries

Optimization theory plays an important role in many fields, as mechanics, physics, economics and engineering sciences (see [2, 3, 21]). After the 60's, variational inequalities was intensively studied by many authors (see [2-13, 18-22]) as a branch of optimization. It is well-known that variational inequalities are closely related to optimization problems, who were also investigated by authors like Preda (see [14-17]).

An important research direction in optimization theory is the stability of the solutions set for variational inequalities and equilibrium problems with perturbing parameters. Stability refers to semicontinuity, continuity, Lipschitz continuity or some kinds of differentiability of the optimal solutions set of variational inequalities or optimization problems. Many authors have studied these aspects of stability separately for different types of variational inequalities (see [5-9, 11,13, 19, 20, 22]). Among them, Lalitha and Bathia studied in [11] the continuity of a parametric quasivariational inequality of the Minty type. Also, Stanciu analysed in [19] the continuity of the solutions set of a Minty type invex quasivariational inequality.

This work extends the papers of Lalitha and Bathia (see [11]) and Stanciu (see [19]) to the case of an equilibrium problem. Thus, further on we give the general framework in which we will do the study and the equilibrium problem considered here; then we recall the basic definitions and results that will be used. Making certain assumptions on the set  $A$  and on the applications  $K$ ,  $T$  and  $\psi$ , in Section 2 we will give sufficient conditions for the upper semicontinuity of the solutions set of the problem  $(\psi MVI(x_0))$ .

Section 3 focuses on the lower semicontinuity of the solutions set of the same problem. In Section 4 we analyse the stability of the  $\varepsilon$ -solutions set of the problem  $(\psi MVI(x_0))$ . In the last section the study is extended to the upper and lower semicontinuity of a nested optimization problem having equilibrium problem constraints.

Throughout the paper, we consider the following equilibrium problem, corresponding to a parameter  $x_0 \in X$ :

$$(\psi MVI(x_0)) \text{ Find } u_0 \in K(x_0, u_0) \cap A \text{ such that} \\ \psi(t, u_0, v) \leq 0, \forall v \in K(x_0, u_0), \forall t \in T(x_0, v),$$

where  $X \subset \mathbb{R}^n$  is a nonempty, closed set,  $A \subset Y = \mathbb{R}^m$  is a nonempty, closed and convex set,  $K : X \times Y \rightarrow 2^Y$  is a closed application,  $T : X \times Y \rightarrow 2^Y$  and  $\psi : Y \times Y \times Y \rightarrow Y$ . Suppose that  $domK = domT = X \times Y$ , where  $domK = \{(x, y) \in X \times Y \mid K(x, y) \neq \emptyset\}$ .

We denote the solutions set of  $(\psi MVI(x_0))$  by  $M_\psi(x_0)$ , that is

$$M_\psi(x_0) = \{u_0 \in K(x_0, u_0) \cap A \mid \\ \psi(t, u_0, v) \leq 0, \forall v \in K(x_0, u_0), \forall t \in T(x_0, v)\}.$$

For  $\psi(t, u, v) = \langle t, \eta(u, v) \rangle$ , where  $\eta : Y \times Y \rightarrow Y$  we obtain the quasivariational inequality considered by Stanciu in [19]. For  $\psi(t, u, v) = \langle t, u - v \rangle$  we obtain the inequality considered by Lalitha and Bhatia in [11].

Now, let us recall some basic definitions and their properties.

For each  $\varepsilon > 0$  and  $x_0 \in Y$ , denote by  $B(x_0, \varepsilon)$  the open ball with center in  $x_0$  and radius  $\varepsilon$ , that is  $B(x_0, \varepsilon) := \{x \in Y \mid \|x_0 - x\| < \varepsilon\}$ , and with  $U(A, \varepsilon)$  an open  $\varepsilon$ -neighborhood of a subset of  $A \subseteq Y$  defined by  $U(A, \varepsilon) := \{x \in Y \mid \text{there exists } a \in A \text{ such that } \|a - x\| < \varepsilon\}$ .

Now let  $F : X \rightarrow 2^Y$  be a set-valued map with  $domF = X$ , where  $X$  is a nonempty and closed subset of  $\mathbb{R}^n$  and  $Y = \mathbb{R}^m$ .

**Definition 1.1.** (see [10]) *The application  $F$  is said to be:*

(i) *Upper semicontinuous in the sense of Berge (in short, B-usc) at  $x_0 \in X$  if for every open set  $N$  satisfying  $F(x_0) \subset N$ , there exists a  $\delta > 0$ , such that for every  $x \in B(x_0, \delta)$ ,  $F(x) \subset N$ .*

(ii) *Lower semicontinuous in the sense of Berge (in short, B-lsc) at  $x_0 \in X$  if for every open set  $N$  satisfying  $F(x_0) \cap N \neq \emptyset$ , there exists a  $\delta > 0$ , such that for every  $x \in B(x_0, \delta)$ ,  $F(x) \cap N \neq \emptyset$ .*

The application  $F$  is said to be B-lsc (respectively B-usc) on  $X$  if  $F$  is B-lsc (respectively B-usc) at each point  $x_0 \in X$ .  $F$  is said to be B-continuous on  $X$  if it is both B-lsc and B-usc on  $X$ .

Aubin and Ekeland gave in [1] the following equivalent definition for a B-lower semicontinuous function:

**Remark 1.1.** (see [1])  $F$  is said to be B-lsc at  $x_0 \in X$  if and only if for any sequence  $\{x_n\} \subset X$  converging to  $x_0$  and for any  $y_0 \in F(x_0)$  there exists a sequence  $\{y_n\} \subset F(x_n)$  converging to  $y_0$ .

**Definition 1.2.** (see [1])  $F$  is said to be closed at  $x_0 \in X$ , if for each of the sequences  $\{x_n\} \subset X$  converging to  $x_0$  and  $\{y_n\} \subset Y$  converging to  $y_0$  such that  $y_n \in F(x_n)$ , we have  $y_0 \in F(x_0)$ .  $F$  is said to be closed on  $X$  if it is closed at each  $x_0 \in X$ .

**Remark 1.2.** (see [1]) It is well-known that, if  $F$  is B-usc at  $x_0 \in X$  and  $F(x_0)$  is closed, then  $F$  is closed at  $x_0$ .

**Definition 1.3.** Let  $F : X \rightarrow 2^Y$  be a set-valued map with  $\text{dom}F = X$  and  $\psi : Y \times Y \times Y \rightarrow Y$ . We say that the application  $F$  is:

- (i)  $\psi$  -pseudomonotone on  $X$  iff for any  $x, x_0 \in X$ ,  
 $\psi(\xi, x, x_0) \geq 0$ , for some  $\xi \in F(x_0) \implies \psi(\mu, x, x_0) \geq 0$ , for any  $\mu \in F(x)$ ;
- (ii)  $\psi$  -quasimonotone on  $X$  iff for any  $x, x_0 \in X, x \neq x_0$ ,  
 $\psi(\xi, x, x_0) > 0$ , for some  $\xi \in F(x_0) \implies \psi(\mu, x, x_0) > 0$ , for any  $\mu \in F(x)$ .

## 2. Upper semicontinuity of the solutions set

In this section we derive sufficient conditions that ensure the upper semicontinuity of the application  $M_\psi : X \rightarrow 2^Y$ , where  $M_\psi(x)$  is the solutions set of problem  $(\psi MVI(x))$ , making certain assumptions on the applications  $T$  and  $K$ .

**Theorem 2.1.** Suppose that for  $x_0 \in X$ , the following conditions are satisfied:

- (i)  $K$  is closed and B-lsc on  $\{x_0\} \times Y$ ;
- (ii)  $T$  is B-lsc on  $\{x_0\} \times Y$ ;
- (iii)  $A$  is a compact subset of  $Y$ ;
- (iv)  $\psi(\cdot, \cdot, \cdot)$  is continuous in all the arguments.

Then  $M_\psi$  is B-usc at  $x_0$ . Moreover,  $M_\psi(x_0)$  is compact and  $M_\psi$  is closed at  $x_0$ .

**Proof.** Suppose, on the contrary, that  $M_\psi$  be not B-usc at  $x_0$ . Then there exists an open set  $N$  containing  $M_\psi(x_0)$ , such that for every sequence  $x_n \rightarrow x_0$ , there exists  $u_n \in M_\psi(x_n)$ ,  $u_n \notin N$ , for every  $n$ . Using the line of [11], is sufficient to show that  $u_0 \in M_\psi(x_0)$ . If  $u_0 \notin M_\psi(x_0)$ , then there exist  $v_0 \in K(x_0, u_0)$  and  $t_0 \in T(x_0, v_0)$ , such that

$$\psi(t_0, u_0, v_0) > 0. \quad (2.1)$$

Since  $K$  is B-lsc at  $(x_0, u_0)$ ,  $v_0 \in K(x_0, u_0)$  and  $(x_n, u_n) \rightarrow (x_0, u_0)$ , there exists  $v_n \in K(x_n, u_n)$ , such that  $v_n \rightarrow v_0$ . Similarly, since  $T$  is B-lsc at  $(x_0, v_0)$ ,  $t_0 \in T(x_0, v_0)$  and  $(x_n, v_n) \rightarrow (x_0, v_0)$ , it follows that there exists  $t_n \in T(x_n, v_n)$ , such that  $t_n \rightarrow t_0$ .

From  $u_n \in M_\psi(x_n)$  we have

$$\psi(t, u_n, v) \leq 0, \quad \forall v \in K(x_n, u_n), \forall t \in T(x_n, v). \quad (2.2)$$

Taking  $v = v_n$  and  $t = t_n$  in (2.2) and taking limits as  $n \rightarrow \infty$ , we have  $\psi(t_0, u_0, v_0) \leq 0$ , which contradicts relation (2.1). Therefore  $M_\psi$  is B-usc at  $x_0$ .

Proceeding like in [11], we can show that  $M_\psi(x_0)$  is a closed set. Moreover, as  $M_\psi(x_0) \subset A$  and  $A$  is compact, it follows that  $M_\psi(x_0)$  is compact. It is well-known that, if  $M_\psi$  is B-usc at  $x_0 \in X$  and  $M_\psi(x_0)$  is closed, then  $M_\psi$  is closed at  $x_0$  (see [1]).  $\square$

**Example 2.1.** Let  $X = [-1, 1]$ ,  $Y = \mathbb{R}$  and  $A = [0, 2]$ . Define  $\psi : Y \times Y \times Y \rightarrow Y$  as  $\psi(t, u, v) = t^2(u^2 - v^2)$  and the set-valued maps  $K : X \times Y \rightarrow 2^Y$  and  $T : X \times Y \rightarrow 2^Y$  as follows

$$K(x, u) = \begin{cases} [u, 2], & \text{if } u \leq 2 \\ [2, u], & \text{if } u > 2 \end{cases}, \quad T(x, u) = \begin{cases} \{0\}, & \text{if } x = 0 \\ [0, 1], & \text{if } x < 0 \\ [0, |x + u|], & \text{if } x > 0 \end{cases}.$$

We have  $M_\psi(x) = [0, 2]$ ,  $\forall x \in X$ . Here there are satisfied all the assumptions of the above theorems. Therefore  $M_\psi$  is B-usc at  $x_0$ ,  $M_\psi(x_0)$  is compact and  $M_\psi$  is closed at  $x_0$ .

The following two examples illustrate the fact that the conditions of closedness on the map  $K$  and of compactness on the set  $A$ , respectively, cannot be relaxed in Theorem 2.1.

**Example 2.2.** Let  $X = [-2, 2]$ ,  $Y = \mathbb{R}$  and  $A = [0, 2]$ . Define  $\psi : Y \times Y \times Y \rightarrow Y$  as  $\psi(t, u, v) = (t^2 + t)(u - v)$  and the set-valued maps  $K : X \times Y \rightarrow 2^Y$  and  $T : X \times Y \rightarrow 2^Y$  as follows

$$K(x, u) = \begin{cases} \{0\}, & \text{if } x = 1 \\ [0, \frac{1}{2}], & \text{if } x \neq 1 \end{cases}, \quad T(x, u) = \begin{cases} \{0\}, & \text{if } u \leq 1 \\ [0, u], & \text{if } u > 1 \end{cases}.$$

For  $x_0 = 1$ ,  $M_\psi(x_0) = \{0\}$  and for  $x \neq x_0$ ,  $M_\psi(x) = [0, \frac{1}{2}]$ . We can see that the maps  $K$  and  $T$  are B-lsc on  $\{x_0\} \times Y$  but  $K$  is not closed on  $\{x_0\} \times Y$ . It can be observed that  $M_\psi$  is neither B-usc at  $x_0$  or closed at  $x_0$ .

**Example 2.3.** Let  $X = [-2, 2]$  and  $Y = A = \mathbb{R}$ . Define  $\psi : Y \times Y \times Y \rightarrow Y$  as  $\psi(t, u, v) = t(u + v + t)$  and the applications  $K : X \times Y \rightarrow 2^Y$  and  $T : X \times Y \rightarrow 2^Y$  as follows

$$K(x, u) = \begin{cases} [u, 0], & \text{if } u \leq 0 \\ [0, u], & \text{if } u > 0 \end{cases}, \quad T(x, u) = \begin{cases} \{0\}, & \text{if } x = 1 \\ [0, 1], & \text{if } x \neq 1 \end{cases}.$$

For  $x_0 = 1$ ,  $M_\psi(x_0) = \mathbb{R}$  and for  $x \neq x_0$ ,  $M_\psi(x) = (-\infty, 0]$ . In this example

the first two conditions and fourth are satisfied, but  $A$  is not compact. It can be observed that  $M_\psi$  is B-usc at  $x_0$  and closed at  $x_0$ , but  $M_\psi(x_0)$  is not compact.

**Theorem 2.2.** *Suppose that for  $x_0 \in X$ , the following conditions are satisfied:*

- (i)  $K$  is closed on  $\{x_0\} \times Y$ ;
- (ii)  $A$  is a compact subset of  $Y$ ;
- (iii)  $\forall u_0 \in K(x_0, u_0) \cap A, \forall (x_n, u_n) \rightarrow (x_0, u_0)$  and

$$\psi(t_0, u_0, v_0) > 0, \text{ for some } v_0 \in K(x_0, u_0), t_0 \in T(x_0, v_0) \quad (2.3)$$

*implies that there exists a positive integer  $n$ , such that  $\psi(t, u_n, v) > 0$ , for some  $v \in K(x_n, u_n), t \in T(x_n, v)$ ;*

- (iv)  $\psi(\cdot, \cdot, \cdot)$  is continuous in all the arguments.

*Then  $M_\psi$  is B-usc at  $x_0$ . Moreover,  $M_\psi(x_0)$  is compact and  $M_\psi$  is closed at  $x_0$ .*

**Proof.** Suppose, on the contrary, that  $M_\psi$  be not B-usc at  $x_0$ . Then there exists an open set  $N$  containing  $M_\psi(x_0)$ , such that for every sequence  $x_n \rightarrow x_0$ , there exists  $u_n \in M_\psi(x_n), u_n \notin N$ , for every  $n$ . As in [11], we arrive to a contradiction and hence  $M_\psi$  is B-usc at  $x_0$ . The fact that  $M_\psi(x_0)$  be compact and  $M_\psi$  be closed at  $x_0$ , follows as in Theorem 2.1.  $\square$

It can be observed that in the above theorem the conditions of B-lower semicontinuity of  $K$  and  $T$  from Theorem 2.1 are replaced with a weaker one. The advantage of assumption (iii) can be illustrated by means of the following example, wherein the solutions set is B-usc and compact at  $x_0$ , even though the map  $T$  is not B-lsc on  $\{x_0\} \times Y$ , where  $x_0 = 0$ .

**Example 2.4.** Let  $X = [-1, 1], Y = \mathbb{R}$  and  $A = [0, 2]$ . Define  $\psi : Y \times Y \times Y \rightarrow Y$  as  $\psi(t, u, v) = u(v - t)$  and the set-valued maps  $K : X \times Y \rightarrow 2^Y$  and  $T : X \times Y \rightarrow 2^Y$  as follows

$$K(x, u) = \begin{cases} \{0, 1\}, & \text{if } u = 0 \\ [0, |u| + 3], & \text{if } u \neq 0 \end{cases}, \quad T(x, u) = \begin{cases} [1, 2], & \text{if } x = 0 \\ \{3\}, & \text{if } x \neq 0 \end{cases}.$$

We have  $M_\psi(x) = \{0\}, \forall x \in X$ .

**Example 2.5.** Let  $X = [-1, 1], Y = \mathbb{R}$  and  $A = [0, 2]$ . Define  $\psi : Y \times Y \times Y \rightarrow Y$  as  $\psi(t, u, v) = u + v - 2t$  and the set-valued maps  $K : X \times Y \rightarrow 2^Y$  and  $T : X \times Y \rightarrow 2^Y$  as follows:  $K(x, u) = \{0, 1\}$ ,

$$T(x, u) = \begin{cases} [0, \frac{1}{2}], & \text{if } x = 0 \\ \{1\}, & \text{if } x \neq 0 \end{cases}.$$

For  $x_0 = 0, M_\psi(x_0) = \{0\}$  and for  $x \neq x_0, M_\psi(x) = \{0, 1\}$ . We can see that for  $x_0 = 0$  and  $(x_n, u_n) = (\frac{1}{n}, \frac{1}{n})$ , condition (iii) of Theorem 2.2 fails to hold and  $M_\psi$  is not B-usc at  $x_0$ . So, wherein the conclusion of Theorem 2.2 fails to hold in the absence of condition (iii).

### 3. Lower semicontinuity of the solutions set

Since continuity implies both upper as well as lower semicontinuity, in this section we establish conditions ensuring the lower semicontinuity of the application  $M_\psi$  defined as in Section 2.

**Theorem 3.1.** *Suppose that for  $x_0 \in X$ , the following conditions are satisfied:*

- (i)  $\bar{K}$  is B-lsc at  $x_0$ , where  $\bar{K}(x) = \{u \in A : u \in K(x, u)\}$ ;
- (ii)  $\forall u_0 \in K(x_0, u_0) \cap A, \forall (x_n, u_n) \rightarrow (x_0, u_0)$  and

$$\psi(t, u_0, v) \leq 0, \forall v \in K(x_0, u_0), \forall t \in T(x_0, v) \quad (3.1)$$

*implies that there exists a positive integer  $n$ , such that  $\psi(t, u_n, v) \leq 0, \forall v \in K(x_n, u_n), \forall t \in T(x_n, v)$ ;*

- (iii)  $\omega(\cdot, \cdot, \cdot)$  is continuous in all the arguments.

*Then  $M_\psi$  is B-lsc at  $x_0$ .*

**Proof.** Suppose, on the contrary, that  $M_\psi$  be not B-lsc at  $x_0$ . From Remark 2.1, there exists a sequence  $\{x_n\}$  in  $X$  converging to  $x_0$  and  $u_0 \in M_\psi(x_0)$ , such that for every sequence  $y_n \in M_\psi(x_n), y_n \not\rightarrow u_0$ . Since  $x_n \rightarrow x_0$  and  $u_0 \in \bar{K}(x_0)$ , from assumption (i) it follows that there exists a sequence  $u_n \in \bar{K}(x_n)$  such that  $u_n \rightarrow u_0$ . It follows that  $u_n \notin M_\psi(x_n)$  and then

$$\psi(t_n, u_n, v_n) > 0, \text{ for some } v_n \in K(x_n, u_n), t_n \in T(x_n, v_n). \quad (3.2)$$

Since  $u_0 \in M_\psi(x_0)$ , it follows that relation (3.1) holds and hence by condition (ii) of the hypothesis, there exists  $n \in \mathbb{N}$ , such that

$$\psi(t, u_n, v) \leq 0, \forall v \in K(x_n, u_n), \forall t \in T(x_n, v),$$

which contradicts (3.2). Therefore  $M_\psi$  is B-lsc at  $x_0$ .  $\square$

Now we give an example where there are satisfied all the conditions of the above theorem and, therefore, the conclusion is valid.

**Example 3.1.** Let  $X = [-2, 2], Y = \mathbb{R}$  and  $A = [0, +\infty)$ . Define  $\psi : Y \times Y \times Y \rightarrow Y$  as  $\psi(t, u, v) = tv - u$  and the set-valued maps  $K : X \times Y \rightarrow 2^Y$  and  $T : X \times Y \rightarrow 2^Y$  as follows:

$$K(x, u) = \begin{cases} [2u - 1, 1 + |x|], & \text{if } u < 1 \\ [1, u], & \text{if } u \geq 1 \end{cases}, \quad T(x, u) = \begin{cases} [0, 1], & \text{if } x = 0 \\ \{0\}, & \text{if } x \neq 0 \end{cases}.$$

For  $x_0 = 0, M_\psi(x_0) = [1, +\infty)$  and for  $x \neq x_0, M_\psi(x) = [0, +\infty)$ .

If in Example 3.1 we take  $T(x, u) = \begin{cases} [0, 1], & \text{if } x \neq 0 \\ \{0\}, & \text{if } x = 0 \end{cases}$ , condition (ii) of

Theorem 3.1 fails to hold for  $u_0 = 0, x_0 = 0$  and  $(x_n, u_n) = (\frac{1}{n}, \frac{1}{n})$  (we take  $t = 0$ ). It can be easily verified that  $M_\psi(x_0) = [0, +\infty)$  and for  $x \neq x_0$ ,

$M_\psi(x) = [1, +\infty)$ ; hence  $x \neq x_0$ ,  $M_\psi$  is not B-lsc at  $x_0$ . So, if the condition (ii) is not satisfied, then the conclusion of Theorem 3.1 fails to hold.

As lower semicontinuity is stronger than upper semicontinuity, the following two results give sufficient conditions ensuring the lower semicontinuity, by imposing additional conditions to those of Theorem 2.1 regarding the upper semicontinuity.

**Theorem 3.2.** *Suppose that conditions of Theorem 2.1 are satisfied and that for  $x_0 \in X$ , we have:*

(i) *for every  $u_0 \in M_\psi(x_0)$ ,  $\psi(t, u_0, v) < 0$ ,  $\forall v \in M_\psi(x_0) \setminus \{u_0\}$  and for some  $t \in T(x_0, v)$ ;*

(ii)  *$-T$  is  $\psi$ -quasimonotone on  $\{x_0\} \times Y$ .*

*Then  $M_\psi$  is B-lsc at  $x_0$ .*

**Proof.** Suppose, on the contrary, that  $M_\psi$  be not B-lsc at  $x_0$ . From Remark 1.2, there exists a sequence  $\{x_n\}$  in  $X$  with  $x_n \rightarrow x_0$  and  $u_0 \in M_\psi(x_0)$ , such that for every sequence  $y_n \in M_\psi(x_n)$ ,  $y_n \not\rightarrow u_0$ . As  $u_n \in A$  and  $A$  is a compact subset of  $Y$ , without loss of generality, we can assume that  $y_n \rightarrow u' \in A$ ,  $u' \neq u_0$ . By Theorem 2.1, it follows that  $u' \in M_\psi(x_0)$ . As  $u_0, u' \in M_\psi(x_0)$ , from (i) it follows that

$$\psi(t', u_0, u') > 0, \text{ for some } t' \in -T(x_0, u')$$

and

$$\psi(t_0, u_0, u') > 0, \text{ for some } t_0 \in -T(x_0, u_0),$$

which contradicts the  $\psi$ -quasimonotony of  $-T$ .  $\square$

If in Example 3.1 we take  $T(x, u) = \begin{cases} [0, 1], & \text{if } x \neq 0 \\ \{0\}, & \text{if } x = 0 \end{cases}$ , there are satisfied the conditions of Theorem 2.1,  $-T$  is  $\psi$ -quasimonotone on  $\{x_0\} \times Y$ , but for  $u_0 = 0$  and  $v = 1 \in M_\psi(x_0)$  it can be seen that  $\psi(t, u_0, v) = 0$  for any  $t \in T(x_0, v) = \{0\}$  and hence condition (i) of the above theorem is not satisfied. Since  $M_\psi(x_0) = [0, +\infty)$  and for  $x \neq x_0$ ,  $M_\psi(x) = [1, +\infty)$ ,  $M_\psi$  is not B-lsc at  $x_0$ . So, if the condition (i) is not satisfied, then the conclusion of Theorem 3.2 fails to hold.

By using the same technique as above, we can establish the following result:

**Theorem 3.3.** *Suppose that conditions of Theorem 2.1 are satisfied and that for  $x_0 \in X$ , we have:*

(i) *for every  $u_0 \in M_\psi(x_0)$ ,  $\psi(t, u_0, v) \leq 0$ ,  $\forall v \in M_\psi(x_0)$  and for some  $t \in T(x_0, v)$ ;*

(ii)  *$-T$  is  $\psi$ -pseudomonotone on  $\{x_0\} \times Y$ ;*

(iii)  *$\psi(t, u, v) = 0$ , for  $t \in T(x_0, u) \cup T(x_0, v) \Rightarrow u = v$ ;*

(iv)  *$\psi(t, u, v) = -\psi(t, v, u)$ ,  $\forall t, u, v \in K$ .*

*Then  $M_\psi$  is B-lsc at  $x_0$ .*

**Proof.** Suppose, on the contrary, that  $M_\psi$  be not B-lsc at  $x_0$ . From Remark 2.1, there exists a sequence  $\{x_n\}$  in  $X$  with  $x_n \rightarrow x_0$  and  $u_0 \in M_\psi(x_0)$ , such that for every sequence  $y_n \in M_\psi(x_n)$ ,  $y_n \not\rightarrow u_0$ . As  $u_n \in A$  and  $A$  is a compact subset of  $Y$ , without loss of generality, we can assume that  $y_n \rightarrow u' \in A$ ,  $u' \neq u_0$ . By Theorem 2.1, it follows that  $u' \in M_\psi(x_0)$ . As  $u_0, u' \in M_\psi(x_0)$ , from (i) it follows that

$$\psi(t', u_0, u') \geq 0, \text{ for some } t' \in -T(x_0, u') \quad (3.3)$$

and

$$\psi(t_0, u', u_0) \geq 0, \text{ for some } t_0 \in -T(x_0, u_0). \quad (3.4)$$

Since  $-T$  is  $\psi$ -pseudomonotone, it follows from (3.3) that  $\psi(t_0, u_0, u') \geq 0$  and, together with (3.4), we obtain  $\psi(t_0, u_0, u') = 0$ . By (iii) we have  $u' = u_0$ , which is a contradiction.  $\square$

If the hypothesis of Theorem 3.2 or Theorem 3.3 hold, the application  $M_\psi : X \rightarrow 2^Y$  is B-continuous at  $x_0$ .

#### 4. Continuity of the $\varepsilon$ -solutions set

In this section we extend the study of upper and lower semicontinuity to  $\varepsilon$ -solutions set and to modified  $\varepsilon$ -solutions set of problem  $(\psi MVI(x_0))$ . For a fixed  $\varepsilon \geq 0$ , we define an  $\varepsilon$ -solution of problem  $(\psi MVI(x_0))$  to be an  $u_0 \in K(x_0, u_0) \cap A$  such that  $\forall v \in K(x_0, u_0), \forall t \in T(x_0, v), \psi(t, u_0, v) \leq \varepsilon$ . The set of all  $\varepsilon$ -solutions of  $(\psi MVI(x_0))$  is denoted by  $M_\psi^\varepsilon(x_0)$ . We can easily see that if  $\varepsilon = 0$ , we have  $M_\psi^\varepsilon(x_0) = M_\psi(x_0)$ . Motivated by [13] we introduce the set of modified  $\varepsilon$ -solutions of  $(\psi MVI(x_0))$  as

$$\widetilde{M}_\psi^\varepsilon(x) = \begin{cases} M_\psi(x_0), & \text{if } x = x_0 \\ M_\psi^\varepsilon(x), & \text{if } x \neq x_0 \end{cases}.$$

The proofs of the following two theorems are analogous to those of Theorems 2.1, 2.2 respectively.

**Theorem 4.1.** *Suppose that for  $x_0 \in X$  the conditions of Theorem 2.1 hold. Then  $M_\psi^\varepsilon$  is B-usc at  $x_0$ , for any  $\varepsilon \geq 0$ . Moreover,  $M_\psi^\varepsilon(x_0)$  is compact and  $M_\psi^\varepsilon$  is closed at  $x_0$ , for any  $\varepsilon \geq 0$ .*

**Theorem 4.2.** *Suppose that for  $x_0 \in X$  the conditions (i)–(ii) and (iv) of Theorem 2.1 hold and, in addition,  $\forall u_0 \in K(x_0, u_0) \cap A, \forall (x_n, u_n) \rightarrow (x_0, u_0)$  and*

$$\psi(t_0, u_0, v_0) > \varepsilon, \text{ for some } v_0 \in K(x_0, u_0), t_0 \in T(x_0, v_0)$$

*implies that there exists a positive integer  $n$ , such that  $\psi(t, u_n, v) > \varepsilon$ , for some  $v \in K(x_n, u_n), t \in T(x_n, v)$ .*

*Then  $M_\psi^\varepsilon$  is B-usc at  $x_0$ . Moreover,  $M_\psi^\varepsilon(x_0)$  is compact and  $M_\psi^\varepsilon$  is closed at  $x_0$ .*



Next we show that if problem  $(\psi MVI(x_0))$  is well-posed, then  $M_\psi$  is B-upper semicontinuous.

**Definition 4.1.** A sequence  $\{u_n\}$  is said to be an approximating sequence for the problem  $(\psi MVI(x_0))$  iff there exists a sequence  $\{x_n\}$  in  $X$ , such that  $x_n \rightarrow x_0$  and there exists a sequence  $\{\varepsilon_n\}$  in  $\mathbb{R}$ ,  $\varepsilon_n > 0$  with  $\varepsilon_n \rightarrow 0$ , such that  $u_n \in M_\psi^{\varepsilon_n}(x_n)$ ,  $\forall n \in \mathbb{N}$ .

**Definition 4.2.** We say that the equilibrium problem  $(\psi MVI(x_0))$  is well-posed iff

- (i) the solution set  $M_\psi(x_0)$  of  $(\psi MVI(x_0))$  is nonempty;
- (ii) every approximating sequence for  $(\psi MVI(x_0))$  has a subsequence, which converges to some point of  $M_\psi(x_0)$ .

**Remark 4.1.** Well-posedness of  $(\psi MVI(x_0))$  implies that the solution set  $M_\psi(x_0)$  is a nonempty compact set.

**Theorem 4.3.** If  $(\psi MVI(x_0))$  is well-posed, then  $M_\psi$  is B-usc at  $x_0$ .

**Proof.** Suppose, on the contrary, that  $M_\psi$  be not B-usc at  $x_0$ . Then there exists an open set  $N$  containing  $M_\psi(x_0)$ , such that for every sequence  $x_n \rightarrow x_0$ , there exists  $u_n \in M_\psi(x_n)$  but  $u_n \notin N$ . As  $x_n \rightarrow x_0$  and  $u_n \in M_\psi(x_n)$ , it follows that  $\{u_n\}$  is an approximating sequence for  $(\psi MVI(x_0))$ . Since  $u_n \notin N$  and  $M_\psi(x_0) \subset N$ , none of its subsequences converge to a point of  $M_\psi(x_0)$ , thereby leading to a contradiction to the fact that  $(\psi MVI(x_0))$  is well-posed. So,  $M_\psi$  is B-usc at  $x_0$ .  $\square$

The converse of the above result may fail to hold. For the problem  $(\psi MVI(x_0))$  considered in Example 2.3, if we choose the sequences  $\{x_n\}$  and  $\{u_n\}$  as  $x_n = 1 + \frac{1}{n}$  and  $u_n = -n$  for every  $n$ , then it can be observed that  $\{u_n\}$  is an approximating sequence for the problem  $(\psi MVI(x_0))$ , but it possesses no convergent subsequence, thereby implying that  $(\psi MVI(x_0))$  is not well-posed.

Regarding semicontinuity of the modified  $\varepsilon$ -solutions set of problem  $(\psi MVI(x_0))$ , we have the following result:

**Theorem 4.4.** Suppose that for  $x_0 \in X$ , the following conditions are satisfied:

- (i)  $K$  is B-usc with compact values on  $\{x_0\} \times Y$ ;
- (ii)  $\bar{K}$  is B-lsc at  $x_0$ , where  $\bar{K}(x) = \{u \in A : u \in K(x, u)\}$ ;
- (iii)  $T$  is B-usc with compact values on  $\{x_0\} \times Y$ ;
- (iv)  $\psi(\cdot, \cdot, \cdot)$  is continuous in all the arguments.

Then  $\widetilde{M}_\psi^\varepsilon$  is B-lsc at  $x_0$ , for each  $\varepsilon > 0$ .

**Proof.** Suppose, on the contrary, that  $\widetilde{M}_\psi^\varepsilon$  be not B-lsc at  $x_0$ . Using the line of [11], there exists a sequence  $\{x_n\}$  in  $X$  converging to  $x_0$  and  $u_0 \in \widetilde{M}_\psi^\varepsilon(x_0)$ , such that for every sequence  $y_n \in \widetilde{M}_\psi^\varepsilon(x_n)$ , we have  $y_n \not\rightarrow u_0$ .

Since  $\overline{K}$  is B-lsc at  $x_0$ ,  $x_n \rightarrow x_0$  and  $u_0 \in \overline{K}(x_0)$ , there exists a sequence  $\{u_n\} \subset \overline{K}(x_n)$  converging to  $u_0$ . It follows that  $u_n \notin \widetilde{M}_\psi^\varepsilon(x_n)$ , that is

$$\psi(t_n, u_n, v_n) > \varepsilon, \text{ for some } v_n \in K(x_n, u_n), t_n \in T(x_n, v_n). \quad (4.1)$$

As  $K$  is B-usc and compact valued at  $(x_0, u_0)$  and  $v_n \in K(x_n, u_n)$ , there exists  $v_0 \in K(x_0, u_0)$ , such that  $v_n \rightarrow v_0$ . Also, since  $T$  is B-usc and compact valued at  $(x_0, v_0)$  and  $t_n \in T(x_n, v_n)$ , there exists some  $t_0 \in T(x_0, v_0)$ , such that  $t_n \rightarrow t_0$ . Taking limit as  $n \rightarrow \infty$  in relation (4.1) we have  $\psi(t_0, u_0, v_0) \geq \varepsilon$ , a contradiction to  $u_0 \in \widetilde{M}_\psi^\varepsilon(x_0)$ .  $\square$

## 5. The case of a nested optimization problem

Consider the following nested optimization problem:

$$(P) \min f(x, u) + \max_{1 \leq i \leq k} c_i(x), \text{ with } u \in M_\psi(x), x \in \widetilde{X},$$

where  $f : X \times Y \rightarrow \mathbb{R}$ ,  $M_\psi(x)$  is the solutions set of the equilibrium problem ( $\psi MVI(x)$ ) defined in Section 1,  $\widetilde{X} := \{x \in X \mid a(x) \leq 0\}$ , with  $c_i : \mathbb{R}^n \rightarrow \mathbb{R}$  for  $1 \leq i \leq k$  and  $a : \mathbb{R}^n \rightarrow \mathbb{R}^p$  being convex and continuous functions.

This problem is equivalent with

$$(P') \min g(x, u), \text{ with } u \in M_\psi(x), x \in \widetilde{X},$$

where the function  $g : X \times Y \rightarrow \mathbb{R}$  is defined such that for every  $(x, u) \in X \times Y$ , we have  $g(x, u) = f(x, u) + \max_{1 \leq i \leq k} c_i(x)$ .

We denote by  $\Omega$  the set of solutions of problem (P), that is

$$\Omega := \left\{ (x, u) \in \widetilde{X} \times Y \mid u \in K(x, u) \cap A, g(x, u) \leq \inf_{y \in \widetilde{X}, v \in M_\psi(y)} g(y, v) \right. \\ \left. \text{and } \psi(t, u, v) \leq 0, \forall v \in K(x, u), \forall t \in T(x, v) \right\}.$$

For  $\varepsilon \geq 0$ , we define a parametric form (P( $\varepsilon$ )) of the optimization problem (P), as follows:

$$(P(\varepsilon)) \min f(x, u) + \max_{1 \leq i \leq k} c_i(x), \text{ with } u \in M_\psi^\varepsilon(x), x \in \widetilde{X},$$

where  $M_\psi^\varepsilon(x)$  is the  $\varepsilon$ -solutions set of problem ( $\psi MVI(x)$ ). For  $\varepsilon = 0$ , the problem reduces to problem (P).

For  $\delta, \varepsilon \geq 0$ , define the  $\varepsilon$ -solutions set for the problem (P( $\varepsilon$ )) as

$$\Omega^\delta(\varepsilon) := \left\{ (x, u) \in \widetilde{X} \times Y \mid u \in K(x, u) \cap A, \right. \\ \left. g(x, u) \leq \inf_{y \in \widetilde{X}, v \in M_\psi(y)} g(y, v) + \delta \right. \\ \left. \text{and } \psi(t, u, v) \leq \varepsilon, \forall v \in K(x, u), \forall t \in T(x, v) \right\}.$$

This section gives sufficient conditions for continuity of  $\varepsilon$ -solutions set of problem  $(P(\varepsilon))$ , that is of the application  $\Omega^\delta : \mathbb{R}_+ \rightarrow 2^{X \times Y}$ , with  $\Omega^\delta(\varepsilon)$  being the  $\varepsilon$ -solutions set of problem  $(P(\varepsilon))$ .

We see that if  $\delta = 0$ , then  $\Omega^\delta(0) = \Omega$ . Also for any  $\delta \geq 0$ ,  $\Omega^\delta(0) \subseteq \Omega^\delta(\varepsilon)$ ,  $\forall \varepsilon \geq 0$ , from which we deduce that  $\Omega^\delta$  is B-lsc at  $\varepsilon = 0$ . Hence, to obtain the continuity of the application  $\Omega^\delta$  at  $\varepsilon = 0$ , it is enough to establish conditions for the upper semicontinuity. The conditions that ensure the upper semicontinuity of  $\Omega^\delta$  at  $\varepsilon = 0$  are similar to those given by Lalitha in [11] for the special case where  $\psi(t, u, v) = \langle t, u - v \rangle$ , for any  $u, v \in Y$ .

**Theorem 5.1.** *Suppose that the conditions of Theorem 2.1 hold and*

(i)  *$X$  is a bounded subset of  $\mathbb{R}^n$ ;*

(ii)  *$f$  is lower semicontinuous.*

*Then for every  $\delta \geq 0$ ,  $\Omega^\delta$  is B-usc at  $\varepsilon = 0$ .*

**Corollary 5.1.** *Suppose that conditions (i), (ii) and (iv) of Theorem 2.1 hold and*

(i) *there exist  $\varepsilon', \delta' > 0$  such that  $\Omega^{\delta'}(\varepsilon')$  is bounded;*

(ii)  *$f$  is lower semicontinuous.*

*Then for every  $\delta \leq \delta'$ ,  $\Omega^\delta$  is B-usc at  $\varepsilon = 0$ .*

**Theorem 5.2.** *Suppose that the conditions of Theorem 2.2 hold and*

(i)  *$X$  is a bounded subset of  $\mathbb{R}^n$ ;*

(ii)  *$f$  is lower semicontinuous.*

*Then for every  $\delta \geq 0$ ,  $\Omega^\delta$  is B-usc at  $\varepsilon = 0$ .*

**Remark 5.1.** Since  $\Omega^\delta$  is B-usc at  $\varepsilon = 0$ , if the conditions of one of Theorems 5.1 or 5.2 hold, then  $\Omega^\delta$  is B-continuous at  $\varepsilon = 0$ .

The following optimization problem satisfies the conditions of Theorem 5.1.

**Example 5.1.** Let  $X = [-2, 2]$ ,  $Y = \mathbb{R}$  and  $A = [0, 2]$ . Define  $\psi : Y \times Y \times Y \rightarrow Y$  as  $\psi(t, u, v) = t^2(u^3 - v^3)$  and the applications  $K : X \times Y \rightarrow 2^Y$  and  $T : X \times Y \rightarrow 2^Y$  as follows:  $K(x, u) = [0, |u|]$ ,

$$T(x, u) = \begin{cases} \{1\}, & \text{if } x = 0 \\ \{0, 1\}, & \text{if } x \neq 0 \end{cases}.$$

For any  $x \in X$  we have  $M_\psi(x) = \{0\}$  and  $M_\psi^\varepsilon(x) = [0, \sqrt[3]{\varepsilon}]$ .

Now consider the problem

$$\inf f(x, u) + \max_{1 \leq i \leq k} c_i(x), \text{ with } u \in M_\psi(x), x \in \tilde{X},$$

where  $f(x, u) = |x - u|$ ,  $k = 10$  and  $c_i(x) = (i - 1)x + 1$ ,  $i = \overline{1, 10}$ . Let  $M_\psi(x)$  be the solutions set of the inequality considered above and  $a(x) = x - 1$ .

It can be verified that  $\tilde{X} = [-2, 1]$ ,  $\max_{1 \leq i \leq 10} c_i(x) = \begin{cases} c_1(x), & \text{if } x \leq 0 \\ c_{10}(x), & \text{if } x > 0 \end{cases}$  and  $\Omega = \{(0, 0)\}$ . Also,

$$\begin{aligned} \Omega^\delta(\varepsilon) &:= \left\{ (x, u) \in \tilde{X} \times Y \mid u \in K(x, u) \cap A, \right. \\ &\quad \left. f(x, u) + \max_{1 \leq i \leq 10} c_i(x) \leq \right. \\ &\quad \left. \leq \inf_{y \in \tilde{X}, v \in M_\psi(y)} f(y, v) + \max_{1 \leq i \leq 10} c_i(y) + \delta, u \in [0, \sqrt[3]{\varepsilon}] \right\} \\ &= \left\{ (x, u) \in \tilde{X} \times Y \mid u \in K(x, u) \cap A, \right. \\ &\quad \left. |x - u| + \max_{1 \leq i \leq 10} c_i(x) \leq 1 + \delta, u \in [0, \sqrt[3]{\varepsilon}] \right\} \\ &= \left\{ (x, u) \in \tilde{X} \times Y \mid |x - u| + \max_{1 \leq i \leq 10} c_i(x) \leq 1 + \delta, u \in [0, \min\{\sqrt[3]{\varepsilon}, 2\}] \right\} \end{aligned}$$

and hence  $\Omega^\delta(0) = \{(x, 0) \mid |x| + \max_{1 \leq i \leq 10} c_i(x) \leq 1 + \delta, x \in \tilde{X}\}$ . Therefore  $\Omega^\delta(0) = \{(x, 0) \mid x \in [\max\{-2, -\delta\}, \min\{1, \frac{\delta}{10}\}]\}$ . It can be observed that  $\Omega^\delta$  is B-usc and hence, B-continuous at  $\varepsilon = 0$ .

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