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Convergence theorems for semigroup of asymptotically nonexpansive mappings

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Abstract - In this paper, we prove the following results: Let K be a closed convex subset of a real Banach space E. Let $\mathcal{T} := \{T(t) \,|\, t \in \mathbb{R}_+\}$ be strongly continuous semigroup of asymptotically nonexpansive mappings from K into K such that $F(\mathcal{T}) := \cap_{t\in\mathbb{R}_+} F(T(t)) \neq \emptyset$, where $F(T(t)) = \{x \in K \,|\, T(t)x = x\}$ and $\mathbb{R}_+$ denotes the set of nonnegative real numbers. Then for arbitrary $x_0 \in K$, the implicit iteration $\{x_n\}$ given by $x_n = \alpha_n x_{n-1} + (1-\alpha_n)(T(t_n))^n x_n$, $n \geq 0$ converges weakly (strongly) to an element of $F(\mathcal{T})$, where $\{\alpha_n\}$, $\{t_n\}$ are sequences of real numbers satisfying certain conditions.

Key words and phrases : Asymptotically nonexpansive semigroup, Opial's condition, implicit iteration.

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1. Introduction

Let E be a real Banach space, K be a nonempty closed convex subset of E. A mapping $T : K \to K$ is said to be

(1) nonexpansive, if

$$\|Tx - Ty\| \leq \|x - y\|$$

for all $x, y \in K$,

(2) asymptotically nonexpansive [5], if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|T^n x - T^n y\| \leq k_n \|x - y\|$$

for all $x, y \in K$ and $n \in \mathbb{N}$.

The class of asymptotically nonexpansive mapping is an important generalization of the class of nonexpansive mappings.

Construction of fixed points of nonexpansive mappings has important applications, in particular, in image recovery and in signal processing, see e.g. [2, 8]. In 1967, Browder (see [1]) proved the following result.

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Theorem 1.1. *Let K be a closed convex subset of a Hilbert space H and T be a nonexpansive mapping with its values in K with a fixed point. Let $\{\alpha_n\}$ be a sequence in $(0,1)$ converging to 0. Fix $u \in K$ and define a sequence $\{x_n\}$ by*

$$x_n = \alpha_n u + (1-\alpha_n)Tx_n\,, \ \ n \in \mathbb{N}\,.$$

Then $\{x_n\}$ converges strongly to the element of $F(T)$ nearest to u, where $F(T)$ denotes the set of fixed points of T.

Closely related to the class of nonexpansive mappings and asymptotically nonexpansive mappings are strongly continuous semigroup of nonexpansive mappings and strongly continuous semigroup of asymptotically nonexpansive mapping which are directly linked to solutions of differential equations and which have been studied by several authors see e.g. [3, 10, 11].

Let K be a closed convex subset of a Banach space E. $\mathcal{T} := \{T(t) \,|\, t \in \mathbb{R}_+\}$, where $\mathbb{R}_+$ denotes the set of nonnegative real numbers, is said to be *strongly continuous semigroup of asymptotically nonexpansive mappings* from K into K if the following conditions are satisfied:

(1) $T(0)x = x$ for all $x \in K$;

(2) $T(s+t) = T(s) \circ T(t)$ for all $s, t \in \mathbb{R}_+$;

(3) for each $t \in \mathbb{R}_+$, $T(t)$ be an asymptotically nonexpansive mapping on K,

(4) for each $x \in K$, the mapping $T(\cdot)x$ from $\mathbb{R}_+$ into K is continuous.

If in the above definition, condition (3) is replaced by the following condition:

(3)* for each $t \in \mathbb{R}_+$, $T(t)$ be an nonexpansive mapping on K,

then $\mathcal{T}$ is called strongly continuous semi-group of nonexpansive mapping on K.

Let $F(\mathcal{T}) := \cap_{t\in\mathbb{R}_+} F\left(T(t)\right) \neq \emptyset$, where $F\left(T(t)\right) = \{x \in K \,|\, T(t)x = x\}$.

Construction of common fixed points of nonexpansive semigroups is an important subject in the theory of nonexpansive semigroup mappings and its applications.

Shioji and Takahashi (see [10]), in Hilbert space, introduced the following implicit iteration

$$u_n = (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)u_n ds + \alpha_n u\,, u \in K, n \geq 1\,,$$

where $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $\lim_n \alpha_n = 0$, $t_n > 0$ and $\lim_n t_n = \infty$.

Suzuki [11] was the first who introduced an iteration, where iterate u_n at step n is constructed directly from the semigroup, i.e. from $T(t_n)$. Suzuki [11] proved the following result.

Theorem 1.2. *Let K be a closed convex subset of a Hilbert space H. Let $\{T(t) \mid t \in \mathbb{R}_+\}$ be a strongly continuous semigroup of nonexpansive mappings on K such that $F(\mathcal{T}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequences of real numbers satisfying $0 < \alpha_n < 1$, $t_n > 0$ and $\lim_n t_n = \lim_n \alpha_n/t_n = 0$. Fix $u \in K$ and define a sequence $\{u_n\}$ in K by*

$$u_n = (1-\alpha_n)T(t_n)u_n + \alpha_n u$$

for $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to the element of $F(\mathcal{T})$ nearest to u.

Xu (see [13]) established a Banach space version of the Theorem 1.2. More recently Thong (see [12]), consider an implicit iteration for strongly continuous semigroup of nonexpansive mappings and proved weak and strong convergence results.

When we study iterative schemes for asymptotically nonexpansive mappings, it looks more complicated than the schemes for nonexpansive mappings. T is not always nonexpansive (mapping which do not increase distance), i.e. T may increase distances. In order to overcome difficulties caused by increasingness of T, one need to adjust, the defining mapping at each iteration step in the iteration schemes, i.e., we have to use T^n (instead of T) at step n as the defining mapping. Though T^n may still increase distance (as $k_n \geq 1$), however since $k_n \to 1$, eventually T^n would increase distance marginally. Schu (see [9]) was first to use the above idea and defined iteration scheme for asymptotically nonexpansive mapping.

Motivated by the above works, in this paper we propose an implicit iteration for semigroup of asymptotically nonexpansive mappings and prove weak and strong convergence results in Banach spaces.

In the sequel, we will need the following:

A Banach space E is said to satisfy Opial's condition [7] if for any sequence $\{x_n\} \in E$, $x_n \rightharpoonup x$ as $n \to \infty$ implies

$$\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\| , \quad \forall\, y \in E \text{ with } x \neq y\,.$$

Lemma 1.1. ([6]) *Let the nonnegative number sequences $\{a_n\}$, $\{u_n\}$ satisfy*

$$a_{n+1} \leq (1+u_n)a_n\,, \quad n = 0,1,2,\cdots\,, \ \sum_{n=0}^{\infty} u_n < \infty\,.$$

Then $\lim_{n\to\infty} a_n$ exists.

2. Main results

We now study an implicit iteration for semigroup of asymptotically nonexpansive mappings, which generates a sequence $\{x_n\}$ implicitly by,

$$x_n = \alpha_n x_{n-1} + (1-\alpha_n)\left(T(t_n)\right)^n x_n \tag{2.1}$$

Theorem 2.1. *Let E be a real Banach space and K be a closed convex subset of E. Let $\{T(t) \mid t \geq 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings on K such that $F(\mathcal{T}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequence of real numbers satisfying*

(i) $0 < \alpha_n < 1$, $t_n > 0$, $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$

(ii) $\sum_{n=1}^{\infty}(k_n - 1) < \infty$.

Let $\{x_n\}$ be defined by (2.1). *Then*

(a) $\lim_{n\to\infty} \|x_n - p\|$ *exists for each* $p \in F(\mathcal{T})$.

(b) $\|T(t_n)x_n\|$ *is bounded.*

Proof. For each $p \in F(\mathcal{T})$, we have

$$\begin{aligned} \|x_n - p\| &= \|\alpha_n(x_{n-1} - p) + (1-\alpha_n)((T(t_n))^n x_n - p)\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n) \|(T(t_n))^n x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n)k_n \|x_n - p\| \\ &\leq \alpha_n \|x_{n-1} - p\| + (1-\alpha_n+\mu_n) \|x_n - p\| \end{aligned}$$

where $\mu_n = k_n - 1$, and by (ii) we have, $\sum_{n=1}^{\infty} \mu_n < \infty$. After simplification, above inequality gives

$$\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{\mu_n}{\alpha_n} \|x_n - p\|$$

From condition $\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{\alpha_n}{t_n} = 0$, we have $\lim_{n\to\infty} \alpha_n = 0$. It follows that, for $b \in (0,1)$, there exists a positive integer n_0 such that $\alpha_n \in (0, b]$ for all $n \geq n_0$. So we have

$$\|x_n - p\| \leq \|x_{n-1} - p\| + \frac{\mu_n}{1-b} \|x_n - p\|$$

i.e.

$$\begin{aligned} \|x_n - p\| &\leq \frac{1-b}{1-b-\mu_n} \|x_{n-1} - p\| \\ &= \left(1 + \frac{\mu_n}{1-b-\mu_n}\right) \|x_{n-1} - p\| \end{aligned}$$

since $\sum_{n=1}^{\infty} \mu_n < \infty$, we have $\mu_n \to 0$, so there exists n_0 such that $\mu_n \leq \frac{1-b}{2}$ for all $n \geq n_0$, so

$$\|x_n - p\| \leq \left(1 + \frac{2\mu_n}{1-b}\right) \|x_{n-1} - p\|$$

so $\lim_{n\to\infty}\|x_n-p\|$ exists and hence $\{x_n\}$ is bounded.
By (2.1), we have

$$\begin{aligned}\|(T(t_n))^n x_n\| &= \left\|\frac{1}{1-\alpha_n}\,x_n-\frac{\alpha_n}{1-\alpha_n}\,x_{n-1}\right\|\\ &\le \frac{1}{1-\alpha_n}\,\|x_n\|+\frac{\alpha_n}{1-\alpha_n}\,\|x_{n-1}\|\\ &\le \frac{1}{1-b}\,\|x_n\|+\frac{b}{1-b}\,\|x_{n-1}\|\end{aligned}$$

which implies that $\|T(t_n)x_n\|$ is bounded. □

2.1. Weak convergence theorem

Theorem 2.2. *Let E be a real reflexive Banach space satisfying Opial's condition and K be a nonempty closed convex subset of E. Let $\{T(t)\,|\,t\ge 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings on K such that $F(\mathcal{T})\ne\emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequence of real numbers as in Theorem* 2.1. *Then the $\{x_n\}$ defined by* (2.1) *converges weakly to an element of $F(\mathcal{T})$.*

Proof. By Theorem 2.1, for each $p\in F(\mathcal{T})$ the $\lim_{n\to\infty}\|x_n-p\|$ exists and $\{x_n\}$, $\{\|T(t_n)x_n\|\}$ are bounded. Since E is reflexive, K is closed and convex, and $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ which converges weakly to $x^*\in K$. We prove $x^*\in F(\mathcal{T})$. Put $u_j=x_{n_j}$, $\beta_j=\alpha_{n_j}$, $s_j=t_{n_j}$ for $j\in\mathbb{N}$. Fix $t>0$.
Since

$$\begin{aligned}\left\|u_j-(T(t))^j x^*\right\| &\le \sum_{k=0}^{\left[\frac{t}{s_j}\right]-1}\left\|(T((k+1)s_j))^j u_j-(T(ks_j))^j u_j\right\|\\ &\quad+\left\|\left(T\left(\left[\frac{t}{s_j}\right]s_j\right)\right)^j u_j-\left(T\left(\left[\frac{t}{s_j}\right]s_j\right)\right)^j x^*\right\|\\ &\quad+\left\|\left(T\left(\left[\frac{t}{s_j}\right]s_j\right)\right)^j x^*-(T(t))^j x^*\right\|\\ &\le \left[\frac{t}{s_j}\right]k_j\left\|(T(s_j))^j u_j-u_j\right\|+k_j\|u_j-x^*\|\\ &\quad+k_j\left\|\left(T\left(t-\left[\frac{t}{s_n}\right]s_j\right)\right)^j x^*-x^*\right\|\\ &= tk_j\left(\frac{\beta_j}{s_j}\right)\left\|(T(s_j))^j u_j-u_{j-1}\right\|+k_j\|u_j-x^*\|\\ &\quad+k_j\max\left\{\left\|(T(s))^j x^*-x^*\right\|\,|\,0\le s\le s_j\right\}\end{aligned}$$

for all $j \in \mathbb{N}$, we have

$$\limsup_{j\to\infty} \left\| u_j - (T(t))^j x^* \right\| \le \limsup_{j\to\infty} \|u_j - x^*\| .$$

By Opial's condition this implies that $\lim_{j\to\infty} (T(t))^j x^* = x^*$. Also

$$x^* = \lim_{j\to\infty} (T(t))^j x^* = \lim_{j\to\infty} (T(t))^{j+1} x^* = T(t) \left(\lim_{j\to\infty} (T(t))^j x^* \right) = T(t)x^*.$$

Therefore $x^* \in F(\mathcal{T})$. We now prove that $\{x_n\}$ converging weakly to x^*. Suppose that there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ converges weakly to $\bar{x}$ and $x^* \neq \bar{x}$. In a similar manner we can also show that $\bar{x} \in F(\mathcal{T})$. Further both limits $\lim_{n\to\infty} \|x_n - x^*\|$ and $\lim_{n\to\infty} \|x_n - \bar{x}\|$ exists.
We have

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - x^*\| &= \limsup_{j\to\infty} \left\|x_{n_j} - x^*\right\| < \limsup_{j\to\infty} \left\|x_{n_j} - \bar{x}\right\| \\
&= \lim_{n\to\infty} \|x_n - \bar{x}\| = \limsup_{i\to\infty} \|x_{n_i} - \bar{x}\| \\
&< \limsup_{i\to\infty} \|x_{n_i} - x^*\| = \lim_{n\to\infty} \|x_n - x^*\| .
\end{aligned}$$

A contradiction, hence $x^* = \bar{x}$ and hence $x_n \rightharpoonup x^*$.
This completes the proof. □

2.2. Strong convergence theorem

Theorem 2.3. *Let E be a real Banach space and K be a nonempty compact convex subset of E. Let $\{T(t) \,|\, t \ge 0\}$ be a strongly continuous semigroup of asymptotically nonexpansive mappings on K such that $F(\mathcal{T}) \neq \emptyset$. Let $\{\alpha_n\}$ and $\{t_n\}$ be sequence of real numbers as in Theorem* 2.1*. Then the sequence $\{x_n\}$ defined by* (2.1) *converges strongly to an element of $F(\mathcal{T})$.*

Proof. By Theorem 2.1, for each $p \in F(\mathcal{T})$ the $\lim_{n\to\infty} \|x_n - p\|$ exists and $\{x_n\}$, $\{\|T(t_n)x_n\|\}$ are bounded. By (2.1), we have

$$\|x_n - (T(t_n))^n x_n\| = \alpha_n \|x_{n-1} - (T(t_n))^n x_n\| \to 0 \text{ as } n \to \infty \tag{2.2}$$

Next we show that $\lim_{n\to\infty} \|x_n - (T(t))^n x_n\| = 0$ for each $t > 0$.
For all $n \in \mathbb{N}$, we have

$$\begin{aligned}
\|x_n - (T(t))^n x_n\| \le & \sum_{k=0}^{\left[\frac{t}{t_n}\right]-1} \|(T((k+1)t_n))^n x_n - (T(kt_n))^n x_n\| \\
& + \left\| \left(T\left(\left[\frac{t}{t_n}\right] t_n \right) \right)^n x_n - (T(t))^n x_n \right\|
\end{aligned}$$

$$\leq \left[\frac{t}{t_n}\right] k_n \left\|(T(t_n))^n x_n - x_n\right\|$$
$$+ k_n \left\|\left(T\left(t - \left[\frac{t}{s_n}\right] t_n\right)\right)^n x_n - x_n\right\|$$
$$= t k_n \left(\frac{\alpha_n}{t_n}\right) \left\|(T(t_n))^n x_n - x_{n-1}\right\|$$
$$+ k_n \max\left\{\left\|(T(s))^n x_n - x_n\right\| \mid 0 \leq s \leq t_n\right\}$$

therefore

$$\lim_{n\to\infty} \left\|x_n - (T(t))^n x_n\right\| = 0\,. \tag{2.3}$$

We now show that $\{x_n\}$ converges strongly to an element of $F(\mathcal{T})$.
The compactness of K suffices for the existence of a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that $x_{n_j} \to x^* \in K$.
Fix $t > 0$. By continuity of the mapping $T(t)$ and the norm $\|\cdot\|$, together with (2.3), we have

$$\left\|x^* - T(t)^j x^*\right\| \leq \left\|x_{n_j} - x^*\right\| + \left\|x_{n_j} - (T(t))^j x_{n_j}\right\|$$
$$+ \left\|(T(t))^j x_{n_j} - (T(t))^j x^*\right\|$$
$$\leq (1 + k_j) \left\|x_{n_j} - x^*\right\| + \left\|x_{n_j} - (T(t))^j x_{n_j}\right\|.$$

Letting $j \to \infty$, we get

$$\lim_{j\to\infty} (T(t))^j x^* = x^*\,.$$

This gives that

$$x^* = \lim_{j\to\infty} (T(t))^j x^* = \lim_{j\to\infty} (T(t))^{j+1} x^* = T(t)\left(\lim_{j\to\infty} (T(t))^j x^*\right) = T(t)x^*,$$

hence $x^* \in F(\mathcal{T})$. Because $\lim_{n\to\infty} \|x_n - p\|$ exists for all $p \in F(\mathcal{T})$ by Theorem 2.1, we obtain

$$\lim_{n\to\infty} \|x_n - x^*\| = \lim_{j\to\infty} \left\|x_{n_j} - x^*\right\| = 0.$$

This completes the proof. □

Remark 2.1. Theorem 2.2 and Theorem 2.3 generalize main result of Thong [12] for larger class of strongly continuous semigroup of asymptotically non-expansive mappings.

Remark 2.2. Theorem 2.2 and Theorem 2.3 generalize main results of [4], [10] and [11] in the sense that our result is applicable to more general class of strongly continuous semigroup of asymptotically noexpansive mappings and the more general implicit iteration as well as to more general Banach spaces.

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